MA575
MWF 9-9:50pm
CB 343
Fall 2006

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Due Wednesday, 25 October 2006. (Corrected.)

1. (Beals p. 66) Determine the sum of the series

$$\sum_{n=0}^{\infty}(n+1)(n+2)z^n.$$

2. (Beals, p. 66) Prove that for any positive integer k,

$$\sum_{n=0}^{\infty}\binom{n+k}{n}z^n = \frac{1}{(1-z)^{k+1}}, \qquad \text{if } |z|<1.$$

(For the first two problems, you may assume that you know the derivatives of $1/(1-z)^k$.)

3. (Beals p. 66) Suppose that the function f is defined by a convergent power series and suppose that $f(z+w) = f(z)f(w)$ for all z and w in $\mathbf{C}$.
   (a) Prove directly from this assumption that there is a constant $a \in \mathbf{C}$ so that $f'(z) = af(z)$. In fact, $a = f'(0)$.
   (b) Use part (a) to prove that $f(z) = \sum_{n=0}^{\infty}\frac{(az)^n}{n!}$, provided $a \neq 0$.

4. Find all power series $f(z)$ with positive radius of convergence and which satisfy $f'(z) + 2zf(z) = 0$ and $f(0) = 1$. Hint: Find the derivative $f'(0)$ and then find a recursion relation for the coefficients of the power series for f. Can you identify the function f from its power series?

5. (Beals, p. 70) Prove that e is irrational. If $s_n = \sum_{k=0}^{n}\frac{1}{k!}$, then

$$0 < e - s_n < \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right) = \frac{1}{n!n}.$$

If e were a rational number, p/q, then $q!e$ and $q!s_q$ would be distinct integers. Hint: For this result, I think we need to use a proof by contradiction.

6. (U. Minnesota, written exam in topology, 1983) Determine if each of the following functions is a metric on $\mathbf{R}$.
   (a) $d(x,y) = \dfrac{|x-y|}{1+|x-y|}$

(b) $\delta(x, y) = \sqrt{|x - y|}$

(c) $D(x, y) = |x - y|^2$

Additional questions

1. Suppose f is a continuous function defined on the complex plane and $f(z + w) = f(z)f(w)$ for all z and w in $\mathbf{C}$. Do we have $f(z) = \exp(az)$ for some a in $\mathbf{C}$?

2. Suppose f is a function defined on the complex plane and $f(z + w) = f(z)f(w)$ for all z and w in $\mathbf{C}$. Do we have $f(z) = \exp(az)$ for some a in $\mathbf{C}$?

October 23, 2006