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Exercise set 7.

1. Let G be an open set in $\mathbf{R}^n$. Is the indicator function χ_G upper or lower semi-continuous on $\mathbf{R}^n$?

2. Eventually, we will learn how to compute the Lebesgue measure of the unit circle. For the moment, let $\pi = |D|$ where $D = \{x : x \in \mathbf{R}^2,\ |x| \leq 1\}$. Use our results on linear transformations to find the area of an ellipse, $\{x \in \mathbf{R}^2 : x_1^2/a^2 + x_2^2/b^2 \leq 1\}$.

3. (Repeated from set 6.) Let f be the Cantor-Lebesgue function.
   (a) Show that $f(C) = [0,1]$ where C is the Cantor set.
   (b) Find a set $A \subset C$ so that $f(A) = N$, where the set N is a non-measurable set of the unit interval.
   (c) Why is A measurable?

4. Let $f : \mathbf{R} \to \mathbf{R}$ be measurable. Suppose that the derivative $f'(x)$ exists for a.e. $x \in \mathbf{R}$. Show f' is measurable.

5. (Hard?) Let $f : \mathbf{R} \to \mathbf{R}$ and suppose that the derivative $f'(x)$ exists for a.e. $x \in \mathbf{R}$. Show f is Lebesgue measurable.

6. (Hard?) Let $f : \mathbf{R} \to \mathbf{R}$ be Lebesgue measurable and let E be the set where the derivative exists. Is E Lebesgue measurable?

Problem set 7.

These problems should be handed in on Friday, 9 March 2007.

1. (Stein) Let $E \subset \mathbf{R}^n$ be a set with $|E|_e > 0$. Let $0 < \alpha < 1$. Show that there is an interval I so that $|E \cap I|_e \geq \alpha |I|_e$. Hints: We may assume that $|E|_e < \infty$. Find an open set G so that $G \supset E$ and $|E|_e \geq \alpha |G|$. Write G as a union of non-overlapping intervals.

2. (Stein) Let E be a measurable subset of $\mathbf{R}$ with $|E| > 0$. Follow the following outline to show that the set of differences $E - E = \{x - y : x \in E,\ y \in E\}$ contains an interval.

(a) According to the previous problem, there is an interval I so that $|I \cap E| \geq \frac{3}{4}|I|$. Let $E_0 = E \cap I$. Let $a \in \mathbf{R}$ and consider E_0 and $E_0 + a$. These sets lie in $I \cup (I + a)$. Use the additivity of measure to show that these sets cannot be disjoint if $|a|$ is small. You should be able to find $\beta > 0$ so that the the sets are not disjoint if $|a| < \beta|I|$.

(b) Conclude that $E - E$ contains an interval.

March 5, 2007, corrected