MA677 | Instructor: Russell Brown
MWF 10-10:50pm | Office: POT741
CB 345 | Phone: 859 257 3951
Fall 2007 | russell.brown@uky.edu

The following problems will be due on 24 October.

1. Let $g_n(x) = n\chi_{[0,n^{-2}]}(x)$ for $n = 1, 2, 3, \ldots$ as elements of $L^2([0, 1])$. Show that $\|g_n\|_2 = 1$ and that the sequence $\{g_n\}$ converges weakly to 0.

2. (a) If g is in L^p and f is in $L^{p'}$, show that $f * g$ is continuous on the real line.

   (b) If A and B are sets, let $A + B = \{z : z = x + y \text{ with } x \in A \text{ and } y \in B\}$. Show that if $|A| > 0$ and $|B| > 0$, then $A + B$ contains an interval.

3. (a) Throughout this problem, f and g are measurable. Define

$$\|f\|_{L^p(ds/s)} = \left(\int_0^\infty |f(s)|^p \frac{ds}{s} \right)^{1/p}$$

   and put $\|f\|_{L^\infty(ds/s)} = \text{ess-sup}|f|$.
   Let $1 \le p \le \infty$. Prove the following variant of Hölder's ineqality

$$\left| \int_0^\infty f(t)g(t)\frac{dt}{t} \right| \le \|f\|_{L^p(ds/s)}\|g\|_{L^{p'}(ds/s)}.$$

   (b) We may define a convolution operation on the multiplicative group $(0, \infty)$ by

$$f \star g(x) = \int_0^\infty f(t/s)g(s)\frac{ds}{s}.$$

   Show that for $1 \le p < \infty$, we have

$$\|f \star g\|_{L^p(ds/s)} \le \|f\|_{L^p(ds/s)}\|g\|_{L^1(ds/s)}.$$

   Remark: Of course, this is also true if $p = \infty$. However, we omit $p = \infty$ to shorten the homework assignment.

   (c) Let $1 < p < \infty$. If

$$Hf(t) = \frac{1}{t}\int_0^t f(s)ds,$$

   show that

$$\|Hf\|_{L^p((0,\infty))} \le C_p\|f\|_{L^p((0,\infty))}.$$

   Hint: Write

$$t^{1/p}Hf(t) = \int_0^t (s/t)^{1-1/p}s^{1/p}f(s)\,ds/s$$

   use part a).

(d) Show that there is no constant C so that we have

$$\|Hf\|_1 \leq C\|f\|_1$$

for all functions f in $L^1(\mathbf{R})$.

4. (Wheeden and Zygmund, p. 160) Let U and W be bounded open sets in $\mathbf{R}^n$ and suppose that $\bar{U} \subset W$. Show that we may find a function ϕ in $C_c^\infty(\mathbf{R}^n)$ so that $\phi(x) = 1$ for $x \in U$ and $\operatorname{supp}\phi \subset W$. Hint: Find a third open set V with $\bar{V} \subset W$ and $\bar{U} \subset V$ and let $\phi = K * \chi_V$ for an appropriate K. Why does such a set V exist?

5. (Wheeden and Zygmund, p. 160) Suppose f is in $L^2(\mathbf{R})$ and consider

$$u(x,y) = \frac{1}{\pi}\int_{\mathbf{R}} f(t)\frac{y}{y^2+(x-t)^2}\,dt$$

for (x, y) with $y > 0$. Provide a careful proof that $D_x u(x,y)$ exists.

October 25, 2007