This set of problems will serve as our final exam. Please work on these problems individually and see me if you have questions.

1. Consider the collection of real-valued functions f where f is absolutely continuous on $[0,1]$, $f(0) = f(1) = 0$ and f' is in $L^2([0,1])$. Let $\mathcal{H}$ denote this collection and for f and g in $\mathcal{H}$, let

$$\langle f, g\rangle = \int_0^1 f'(t)g'(t)\,dt.$$

Set $\|f\| = \langle f, f\rangle^{1/2}$.

(a) Prove that for each $x \in [0,1]$,

$$\sup_{x\in[0,1]} |f(x)| \leq \|f\|.$$

Hint: Since f is absolutely continuous on $[0,1]$, we can write $f(x) = f(0) + \int_0^x f'(t)\,dt$.

(b) Prove $\mathcal{H}$ is a Hilbert space over $\mathbf{R}$. That is prove that $\langle\ ,\ \rangle$ is an inner product and that $\mathcal{H}$ is complete in the norm given by this inner product. Hint: If a sequence $\{f_k\}$ is Cauchy in this Hilbert space, then $\{f_k'\}$ is Cauchy in $L^2([0,1])$.

(c) Consider

$$I(f) = \int_0^1 \frac{1}{2}(|f'(x)|^2 + p(x)f(x)^2) - f(x)F(x)\,dx$$

where $p \in L^\infty([0,1])$ and is nonnegative and F is in L^1.
Prove that

$$I(f) \geq \frac{1}{2}\|f\|^2 - \|F\|_{L^1}\|f\|.$$

Conclude that $I(f)$ is bounded below on $\mathcal{H}$.

(d) Let f_i be a sequence in $\mathcal{H}$ for which

$$\lim_{i\to\infty} I(f_i) = \inf_{f\in\mathcal{H}} I(f).$$

Prove $\{f_i\}$ converges in $\mathcal{H}$.
Hint: Begin with

$$\frac{1}{2}\|f_k - f_j\|^2 \leq \frac{1}{2}\int_0^1 |f_k' - f_j'|^2 + p(f_k - f_j)^2\,dx.$$

Then, imitate the proof of Lemma 4.1 on p. 175 of Stein and Shakarchi.

(e) Let u be the limit of the sequence $\{f_i\}$ from the previous part and show that

$$I(u) = \inf_{f \in \mathcal{H}} I(f).$$

(f) Compute

$$\left.\frac{d}{dt}\right|_{t=0} I(u + t\phi), \qquad \phi \in \mathcal{H}.$$

Why is this derivative 0?

Hint: This is easy.

(g) Using your answer to the previous question, show that if u is as above and u has two continuous derivatives, then

$$-u'' + pu = F.$$

(h) Show that if u has two continuous derivatives and satisfies

$$-u'' + pu = F,$$

then

$$\int_0^1 u'\phi' + pu\phi - F\phi \, dx = 0, \qquad \phi \in \mathcal{H}.$$

Remarks: If we assume that p and F are bounded and continuous, then it can be shown that the solution u constructed above is twice continuously differentiable. And thus that u is a solution of the ordinary differential equation in the traditional sense.

The real power of this method is that it extends easily to equations in higher dimensions and to equations with variable coefficients which cannot be solved by explicit formulas.