Professor Readdy
Math 322
Fall 2024

Review for the Second Exam (§3.1 through §4.7)[1]

1. **Cofactor definition of determinant:** Compute $\begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix}$.

2. **Properties of determinants:** Suppose $|M| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -4$. Find the determinant of each of the following matrices. In each case, be sure to explain how you are getting your answer.

   (a) $A = \begin{bmatrix} c & b & a \\ f & e & d \\ i & h & g \end{bmatrix}$.

   (b) $B = M^3$

   (c) $C = (2M)^{-1}$.

3. **More properties:**

   (a) Is the matrix in problem 1 invertible?

   (b) Give an example of a 3×3 matrix B whose determinant is 322

   (c) Give an example of a 3×3 matrix C with $|CC^T| = 2$.

4. **Cramer's rule, matrix inverse and volume:**

   (a) Use Cramer's rule to solve $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

   (b) Find the area of the parallelogram with vertices $(-1, 0), (0, 5), (1, -4)$ and $(2, 1)$

   (c) Find the area of the parallelepiped with vertices $(-1, 0, 4), (0, 5, 4), (1, -4, 4)$ and $(2, 1, 4)$

5. **Miscellaneous determinant questions:**
   True or False. Justify your answer.

   (a) If A and B are $n \times n$ matrices with $\det A = 2$ and $\det B = 3$ then $\det(A + B) = 5$.

   (b) $\det AA^T \geq 0$.

   (c) If $A^3 = 0$ then $\det A = 0$.

   (d) Any system of n linear equations and n variables can be solved by Cramer's rule.

6. **Two more determinant questions:**

   (a) Use row operations to show $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b - a)(c - a)(c - b)$.

   (b) Prove the determinant of an $n \times n$ upper triangular matrix is the product of the diagonal entries.

[1] The exam is Friday, November 22, 2024 9:00 am - 9:50 am in our classroom.

7. **Vector spaces and subspaces:**

   Determine if the given set V is a subspace of the vector space W, where

   (a) $V = \{$polynomials of degree at most n with $p(0) = 0\}$ and $W = \{$polynomials of degree at most $n\}$.

   (b) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$ and $W = \mathbb{R}^2$.

   (c) $V =$ span of any two fixed vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ and $W = \mathbb{R}^3$.

   (d) $V =$ the set of all vectors in $\mathbb{R}^2$ with integer coefficients and $W = \mathbb{R}^2$.

   (e) $V = \{$all diagonal $n \times n$ matrices with real entries$\}$ and $W =$ all $n \times n$ matrices with real entries.

8. **Null space, column space, basis, linear independent sets** Suppose a matrix A is row equivalent to the following matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

   a. The Col(A) is a subspace of $\mathbb{R}^a$ (find a).
   b. The Row(A) is a subspace of $\mathbb{R}^b$ (find b).
   c. The null space of A is a subspace of $\mathbb{R}^c$ (find c).
   d. The dimension of the column space, row space and nullspace of A is...

9. **Null space, column space, basis, linear independent sets**

   (a) Find a basis for the null space, column space and row space of the matrix $A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{bmatrix}$

   (b) Let $M_{2\times 2}$ be the vector space of all 2×2 matrices. Define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A^T$.

   (i) Show T is a linear transformation.
   (ii) What is the range of T?
   (iii) Describe the kernel of T.

   (c) Find a basis for all those V in Exercise 7 which are subspaces.

10. **Pivots:**

    Give four reasons why pivots are important. You may state relevant theorems to support your argument.

11. **Coordinate systems:**

    (a) Use coordinate vectors to verify the polynomials $1 + 2t^2$, $4 + t + 5t^2$ and $3 + 2t$ are linearly dependent in $\mathbb{P}_2$ the vector space of polynomials of degree at most 2.

    (b) Show the ordered set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $p(t) = 1 + 3t - 6t^2$ relative to this basis. What polynomial does $[1, 2, 3]^T_{\mathcal{B}}$ correspond to?

12. **Dimension of a vector space:**

    (a) State the dimension of each of the subspaces in problem 7.

    (b) If A is a 3×5 matrix, what are the possible dimensions of the null space of A? For each possibility, find a matrix which has that nullity.

13. **Change of basis:**

Let $\mathcal{A} = \{1, 1+t, 1+t+t^2\}$ and $\mathcal{B} = \{1, 1-t, 2-4t+t^2\}$ be two ordered bases for $\mathbb{P}_2$.

(a) Find the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$.

(b) Use part (a) to find the coordinates of the vector $[x]_{\mathcal{A}} = [1, 1, 1]$ with respect to the new basis $\mathcal{B}$.

Prof. Readdy
Review II

Very Brief Answers

1. -3

2a. 4
b. $(-4)^3$
c. $-1/32$

3. a. Yes
b. $\mathrm{Diag}\{1,1,322\}$
c. Pick any 3x3 matrix C with $|C| = \sqrt{2}$

4. a. $x_1 = -2/3$, $x_2 = 1/3$, $x_3 = 2/3$, $x_4 = 1/3$
b. 14
c. 0

5 a. FALSE. Let $A = \mathrm{diag}\{2, 1, \ldots, 1\}$ and $B = \mathrm{diag}\{3,1, \ldots, 1\}$.
Then $A+B = \mathrm{diag}\{5,2,\ldots, 2\}$ and $\det(A+B) = 5*(2^{n-1})$.

b. TRUE. (Can you prove it?)

c. TRUE. (Can you prove it?)

d. FALSE. If $\det A = 0$ (i.e., A not invertible), then you are stuck.

6. a. Carefully row-reduce (or column reduce) to show this.
b. See class notes.

7. a. IS a subspace
b. NOT a subspace
c. IS a subspace
d. NOT a subspace
e. IS a subspace

8. a. 2
b. 5
c. 5
d. 2, 2, 3

9. a.

$$A \sim \ldots \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So basis for $\mathrm{Col}(A)$ is {columns 1, 2 & 4 of A}.

Basis for $\mathrm{Row}(A)$ is {rows 1, 2, & 3 of row-reduced A}
Basis for $\mathrm{Nul}(A)$ is $[2,-1,1,0]^T$.

b. i. Check $T(A + B) = T(A) + T(B)$ and $T(cA) = c\,T(A)$ (Easy)
ii. Range(T) = all 2x2 matrices with real entries
iii. ker(T) is the zero 2x2 matrix.

c. For a. $\{t, t^2, \ldots, t^n\}$
For c. Assuming the vectors a and b are nonzero and linearly independent, then basis is $\{a,b\}$. If a and b are dependent and a is nonzero, then $\{a\}$. If both are zero, then no basis.
For e, take the matrices $\{E_1, \ldots, E_n\}$ with zeroes everywhere except for a 1 in the (i,i) entry of the matrix E_i.

10. You should be able to come up with many reasons...

11. a. $5(1+2t^2) -2(4+t+5t^2) +1(3+2t) = 0$.
b. Let's row-reduce the basis vectors and solve for the coordinates at the same time:

$a(1-t^2) + b(t-t^2) + c(2-t+t^2) = 1 + 3t - 6t^2$:

becomes

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

Row-reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & -1 & 1 & -6 \end{array}\right] \sim \ldots \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Note the columns have 3 pivots, so they form a basis for P_2.
The coordinates are $[3\ 2\ -1]^T$.

$[1\ 2\ 3]^T$ in the basis B is $7-t$ in the standard basis.

There are many ways to do this!

12. a. a is n; b is not; c depends on the vectors -- could be 0, 1 or 2; d is not; e is n.

b. If A has 0 pivots, then nullity = 5;
If A has 1 pivot, then nullity = 4;
If A has 2 pivots, then nullity = 3;
If A has 3 pivots, then nullity = 2.

Examples are omitted.

13. a. Row reduce

$[\ B \mid A\ ]$ to $[I \mid B^{-1} A]$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

b. Multiply $B^{-1} A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$