Counting project MA162

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Consider the problem of counting the number of outcomes of a complex operation. This hard counting problem is solved by breaking the complex operation up into simpler steps. If we can give the number of outcomes for each simpler step, then we use the multiplication principle to compute the number of outcomes of the complex operation as the product of the number of outcomes for each of the simpler steps. One of the most important applications of the multiplication principle is the derivation of the formula for the number of permutations of n things taken k at a time: $P(n,k) = n!/(n-k)!$. Let us recall how to do this by an example.

Example. How many three letter words can we form from a set of ten letters if repeated letters are not allowed.

Solution. We break the complex operation of choosing three letters into the simpler operations of choosing individual letters. The first letter may be chosen in 10 ways, the second in 9 ways, the third in 8 ways. After each choice, the number of letters available decreases by one because we are not allowing repeated letters. Thus the total number of words is $10 \times 9 \times 8 = 720$. Of course, this agrees with the formula $P(10,3) = 10!/7!$.

In this project, we will apply this strategy to counting poker hands. In each case, our goal is to find a sequence of operations which produce all the hands of a given type and do not produce a hand more than once. In all of the problems below, assume that you are working with a standard deck of 52 cards. This deck is divided into four suits, and each suit has 13 "numbers" 2-10, J, Q, K, A. A poker hand contains five cards. Thus there are a total of $C(52,5) = 2,598,960$ hands.

Example. How many hands contain two pairs?

Solution. We make a table listing one sequence of operations that will produce all poker hands with two pairs.

Operation	Number of outcomes
Choose 2 of 13 suits	C(13,2)
Choose 2 of 4 cards in first suit	C(4,2)
Choose 2 of 4 cards in second suit	C(4,2)
Choose 1 of the 44 cards which are not in the chosen suits	C(44,1)
TOTAL	123,552

Notice that by choosing the suits first and then choosing the last card from the remaining 44 cards, we guarantee that we do not draw a four of a kind or a full house.

Now, we try a much easier example.

Example. How many royal flushes are there? A royal flush is the cards 10, J, Q, K, A in the same suit.

Solution. Four. Once you choose the suit, the hand is completely determined.

Now practice applying the multiplication principle to the problems below. In your answers, you should give the simple operations that you carry out to produce all the hands of a specified type. Try to explain any particular tricky points.

1. Count the number of full houses. A full house consits of two cards from one number and three from another.
2. Count the number of hand which contain a pair. Do not include hands which contain three of a kind or two pairs.
3. Count the number of straight flushes. A straight flush consists of five cards in order from the same suit. For this problem, you may assume that the ace, A, may be high or low.
4. Count the number of straights which are not straight flushes. A straight consists of five cards in order.
5. Count the number of flushes which are not straight flushes. A flush consists of five cards from the same suit.

Work in a group of 2 to 5 to answer these questions. After you have answered these questions, write up your solutions in an orderly fashion. Turn in your solutions on 22 November 1995 at the beginning of lecture. You should include the names and section numbers of each member of your group. Note that the answers to most of these questions may be found in the text or in a book about card games. In the text, look at problems 50–59 in section 5.3 and at the table on page 324. Since you know the answers to these questions, your grade will be determined, primarily, by how clearly you present your solution and explain your reasoning.

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