MA676
MWF 2-2:50pm
CB 347
Spring 2007

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Exercise set 5.

1. If Z is of measure zero in $\mathbf{R}$, what can you say about the measure of $\mathbf{R} \times Z$?

2. (Stein, p. 42) (Borel-Cantelli lemma) Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable sequence of measurable subsets of $\mathbf{R}^n$. Let $E = \limsup_{k\to\infty} E_k = \cap_{j=1}^{\infty}(\cup_{k=j}^{\infty} E_k)$.

   (a) Show that E is measurable.

   (b) If
   $$\sum_{k=1}^{\infty} |E_k| < \infty.$$
   Prove that $|E| = 0$.

3. Can you find a sequence of measurable sets $E_1 \supset E_2 \supset E_3 \ldots$ so that we do not have
   $$\lim_{j\to\infty} |E_j| = |E|?$$
   Here, $E = \cap_{j=1}^{\infty} E_j$.

4. Prove that if E_1 and E_2 are measurable, then
   $$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|.$$
   How does this generalize to three sets?

5. (Rudin) (Challenging) Can you find a σ-algebra which consists of a countably infinite number of sets?

6. (Wheeden and Zygmund) Define the inner measure of a set E by
   $$|E|_i = \sup\{|F| : F \subset E,\ F \text{ is closed}\}.$$
   Show that for a set E with $|E|_e < \infty$, E is measurable if and only if $|E|_e = |E|_i$.

Problem set 5.

The problems will be due on Wednesday, 21 February 2007.

1. (Stein, p. 41) Suppose that $A \subset B \subset C$ and that A and C are measurable. If $|A| = |C| < \infty$, show that B is measurable.

2. Let $\mathcal{C}$ be the σ-algebra generated by the closed sets. Show that $\mathcal{C} = \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra.

3. If $E_j \subset \mathbf{R}$ is Lebesgue measurable for $j = 1, 2$ show that the set $E_1 \times E_2$ is Lebesgue measurable in $\mathbf{R}^2$. Hint: Use that a set E is of measurable if and only if $E = H \cup Z$ where H is an F_σ.

May 1, 2007