MA575
MWF 9-9:50pm
CB 343
Fall 2006

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Due Friday, 22 September 2006

1. (Beals p. 33) Suppose that $x_1 = 1$ and $x_{n+1} = 1 + 1/x_n$. Prove that the sequence converges and find its limit.

2. (Beals p. 35) For each of the following sequences $\{x_n\}_1^\infty$ determine $y_n = l.u.b.\{x_k : k \geq n\}$ and $z_n = g.l.b.\{x_k : k \geq n\}$ and then find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

   (a) $x_n = (-1)^n + \frac{1}{n}$
   (b) $x_n = (-1)^n - \frac{1}{n}$
   (c) $x_n = (-1)^n(1 + \frac{1}{n})$

3. (Rudin, page 81) Fix a positive number α and choose $x_1 > \sqrt{\alpha}$. For $n \geq 2$, define x_n by the recursion formula
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

   (a) Prove that x_n decreases monotonically and that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.
   (b) Put $\epsilon_n = x_n - \sqrt{\alpha}$ and show that
$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}, \qquad n = 1, 2 \ldots.$$
   (c) Show that we have
$$\epsilon_{n+1} < 2\sqrt{\alpha}\left(\frac{\epsilon_1}{2\sqrt{\alpha}}\right)^{2^n}.$$
   (d) The above procedure allows us to approximate $\sqrt{\alpha}$ quickly and efficiently. Use the above results to give an estimate for the errors ϵ_4 and ϵ_5 when $\alpha = 2$ and $x_1 = 3/2$. The best solutions will not require the use of a calculator.

4. Suppose that $\{a_n\}$ and $\{b_n\}$ are bounded real sequences.

   (a) Prove that $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
   (b) Show by example that strict inequality may occur in the result in part (a).
   (c) Is it possible to have strict inequality if one of the sequences, say $\{a_n\}$ is convergent?

5. (Beals p. 38) Supppose that $\{z_n\}$ is a complex sequence with limit z. Define w_n to be the arithmetic mean of the first n terms in the sequence z_n, $w_n = \frac{1}{n}\sum_{j=1}^{n} z_j$. Show that $\lim_{n\to\infty} w_n = z$. Hint: You may reduce to the case $z = 0$ by replacing z_n by $z_n - z$.

6. Can you find a sequence $\{z_n\}$ which is not convergent, but so that the sequence of means $\{w_n\}$ defined in the previous exercise are convergent?

Additional questions.(Not to be collected.)

1. Show that if z and w are complex numbers, then $||z| - |w|| \leq |z - w|$.

2. If z_n is a convergent sequence, can one prove that $|z_n|$ is a convergent sequence?

3. If z_n^2 is a convergent sequence, can one prove that z_n is a convergent sequence?