Professor Readdy
Math 322
Fall 2024

Review for Exam I (§1.1 through §1.9 and §2.1 through §2.5, omit §1.6) [1]

1. **Geometric and algebraic viewpoint of solving a linear system**
   (a) Solve the following system both algebraically and geometrically:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 - x_2 &= 1 \end{aligned}$$

   (b) Repeat with the system

$$\begin{aligned} x_1 + 2x_2 + 0x_3 &= 4 \\ x_1 - x_2 + 0x_3 &= 1 \end{aligned}$$

2. **Elementary row operations, row echelon form and row reduced echelon form**

   (a) Row reduce the following augmented matrix to row echelon form and then row reduced echelon form. Be sure to label all of your steps!

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 3 & 3 \\ 1 & 1 & 0 & 1 & 4 \end{array}\right].$$

   (b) Identity the pivots and free variables, if any.
   (c) Determine if the system has a solution. If so, give it.
   (d) Solve the homogeneous system. Write your solution in parametric vector form.

3. **More systems**
   Consider the following system of linear equations:

$$\begin{aligned} 2x_1 + 3x_2 &= k \\ 8x_1 + hx_2 &= 2 \end{aligned}$$

   Determine all values of h and k so that the solution set of the system

   (a) is empty.

   (b) contains a unique solution.

   (c) contains infinitely-many solutions.

[1] The exam is Friday, October 18, 2024 9:00 am - 9:50 am in our classroom.

4. **Row equivalence and solving homogeneous system**

   (a) Describe all solutions to $A\mathbf{x} = \mathbf{b}$ where this matrix equation is row equivalent to the following augmented matrix:

$$\left[\begin{array}{cccccc|c} 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

   (b) Find all solutions to $A\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector.

5. **Span**

   Give a set of non-zero vectors which span $\mathbb{R}^3$ but which are not linearly independent.

6. **Linear independence**

   Determine the value(s) of a such that the columns of the matrix A are linearly independent.

$$A = \begin{bmatrix} 1 & a \\ a & a+2 \end{bmatrix}.$$

7. **Inverse of the matrix**

$$A = \begin{bmatrix} 2 & 9 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 9 & 4 \end{bmatrix}.$$

8. **Standard matrices, geometric transformations, 1-1, onto**

   Let T be the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which first rotates points $\pi/2$ radians counterclockwise, then reflects points through the horizontal axis $x_2 = 0$.

   (a) Find the standard matrix of T.
   (b) Is T 1-1? Explain why or why not.
   (c) Is T onto? Explain why or why not.
   (d) Describe geometrically what happens after applying the transformation T twice.

9. **Linear transformations**

   Determine whether each of the linear transformations is linear or not.

   (a) $T(x, y, z) = (y, z, z)$
   (b) $T(x, y, z) = (x, x, x)$
   (c) $T(x, y, z) = (x, x)$
   (d) $T(x, y, z) = (x, x^2, x^3)$

10. **Matrix operations**

    Let B be the matrix $B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$

    (a) Compute $B^T B$.
    (b) Find B^{-1}.
    (c) Find $B - I$.

11. **Properties of matrix operations**
Prove that associativity of vector addition holds: $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$. Be sure to give all the gory details and reasoning.

12. **State TRUE or FALSE. Explain your answer**

a. For any two $n \times n$ matrices A and B, the identity $(A-B)(A+B) = A^2 - B^2$ holds.

b. If $\{\mathbf{u}, \mathbf{v}\}$ are linearly independent vectors then the vectors $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}$ are also linearly independent.

c. If $BC = BD$ then $C = D$.

d. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a set of linearly dependent vectors in $\mathbb{R}^3$ then the vectors $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\}$ are also linearly dependent.

13. **LU decomposition**

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & 10 \\ 4 & 9 & 14 \end{bmatrix}.$$

14. **Geometry of linear transformations**
Describe geometrically what $T = A\mathbf{x}$ does when

a. $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$

b. $A = \begin{bmatrix} .1 & 0 \\ 0 & .1 \end{bmatrix}.$

c. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

d. $A = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}.$

e. $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

f. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$

15. **Using linearity**

Let T be a linear transformation that maps $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Find
a. $T(3\mathbf{v})$
b. $T(4\mathbf{w})$
c. $T(3\mathbf{v} - 4\mathbf{w})$

16. **1-1, onto**
    a. Give an example of a linear transformation that is 1-1 but NOT onto.
    b. Give an example of a linear transformation that is onto but NOT 1-1.
    c. Give an example of a linear transformation that is both 1-1 and onto.

17. **Geometry of linear transformations, continued**
    Recall that we can represent the linear transformation which rotates $\mathbb{R}^2$ by θ radians counterclockwise by $T(\mathbf{x}) = M\mathbf{x}$, where M is the matrix

    $$M = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

    a. Multiply the matrix A that rotates $\mathbb{R}^2$ by a radians counterclockwise with the matrix B that rotates $\mathbb{R}^2$ by b radians counterclockwise.
    b. Write down the matric C that rotates $\mathbb{R}^2$ by $(a + b)$ radians counterclockwise.
    c. Is $AB = C$? Why or why not?