Theory of partial differential equations
MWF 9-9:50am
CB343
Spring 2013

Instructor: Russell Brown
Office: POT741
Phone: 257-3951
russell.brown@uky.edu
Office Hours: MWF 10-11
and by appointment.

Homework 1, due on Wednesday, 16 January 2013

1. Suppose that $X$ and $Y$ are normed linear spaces and that $T : X \to Y$ is a linear map. Show that $T$ is continuous if and only if there is a constant $C$ so that $\|Tx\|_Y \leq C\|x\|_X$.

   Is this true if $T$ is not linear?

2. Can you find a (real) linear map $T : \mathbf{R}^n \to \mathbf{R}$ that is not continuous?

3. Let $\phi : [0, 1] \to [0, 1)$ be a continuous function with $\phi(0) = 0$ and $\phi(x) > 0$ if $x > 0$. Suppose that $U = ((-1, 0) \times (-1, 1)) \cup ([0, 1) \times (-1, 0)) \cup V$ where $V = \{(x, y) : 0 \leq x < 1, \phi(x) < y < 1\}$.

   Fix $\beta > 0$ and let $f(x, y)) = \begin{cases} x^{1+\beta}, & (x, y) \in V \\ 0, & U \setminus V \end{cases}$

   (a) Show that $f \in C^1(\bar{U})$.

   (b) If $f$ is in $C^{0,\alpha}(\bar{U})$ for some $\alpha > 0$, what does this tell us about $\phi$?

   (c) Conclude that the inclusion $C^1(\bar{U}) \subset C^{0,\alpha}(\bar{U})$ may fail for any $\alpha$ in $(0, 1]$.

4. If $U \subset \mathbf{R}^n$ is convex, bounded, and open, show that $C^1(\bar{U}) \subset C^{0,1}(\bar{U})$.

5. Speculate as to what assumption on $U$ implies the inclusion

$$C^1(\bar{U}) \subset C^{0,1}(\bar{U}).$$

   According to problem (3), we have this containment in a convex domain. Can you give a more general assumption that also gives this containment?

6. Let $f(x) = (2 - \log|x|)^\alpha$ in $B(0, 1) \subset \mathbf{R}^n$.

   (a) Show that for $n \geq 2$, the ordinary derivative of $f$ in $B(0, 1) \setminus \{0\}$ coincides with the weak derivative.

   (b) For which $\alpha$ is $f$ in $W^{1,n}(B(0, 1))$?

Week 1.

- Banach spaces, Evans, Appendix D, page 719.

- Hölder spaces, Evans, §5.1
- Weak derivatives, Sobolev spaces, §5.2

January 14, 2013