Theory of partial differential equations
MWF 9-9:50am
CB343
Spring 2013

Instructor: Russell Brown
Office: POT741
Phone: 257-3951
russell.brown@uky.edu
Office Hours: MWF 10-11
and by appointment.

Test 1.
Do not discuss these questions with anyone but your instructor. Give unambiguous citations if you consult a written source.
Due Friday, 8 March 2013. But late papers will be accepted until 15 March 2013.

1. Fix $p$ with $1 < p < \infty$. Suppose that $\{u_k\}_{k=1}^\infty$ is a sequence in $W^{1,p}(U)$, that $\{u_k\}$ converges weakly in $L^p(U)$ to $u$ and for $j = 1, \ldots, n$, $\{D_j u_k\}_{k=1}^\infty$ converges weakly in $L^p(U)$ to $f_j$. Show that $u$ is weakly differentiable and $D_j u = f_j$.

2. Let $\{f_k\}$ be a bounded sequence in $L^2([0,1])$ and suppose that $\{f_k\}$ converges weakly to $f$. If we also have that $\lim_{k\to\infty} \|f_k\|_{L^2([0,1])} = \|f\|_{L^2([0,1])}$, show that $\{f_k\}$ converges to $f$ in $L^2([0,1])$. Hint: Use the FOIL method.

3. Evans, p. 308, #18.

4. Let $\eta$ be a smooth function supported in the unit ball with $\int \eta\, dx = 1$, let $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$, and let $u_\epsilon = \eta_\epsilon * u$.

   (a) If $u \in C^{0,\beta}(\mathbf{R}^n)$ with $0 < \beta < 1$. If $\alpha$ is a multi-index with $|\alpha| \geq 1$, show that we have
   $$\sup_{x\in\mathbf{R}^n} |D^\alpha u_\epsilon(x)| \leq C\epsilon^{\beta-|\alpha|}[u]_{\beta;\mathbf{R}^n}$$
   and
   $$\sup_{x\in\mathbf{R}^n} |u(x) - u_\epsilon(x)| \leq C\epsilon^{\beta}[u]_{\beta;\mathbf{R}^n}.$$

   (b) Conversely, suppose that $u$ is a function and for each $\epsilon > 0$ we have $u = u_\epsilon + u^\epsilon$ with
   $$\begin{aligned} \sup |Du_\epsilon(x)| &\leq A\epsilon^{\beta-1} \\ \sup |u^\epsilon(x)| \leq A\epsilon^{\beta}. \end{aligned}$$
   Show that $[u]_{\beta;\mathbf{R}^n}$ is finite and $[u]_{\beta;\mathbf{R}^n} \leq CA$.

5. Let $U$ be a bounded, connected, open subset of $\mathbf{R}^n$. What can you say about the dual of $W_0^{1,p}(U)$?

March 4, 2013