MA677
MWF 9-9:50pm
CB 343
Fall 2011

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Exam

Work all of the following problems. You may ask me about these problems and refer to other sources. However, you should not discuss the exam with anyone but the instructor. The problems will be due on Friday, 14 October 2011.

1. For a measurable set E, We define the $L^p(E)$ norm by

$$\|f\|_p = \left(\int_E |f(x)|^p \, dx\right)^{1/p}.$$

   We define $L^p(E)$ to be the collection of measurable functions on E for which $\|f\|_p < \infty$. As usual, we will identify functions which are equal almost everywhere.

   Let the indices p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and $q > 1$. Prove Hölder's inequality: If $f \in L^p$ and $g \in L^q$

$$\int fg \, dx \leq \|f\|_p \|g\|_q.$$

   Hint: Use Young's inequality from the first homework assignment and imitate the proof of the Cauchy-Schwarz inequality.

2. Observe that if $p > 1$, then $|f+g|^p \leq |f||f+g|^{p-1} + |g||f+g|^{p-1}$. Use this and Hölder's inequality to prove the triangle inequality for $\|\cdot\|_p$ for $p > 1$. The triangle inequality for $\|\cdot\|_p$ is usually called Minkowski's inequality.

3. Is $L^p(\mathbf{R}^d)$ complete for $1 \leq p < \infty$?

4. If f is measurable and extended real valued, we define the essential supremum by

$$\text{ess sup} f = \sup\{\lambda : m(\{f > \lambda\}) > 0\}.$$

   (a) If $f(x) = 1$, for $x \in \mathbf{Q}$ and $f(x) = 0$ otherwise, find

$$\text{ess sup} f.$$

   (b) If f is continuous and real-valued, show that

$$\text{ess sup} f = \sup f.$$

(c) If $f : [0,1] \to [0,\infty]$ is measurable, show that

$$\lim_{p\to\infty} \|f\|_p = \text{ess sup}\,|f|.$$

Hint: Using Hölder's inequality we can show that $\|f\|_p$ is increasing. To get a lower bound, observe that if $\lambda < \text{ess sup}\,|f|$, then the set $\{|f| > \lambda\}$ is of positive measure. Use this to estimate $\|f\|_p$ from below.

The previous exercise motivates the definition of L^∞. We define L^∞ to be the collection of measurable functions for which the norm $\|f\|_\infty = \text{ess sup}\,|f|$ is finite.

Note that endpoint version of Hölder's inequality

$$\int fg\,dx \leq \|f\|_1 \|g\|_\infty$$

holds and is easy to prove. Too easy to be an exam question.

5. For $0 < p, q < \infty$, when do we have $L^p([0,1]) \subset L^q([0,1])$?

October 5, 2011