MA575 | Instructor: Russell Brown
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Due Friday, 1 December 2006.

1. Let
$$f_n(x) = nx^2(1-x).$$
Does the sequence f_n converge uniformly for x in the unit interval $[0,1]$?

2. (Beals p. 94) Define a sequence of polynomials by $P_0(x) = 0$ and
$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n(x)^2}{2}, \qquad n = 0, 1, 2, \ldots$$
Prove that this sequence converges uniformly to $|x|$ on the interval $[-1, 1]$.

3. Let $f : A \to \mathbf{R}$ be real-valued function defined on a subset A of a metric space X and suppose that $f(x) \geq 0$ for all x in A. Let a be a limit point of A and suppose that $\lim_{x\to a} f(x)$ exists. Prove that $\lim_{x\to a} f(x) \geq 0$.

   If we have $f(x) > 0$ for all x, can we show $\lim_{x\to a} f(x) > 0$?

4. Consider the function
$$f(x) = \begin{cases} x^2 \sin(1/x^a) & x > 0 \\ 0 & x \leq 0 \end{cases}$$
For which values of $a > 0$ is f differentiable at 0?

   For which values of $a > 0$ is f' continuous at 0?

   (For this problem, you may assume standard properties of the sine function.)

5. Let $f : \mathbf{R} \to \mathbf{R}$ have two continuous derivatives. Find
$$\lim_{h\to 0} \frac{f(x-h) + f(x+h) - 2f(x)}{h^2}.$$

6. Let $k \in \mathbf{N}$ and find $\lim_{x\to\infty} \exp(x)/x^k$ and $\lim_{x\to\infty} x^k \exp(-x)$. Hint: Use the definition of $\exp(x)$ as a power series to help find the first limit.

7. Let
$$f(x) = \begin{cases} \exp(-1/x), & x > 0 \\ 0, & x \leq 0 \end{cases}$$
   (a) Show that for $x > 0$, the kth derivative $f^{(k)}(x) = P_k(\frac{1}{x}) \exp(-1/x)$ where P_k is a polynomial in one variable. Give the degree of the polynomial P_k.

(b) Show that $f^{(k)}(0)$ exists and is zero for all $k \in \mathbf{N}$.

8. (Beals p. 104) Suppose that f is a real-valued function on the real line and f' exists everywhere and is a bounded function. Prove that f is uniformly continuous.

9. Prove that every polynomial with real coefficients and of odd degree has a real root.

10. Suppose that $f : [-1, 1] \to \mathbf{R}$ is continuous and that $f''(x)$ exists for all x in $(-1, 1)$. Show that there is a number c in $(-1, 1)$ so that $f''(c) = f(1) + f(-1) - 2f(0)$. Hint: Find a quadratic polynomial $p(x)$ so that $p(j) = f(j)$ for $j = -1, 0, 1$. Let $g(x) = f(x) - p(x)$. Show $g''(c) = 0$ for some c.

Additional problems.

1. (Beals, p. 94) Which of the following sequences converge uniformly on the interval $[0, 1]$?
   (a) $f_n(x) = nx^2(1 - x)$
   (b) $f_n(x) = n^2x(1 - x^2)^n$
   (c) $f_n(x) = n^2x^3e^{-nx^2}$
   (d) $f_n(x) = \dfrac{x^2}{x^2 + (1 - nx)^2}$

2. (Beals p. 107) Suppose that f is a continuous real-valued function on $\mathbf{R}$ and suppose that $f(x)$ is a rational number whenever x is irrational. Must f be constant?

3. (Adam Keach) Let $\phi : [0, \infty) \to [0, \infty)$ be increasing and $\phi(x) = 0$ if and only if $x = 0$. Let d be a metric. Is $\delta(x, y) = \phi(d(x, y))$ a metric?

   Can you find necessary and sufficient conditions on ϕ which guarantee that δ is a metric whenever d is a metric?

4. Does the mean value theorem hold if f is complex valued? ‘

November 27, 2006