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Review I

Brief Answers

1a. Graph the two lines in R^2. The point of intersection is (2,1). I am omitting the row-reduction since it is easy.

b. The row-reduced echelon form of the augmented system

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

This says $x_1 = 2$, $x_2 = 1$ and x_3 is free. This is the equation of a vertical line passing through the point (2, 1, 0).
Notice the original system is to find the intersection of two hyperplanes in R^3.

2. a. Row-reduced form is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right] \begin{array}{l} \\ R2 - R1 \\ R3 - R1 \end{array}$$

and row-reduced echelon is the same!!

b. Pivots in columns 1 and 3. Free variables x_2 and x_4 corresponding
to columns 2 and 4.

c. No solution: Row 3 says $0 = 2$.

d. Homogeneous system immediately has row-reduced echelon form

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution is $x_1 = -x_2 - x_4$
x_2 is free
$x_3 = -2x_4$
x_4 is free

Solution in parametric vector form:

$$a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

where a and b are real parameters.

3. The system row-reduces to

$$\left[\begin{array}{cc|c} 2 & 3 & k \\ 0 & h-12 & 2-4k \end{array}\right]$$

When $h = 12$ and $k = 1/2$, this has infinitely-many solutions.
When $h = 12$ and k not equal to $1/2$, this has no solution.
Unique solution when h is not 12 and k is any real number.

4 a. x_1 is free
$x_2 = - x_3 - x_6 + 1$
x_3 is free
$x_4 = - x_6$
$x_5 = 1$
x_6 is free

(many ways to write this...)

b. x_1 is free
$x_2 = - x_3 - x_6$
x_3 is free
$x_4 = - x_6$
$x_5 = 0$
x_6 is free

5. Any set of more than 3 vectors that span R^3 will not be linearly independent. Take

$$\{e_1, e_2, e_3, v\}$$

where $v = (1,2,3)^T$. Clearly $v = 1 e_1 + 2 e_2 + 3 e_3$, so they are dependent. Also span $\{e_1, e_2, e_3\} = R^3$.

6. Row reduce to

$$\begin{bmatrix} 1 & a \\ 0 & a^2 - a - 2 \end{bmatrix}$$

The matrix equation $Ax = 0$ has only the trivial solution if the above has two pivots. This is when $a^2 - a - 2 = (a-2)(a+1)$

is not equal to zero, that is, a not 2 or -1.

7. You can do this the long way... or realize it is a partitioned matrix:

$$\begin{bmatrix} B & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & C \end{bmatrix}$$

where $C = B^T$, the transpose of B. We know

$$B^{-1} = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}$$

and the inverse of C is the transpose of the inverse of B, that is,

$$C^{-1} = (B^{-1})^T = \begin{bmatrix} -4 & 1 \\ 9 & -2 \end{bmatrix}$$

So the inverse of A is

$$\begin{bmatrix} -4 & 9 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 9 & -2 \end{bmatrix}$$

7. Let R be the rotation by $\pi/2$ radians counterclockwise and H be the reflection through the horizontal axis $x_2 = 0$.

a. $T(e_1) = H(R(e_1) = H(0,1) = (0, -1)$
$T(e_2) = H(R(e_2) = H(-1,0) = (-1, 0)$

so the standard matrix is

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

b. T is 1-1 if and only if $T(v) = 0$ implies $v = (0,0)$, the zero vector. $T(x,y) = (-y, -x) = (0,0)$ implies $x=0$ and $y=0$, so T is 1-1.

c. T is onto since the matrix A has two pivots (exchange two rows) and hence spans R^2.

d. $T(T(x,y)) = T(-y,-x) = (-(-x), -(-y)) = (x,y)$, that is, T applied twice brings you back to the same location.

9. a. linear.

b. linear
c. linear

I will prove linearity for part a. Parts b & c are similar.

T is linear with respect to vector addition:

T((x,y,z) + (x',y',z')) = T(x+x', y+y', z+z') def'n of vector addition
= (y + y', z+z', z+z') def'n of T
= (y,z,z) + (y',z',z') def'n of vector add'n
= T(x,y,z) + T(x',y',z') def'n of T

Let c be any scalar. Then
T(c(x,y,z)) = T(cx,cy,cz) def'n of scalar multiple of a vector
= (cy, cz, cz) def'n of T
= c(y,z,z) property of scalar multiple of a vector
= c T(x,y,z) def'n of T

So T is linear with respect to scalar multiplication of any vector.
Hence T is a linear transformation!

d. not linear:
For example T(2,4,4) = (2, 4, 8)
and T(1,2,2) = (1,1,1), but

2 T(1,2,2) = 2 (1,1,1) is not equal to T(2 (1, 2, 2)) = T(2,4,4)
= (2,4,8),

that is,
2 T(1,2,2) neq T(2,4,4), so
T is not linear!

10. diag{2,2,2,2}
b. Since B^T B = 2 I, we have

1/2 B^T B = I,

that is,
the inverse of B is 1/2 of the transpose of B.

c. [0 0 1 0]
[0 0 0 1]
[-1 0 0 0]
[0 -1 0 0]

11. (u + v) + w = ((u_1, ..., u_n) + (v_1, ..., v_n)) + (w_1, ..., w_n)
(just plugging in the vectors)
Now add: = (u_1 + v_1, ..., u_n + v_n) + (w_1, ..., w_n)
Continue to add
= ((u_1 + v_1) + w_1, ..., (u_n + v_n) + w_n)

Now use associativity of addition for the reals

$$= (u_1 + (v_1 + w_1), \ldots, u_n + (v_n + w_n))$$

Now break apart using def'n of vector addition

$$= (u_1, \ldots, u_n) + ((v_1 + w_1), \ldots, (v_n + w_n))$$

Def'n of vector addition.

$$= u + (v + w)$$

12a. FALSE. (Why?... Find a counterexample or prove it...)
  b. TRUE (Why?... prove it...)
  c. FALSE (You should be able to find a counterexample.)
  d. TRUE (Why?... prove it....)

13.
$$L = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & -1/5 & 1 \end{bmatrix}$$
and
$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

14a. Dilates by a factor of 4.
  b. Contracts by a factor of 1/10.
  c. Reflection over the line $y=x$.
  d. Skew transformation.
  e. Reflection over the hyperplane $y=x$.

15a. $T(3v) = 3\,T(v) = [6\ 3]^T$
  b. $T(4w) = 4\,T(w) = [-4\ 12]^T$
  c. $T(3v - 4w) = 3\,T(v) - 4\,T(w) = [10\ -9]^T$

16. Try!

17a.
$$A = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}$$

$$B = \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

$$AB = \begin{bmatrix} (\cos a)(\cos b) - (\sin a)(\sin b) & (-\cos a)(\sin b) - (\sin a \cos b) \\ (\sin a)(\cos b) + (\cos a)(\sin b) & (-\sin a)(\sin b) + (\cos a \cos b) \end{bmatrix}$$

b.

$$C = \begin{bmatrix} \cos(a+b) & -\sin(a+b) \\ \sin(a+b) & \cos(a+b) \end{bmatrix}$$

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 c.  Yes, AB = C.
     Each entry is an angle-sum formula:
     The (1,1) and (2,2) entries give
         cos(a+b) = cos(a)cos(b) - sin(a)sin(b)
     The (1,2) and (2,1) entries give
         sin(a+b) = cos(a)sin(b) + sin(a)cos(b))
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