MA575
MWF 9-9:50pm
CB 343
Fall 2006

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Homework 6. Due Monday, 6 November 2006.

1. (Beals p. 66) Define a function $d_p(x, y)$ on $\mathbf{R}^n \times \mathbf{R}^n$ by

$$d_p(x, y) = (\sum_{j=1}^{n} |x_j - y_j|^p)^{1/p}.$$

   If we identify points $x = (x_1, x_2)$ in $\mathbf{R}^2$ with complex numbers by $x = x_1 + ix_2$, then it is easy to see that $d_2(x, y) = |x - y|$ and thus, d_2 is a metric (Beals, Chapter 2D, Proposition 2.14 or Lecture 5, Proposition 4).

   (a) Prove that d_1 is a metric on $\mathbf{R}^2$.

   (b) Find constants $A > 0$ and $B > 0$ so that

$$A\, d_1(x, y) \leq d_2(x, y) \leq B\, d_1(x, y).$$

2. Let d_1 and d_2 be two metrics defined on a set X. We say that d_2 is the master of d_1 if for each $\epsilon > 0$ there is an $\eta > 0$ so that for all x and y in X, $d_2(x, y) < \eta$ implies $d_1(x, y) < \epsilon$.

   If d_1 is the master of d_2 and d_2 is the master of d_1, then we say the d_2 and d_1 are equivalent metrics.

   (a) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and suppose that d_2 is the master of d_1. If $\{x_n\}$ converges in the metric space (X, d_2), show that $\{x_n\}$ converges in the metric space (X, d_1).

   (b) Show that the standard metric on $\mathbf{R}$, $d(x, y) = |x - y|$ and the metric $d'(x, y) = |x - y|/(1 + |x - y|)$ are equivalent.

3. Let x be a point in a metric space X and $r > 0$. Prove that the set $F = \{y : d(x, y) \leq r\}$ is closed and that the set $G = \{y : d(x, y) > r\}$ is open.

4. (Beals, p. 81) Prove that the union of two compact sets is again compact.

5. (Beals, p. 81) Prove that a compact set in a metric space is, like, totally bounded.

6. Let A be a subset of a metric space (X, d). We may define distance from a point $x \in X$ to the set A by

$$d_A(x) = \inf\{d(x, y) : y \in A\}.$$

   Prove that $cl(A) = \{x : d_A(x) = 0\}$.

7. Suppose that a sequence $\{x_n\}$ has limit x. Prove that the set $\{x, x_1, x_2, \ldots\}$ is compact.

8. Let A be a subset of a metric space. Do we have $(int(A^c))^c = cl(A)$?

9. (Beals, p. 84) Let $\{x_n\}$ be a sequence in metric space and suppose that $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ is finite, prove that the sequence $\{x_n\}$ is Cauchy.

10. (Beals, p. 84) (Banach fixed point theorem) Suppose that X is a metric space. A function $f : X \to X$ is a *contraction* if there is constant θ with $0 \leq \theta < 1$ so that

$$d(f(x), f(y)) \leq \theta d(x, y), \qquad x, y \in X.$$

Show that if X is complete, then a contraction has a unique fixed point. A fixed point is a solution of the equation $f(x) = x$. Hint: Define a sequence by choosing x_0 and then putting $x_{n+1} = f(x_n)$.

Additional questions

1. If $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ is the sequence $\{x_n\}$ a Cauchy sequence?

2. Let $\{x_n\}$ be a sequence with $x_n < 0$ for every n. If $\lim_{n\to\infty} x_n = x$, must we have $x < 0$?

3. Prove that the metric d_p defined in problem 1, is a metric if $1 \leq p < \infty$.

4. For the metrics d_p defined in problem 1, find $d_\infty(x, y) = \lim_{p\to\infty} d_p(x, y)$. Show that d_∞ is a metric.

5. If $z \in F$ and F is a closed set in a metric space, is z a limit point of F?

6. If F is a finite subset of a metric space, is F closed?

October 27, 2006