Calculus II | Russell Brown
Project 2 | April 26, 1999

We will begin work on Thursday April 1 and hand in the final report on Friday, April 9. You should plan on spending approximately one recitation working on this project. I expect that you will also need to spend some time outside of class working on your report.

The goal of this project is to approximate π using the series for $\tan^{-1}$. Along the way we will illustrate that a bit of cleverness can greatly speed up the evaluation of a function. The first part looks at two different ways to evaluate $\ln 2$. The remainder of the project gives a procedure for evaluating π. The most straightforward approach leads to a series that converges slowly. Clever use of trigonometric identities allows us to find a series which converges more rapidly.

As always, you should work in groups of 1 to 4 students. Hand in one solution for the group. All students should participate in the solution. See the instructor, Brown, if you have any questions about administrative details. See Brown or your TA for questions about math.

If you are curious about other approaches to computing π, point your web browser at `http://www.mathsoft.com/asolve/constant/constant.html` (URL checked on 3/27/1999).

1. Recall that
$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}.$$
Use the partial sums of this series,
$$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}x^k}{k}.$$
Approximate $\ln 2$ in two different ways.

   (a) Use S_n with $x = 1$ and $n = 5$ to find $\ln 2$.

   (b) Use S_n with $x = -1/2$ and $n = 5$ to find $\ln 1/2 = -\ln 2$.

   Which method gives the more accurate answer? Speculate as to why. Note, that we could estimate the difference between partial sum S_n and the sum of the series by using the alternating series test (if $x > 0$) or the integral or comparison test (if $x < 0$). However, you do not need to do this here. You will be asked to carry out similar estimates below.

2. Find the series for $\tan^{-1} x$. To do this, begin with the geometric series for $1/(1-u)$. Substitute $u = -x^2$ to find the series for $1/(1+x^2)$. Finally, recall that if we restrict the domain of $\tan x$ to $-\pi/2 < x < \pi/2$, then the inverse function is given by

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\, dt.$$

For which x does the series for $\tan^{-1} x$ converge?

3. Perhaps the simplest way to approximate π is to note that $\pi = 4\tan^{-1} 1$ and use the partial sums of the series we found in part 3.[1]

Use ten (nonzero) terms of the series to estimate π. How many decimal places are correct?

4. Recall the addition formula for the tangent:

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

Let $\alpha = \tan^{-1} 1/5$ and set $\beta = 4\alpha - \pi/4$. Use the addition formula for the tangent to show

$$\tan \beta = 1/239.$$

Hint: Be persistent, you will need to use the addition formula three times.

Prove that β is in the range of $\tan^{-1}$. That is, prove that $-\pi/2 < \beta < \pi/2$.

[1] Sharp eyed students will note that our theorem on term-wise integration, Theorem 2 on page 604, does not imply that $\tan^{-1}(1)$ is given by the MacLaurin series. Theorem 2 does not discuss what happens at the endpoints of the interval of convergence. However, it is true that the MacLaurin series with $x = 1$ gives $\tan^{-1} 1$. The same holds in part 1. when we used the series for $\ln(1+x)$ at an endpoint of its interval of convergence.

5. Show that we have:

$$\begin{aligned}\pi &= 16\alpha - 4\beta \\ &= 16\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)5^{2k-1}} - 4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)239^{2k-1}}.\end{aligned}$$

6. We may approximate π by considering partial sums of the above series. Define

$$A_n = 16\sum_{k=1}^{n}\frac{(-1)^{k+1}}{(2k-1)5^{2k-1}}, \qquad B_n = 4\sum_{k=1}^{n}\frac{(-1)^{k+1}}{(2k-1)239^{2k-1}}.$$

   Compute $A_3 - B_1$. How many correct decimal places of π does this give?

7. Use the error estimate for sums of alternating series (see page 591 of Stewart) to find n_1 so that

$$|A_{n_1} - 16\tan^{-1}1/5| \leq 10^{-8}$$

   and to find n_2 so that

$$|B_{n_2} - 4\tan^{-1}1/239| \leq 10^{-8}.$$

   For these values we have $|(A_{n_1} - B_{n_2}) - \pi| < 2\cdot 10^{-8}$. Use these values of n_1 and n_2 to compute an approximate value for π.

Thanks to Jerry Muir and K. Stromberg's text, *Classical real analysis*, for the above argument for approximating π.