Theory of partial differential equations
MWF 9-9:50am
CB343
Spring 2013

Instructor: Russell Brown
Office: POT741
Phone: 257-3951
russell.brown@uky.edu
Office Hours: MWF 10-11
and by appointment.

Homework 4, due on Monday, 18 February

1. The goal of this exercise is to give a simpler proof of the Poincaré inequality of Lecture 7, Theorem 5. As a benefit, we see that the Poincaré inequality holds in certain domains of infinite measure.

   Suppose that U is an open set, $U \subset \{x : 0 < x_1 < d\}$, and $1 \leq p < \infty$. Prove that for $u \in W_0^{1,p}(U)$, we have

   $$\|u\|_{L^p(U)} \leq Cd\|Du\|_{L^p(U)}.$$

   The constant C should not depend on d. Show how C depends on n and p.

   Hint: Assume u is smooth and use the fundamental theorem of calculus.

2. Suppose that $k : U \times U \to \mathbf{R}$ is measurable and we have

   $$\sup_{x \in U} \int_U |k(x,y)|\, dy \leq M_0, \qquad \sup_{y \in U} \int_U |k(x,y)|\, dx \leq M_1.$$

   Define a linear operator by

   $$Kf(x) = \int_U k(x,y)f(y)\, dy.$$

   Prove that

   $$\|Kf\|_{L^p(U)} \leq M_0^{1/p'} M_1^{1/p} \|f\|_{L^p(U)}.$$

   Hint: Imitate the proof of Young's convolution inequality.

3. Assume U is an open set of finite measure. Use the representation formula of Lecture 8, Lemma 1 and the previous problem to find yet another proof of the Poincaré inequality for functions in $W_0^{1,p}(U)$, $1 \leq p < \infty$.

4. Use the room and passages construction to give an example of a bounded open set where the inclusion $W^{1,1}(U) \subset L^1(U)$ is not compact.

Reading for February 11–15, 2013

- §5.6, Sobolev embedding, Morrey's inequality
- §5.7, Compactness
- §5.8.1, 5.8.2b, Further topics

February 7, 2013