MA114:003–005 Russell Brown
Project 1 January 22, 1999

The goal of this project is to establish an amusing formula relating π to an infinite product, which is known as Wallis's product. (Of course this infinite product has to be defined as a limit.) The formula is:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots. \tag{1}$$

The right-hand side of this expression can also be written as

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{(2k)^2}{(2k-1)(2k+1)}.$$

Though we have not used the symbol $\prod$ before, you should be able to guess what it means, once I tell you that $\prod$ is related to products in the same way that $\sum$ is related to sums.

Your assignment is to carry out the steps below in order to see why (1) is true. The tools you will use are integration by parts (which is probably the most useful technique for studying integrals) and the comparison properties of integrals on page 304.

You should work in groups of 1 to 3 students. Each group will hand in one report for the group and all group members will receive the same grade. Each of you should participate in preparing the report and on the front of your paper please write "All group members participated in the preparation of this project". Each group member should sign this statement indicating they agree with this statement.

In your writeup, you should not just give the computations without explanation. The best papers will probably not give every detail of every step of the computation but will give a sentence or two describing what is being done and then the result of several steps. When you use results from the text (or from other sources) be sure to give a precise reference by page and equation number.

You should feel free to ask for help–tell us where you got stuck, and we will try to point you in the right direction. You are also free to consult with your fellow students. However, you should not copy directly from another paper, obviously.

You will spend most of recitation on Jan 28 working on this project. The report is due at 10am Friday, 5 February.

1. Derive the reduction formula for cos, #74 inside the back cover of Stewart. Hint: What I really wanted to do was ask you to derive the reduction formula for sin, #73. Unfortunately, this is done for you in section 5.6. You should imitate the argument used in example 6 p. 406 to establish the reduction formula for cos.

2. Use the reduction formula for sin to show that for $n = 2, 3, \ldots,$

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx.$$

3. Show that for $n = 0, 1, 2, \ldots$

$$\int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{2n}{2n+1}$$

and

$$\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n}.$$

Hint: Use induction.

4. Divide the two formulae in part 3 and show

$$\frac{\pi}{2} \frac{\int_0^{\pi/2} \sin^{2n+1} x \, dx}{\int_0^{\pi/2} \sin^{2n} x \, dx} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}.$$

5. To finish, we need to show that

$$\lim_{n \to \infty} \frac{\int_0^{\pi/2} \sin^{2n+1} x \, dx}{\int_0^{\pi/2} \sin^{2n} x \, dx} = 1. \tag{2}$$

To do this, we begin with the inequalities

$$\sin^{2n+2} x \leq \sin^{2n+1} x \leq \sin^{2n} x, \qquad n = 0, 1, 2 \ldots.$$

For which x are these true? Why? Using these inequalities and part 2, show that

$$f(n) \leq \frac{\int_0^{\pi/2} \sin^{2n+1} x \, dx}{\int_0^{\pi/2} \sin^{2n} x \, dx} \leq h(n)$$

where

$$\lim_{n \to \infty} f(n) = \lim_{n \to \infty} h(n) = 1.$$

Now the squeeze theorem tells us that (2) and hence (1) holds. In your report, you should find explicit formulae for the functions f and h.

6. (Extra credit, but no paper will earn more than the maximum score of 20.) Who was Wallis and how did this product arise?

The above is adapted from the book *Calculus*, by Michael Spivak. This is a wonderful book to look at, in your copious spare time.