Calculus II — Russell Brown
Review 3 — April 7, 1999

This test will cover sections 8.1 through 8.9.
In order to be ready for the exam, you should know how to find the MacLaurin series for e^x, $\sin x$, $\cos x$. Using the binomial theorem and integrating, you should be able to find the MacLaurin series $\tan^{-1} x$, $\ln(1+x)$ and $\sin^{-1} x$.

1. Find the limit of the sequences below. For each sequence, assume that $n = 1, 2, 3, \ldots$.
   (a) $a_n = 1/n$,
   (b) $a_n = n^2/(n+1)$
   (c) $a_n = ne^{-n}$
2. For each of the following series, determine if it is convergent or divergent. If the series is convergent, find the sum of the series. Justify your answer.
   (a) $\sum_{n=1}^{\infty} \cos(\pi(n+1)) - \cos((\pi n))$
   (b) $\sum_{n=1}^{\infty} 2^{-n}$
   (c) $\sum_{n=1}^{\infty} 3^n$
   (d) $\sum_{n=1}^{\infty} \frac{1}{n}$
3. Test for convergence. Your answer should show which test you use and how the test is applied.
   (a) $\sum_{n=1}^{\infty} n2^{-n}$
   (b) $\sum_{n=1}^{\infty} (-1)^n/\sqrt{n}$
   (c) $\sum_{n=1}^{\infty} \frac{1}{n!}$
   (d) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$
4. Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$
   Use the integral test to find N so that
$$\left| \sum_{n=1}^{N} \frac{1}{n^{3/2}} - \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \right| \leq 10^{-4}.$$
5. Use the integral test to show that the series
$$\sum_{n=1}^{\infty} \frac{1}{n}$$
   diverges.

6. (a) State the monotonic sequence theorem.
   (b) State the integral test.
   (c) State the alternating series test.
   (d) State the ratio test.
   (e) State Taylor's inequality.
   (f) State the binomial theorem.
   (g) State the theorem on differentiation and integration of power series.

7. Express the repeating decimal
$$21.323232\ldots$$
as a fraction.

8. Give the radius and interval of convergence for the series
   (a) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$
   (b) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$
   (c) $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2}$

9. Find the MacLaurin series at 0 for each of the following functions. Give the interval where the series converges. Be sure to check the endpoints.
   (a) $\dfrac{1}{1+x^2}$
   (b) $\tan^{-1} x$
   (c) $\dfrac{1}{2-x}$
   (d) $\dfrac{1}{\sqrt[3]{1-x^2}}$
   (e) $\dfrac{1}{(1+x)^3}$
   (f) $\sin x^2$
   (g) $\cos x$
   (h) $\dfrac{e^x-1}{x}$.

10. (a) Find the MacLaurin series for e^{-x^2}.
    (b) Use your answer to part a) to approximate
$$\int_0^1 e^{-x^2}\,dx$$
    to within 0.005. Be sure to prove that your answer is correct.

11. Use Taylor's inequality to find the values of x for which

$$|\sin x - x| \leq 0.5.$$

12. Find the indefinite integral as a series.

$$\int \frac{1}{1+x^4}\, dx.$$

For which x will the series converge?

13. Compute the first few derivatives of $f(x)$ where

$$f(x) = (1+x)^{-2/3}.$$

Guess a formula for the nth derivative, $f^{(n)}(x)$. Use the principle of mathematical induction to prove that your answer is correct. Use your formula for the nth derivative to write the MacLaurin series for $f(x)$ at $x = 0$. Check your answer by comparing with the binomial theorem.

14. Consider the sequence given by $a_1 = 2$ and $a_{n+1} = 1/(3 - a_n)$. Use mathematical induction to prove that $0 < a_n$ for all $n = 1, 2, 3, \ldots$ and that $a_n < a_{n+1}$ for $n = 1, 2, 3, \ldots$.

Find the limiting value of the sequence a_n.