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Homework 11.

The following problems will serve as our final exam. They will be due on 13 December 2007. Students should work these problems without discussing them with their fellow students.

1. (Wheeden and Zygmund, page 192 #24) Let $(\mathcal{S}, \Sigma, \mu)$ be a σ-finite measure space and let f be in $L^1(\mathcal{S}, \mu)$. Let Σ_0 be a σ-algebra with $\Sigma_0 \subset \Sigma$ let μ_0 denote the restriction of μ to Σ_0. Assume that $(\mathcal{S}, \Sigma_0, \mu_0)$ is σ-finite.[1] Note that, in general, we cannot expect that f is Σ_0-measurable.

   Show that there is a unique function f_0 in $L^1(\mathcal{S}, \mu_0)$ which is Σ_0-measurable and so that
   $$\int_{\mathcal{S}} fg\, d\mu = \int_{\mathcal{S}} f_0 g\, d\mu$$
   whenever g is a Σ_0-measurable function and the integral on the right is finite.

   The function f_0 is called the conditional expectation of f given Σ_0 and is written as $E(f|\Sigma_0)$. Hint: Consider the additive set function defined for $E \in \Sigma_0$ by $\phi(E) = \int_E f\, d\mu$.

2. Let $\mathcal{L}$ be the Lebesgue σ-algebra on $\mathbf{R}$ and let $\mathcal{A}$ be the σ-algebra generated by the intervals $\{[k, k+1) : k \in \mathbf{Z}\}$. Find $E(f|\mathcal{A})$.

3. (A first step towards polar coordinates in $\mathbf{R}^n$). Let ϕ be a continuous function on $(0, \infty)$. Define a function on $\mathbf{R}^n$ by $\Phi(x) = \phi(|x|)$.

   Let $0 < a < b < \infty$ and show that
   $$\int_{a<|x|<b} \Phi(x) = n\Omega_n \int_a^b \phi(t) t^{n-1}\, dt.$$
   Here, Ω_n is the volume of the unit ball in $\mathbf{R}^n$ (see homework 2).

   Hint: Use Lecture 45, Lemma 8 (Lemma 9.14) or the proof of this Lemma.

4. For α in $\mathbf{R}$ and $R > 0$ find the value of the integral
   $$\int_{|x|<R} |x|^\alpha\, dx.$$
   You may leave your answer in terms of Ω_n.

[1] This assumption does not appear in Wheeden and Zygmund. Is it necessary?

5. (Wheeden and Zygmund, p. 261, #4) Suppose f and g are in $L^2((0,2\pi))$ and are periodic of period 2π and let $S[f] = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ and $S[g] = \sum_{k=-\infty}^{\infty} d_k e^{ikx}$. We define the convolution of f and g by

$$f * g(x) = \frac{1}{2\pi} \int_{(0,2\pi)} f(x-t)g(t)\, dt.$$

Show that $S[f * g] = \sum_{k=-\infty}^{\infty} c_k d_k e^{ikx}$.

Show that $f * g(x) = \sum_{k=-\infty}^{\infty} c_k d_k e^{ikx}$ where the series converges in $L^2((0,2\pi))$.

6. (Wheeden and Zygmund, p. 261, #6) Let $f(x) = \frac{1}{2}(\pi - x)$ for $0 < x < 2\pi$ and assume that f is periodic of period 2π. Find the Fourier series for f and show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Use the previous problem to find the value of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

December 3, 2007