Professor Readdy
Math 322
Fall 2024

Review for Rest of Semester (§5.1 through §6.4) [1]

Eigenvalues and eigenvectors:

1. a. Verify

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

   has eigenvector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

   b. Find an eigenvalue of M.
   c. Quickly find another eigenvalue of M.
   d. Find the characteristic polynomial $p(\lambda)$ of M.
   e. Evaluate $p(M)$.

2. a. Find the eigenvalues and a basis for the eigenspace of

$$C = \begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix}.$$

   b. Diagonalize C.
   c. Compute $\lim_{n \to +\infty} C^n$.

3. a. Find eigenvalues and a basis for each eigenspace for the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix}.$$

   b. Is the matrix diagonalizable? Why or why not?
   c. Find $A = PDP^{-1}$.
   d. Find a Q so that $A = QDQ^{-1}$, where Q is an orthogonal matrix.

4. **Find the algebraic and geometric multiplicity of the eigenvalue(s) of**

$$E = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

5. **Complex eigenvalues**
   Find eigenvalue(s) and eigenvectors for

$$F = \begin{bmatrix} 6 & -13 \\ 1 & 0 \end{bmatrix}.$$

[1] The final exam is Wednesday, December 18, 2024 from 8:00 am to 10:00 am in our classroom. The final is cumulative.

6. **Relation between eigenvalues and determinant**
   a. In each of the previous matrices, compare the determinant of the matrix with the product of all of the eigenvalues (be sure to count multiplicity!).
   b. Using part a, guess a theorem.

7. **Use Gram-Schmidt to find an orthonormal basis**
   a.
$$Z = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \right\}.$$
   b.
$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}.$$
   Why does your answer make sense?

Sketch of answers:

1. a. $M\mathbf{v} = 4\mathbf{v}$, so $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda_1 = 4$.
   b. From part a, $\lambda_1 = 4$.
   c. Even easier, all rows of M are the same, so we automatically have a nontrivial nullspace, that is, $\lambda_2 = 0$.
   d. $p(\lambda) = (\lambda - 4) \cdot \lambda^3$.
   e. Evaluate $p(M) = (M - 4I) \cdot M^3$. You should get the 4×4 zero matrix!

2. a. Eigenvalues $\lambda_1 = 1$ and $\lambda_2 = .6$.
   b.
$$C = PDP^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & .6 \end{bmatrix} \cdot \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & -3/4 \end{bmatrix}.$$
   c. $C^n = PD^nP^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (.6)^n \end{bmatrix} \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & -3/4 \end{bmatrix}$ so $\lim_{n\to+\infty} C = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 3/4 & 3/4 \\ 1/4 & 1/4 \end{bmatrix}$.

3. a. Characteristic polynomial is $(2-\lambda)(\lambda^2 - 12\lambda + 11) = (2-\lambda)(\lambda - 11)(\lambda - 1)$, so have eigenvalues $\lambda = 2$, $\lambda = 1$ and $\lambda = 11$. The corresponding eigenvectors are $[1, 0, 0]^T$, $[0, 2, -1]^T$ and $[0, 1, 2]^T$.
   b. Since the matrix is 3×3 and the characteristic polynomial has 3 distinct roots, we know the eigenvectors are linearly independent.
   c. Hence A is diagonalizable as
$$A = PDP^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/5 & -1/5 \\ 0 & 1/5 & 2/5 \end{bmatrix}$$
   d. The columns of P are mutually orthogonal, but not unit length. So
$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$
   Can check $QDQ^{-1} = A$. (Try!)

4. Since it is a diagonal matrix, easy to see the only eigenvalue is 2 with algebraic multiplicity 3 (since $(\lambda - 2)^3$ is the characteristic polynomial). The geometric multiplicity is 1 since the eigenbasis is one-dimensional and spanned by the vector $[1, 0, 0]^T$.

5. Characteristic polynomial is $\lambda^2 = 6\lambda + 13$, so have eigenvalues $\lambda = 3 \pm 2i$. When $\lambda_1 = 3 + 2i$, eigenvector is $v_1 = [3 + 2i, 1]^T$. We get the other eigenvector by taking the conjugate: $\lambda_2 = \overline{\lambda_1} = 3 - 2i$ has eigenvector $v_2 = \overline{v_1} = [3 - 2i, 1]^T$.

6. a. $\det(M) = 0 = 4 \cdot 0 \cdot 0 \cdot 0$.
   $\det(C) = .6 = 1(.6)$.
   $\det(A) = 22 = 2 \cdot 1 \cdot 11$.
   $\det(E) = 8 = 2^3$.
   $\det(F) = 13 = (3 + 2i) \cdot (3 - 2i)$.
   b. For any $n \times n$ matrix M with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counting multiplicity), we have $\det(M) = \lambda_1 \cdots \lambda_n$.

7. a. $\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{u_3} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, b. $\mathbf{u_1} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.