Math and Magic

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MATH AND MAGIC

RICHARD EHRENBORG AND MARGARET A. READDY

1. Chapter I: Gilbreath's Principle

1.1. **Introduction to cards.** Like many magic tricks, we begin with a standard deck of cards. Each card has two "decorations", namely, a *denomination* from the set

$$\{A, 2, 3, \ldots, 9, 10, J, Q, K\},$$

where A indicates an Ace, J a Jack, Q a Queen and K a King, and a *suit* from the set

$$\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\},$$

where $\clubsuit$ is a club, $\diamondsuit$ is a diamond, $\heartsuit$ is a heart, and $\spadesuit$ is a spade. There are 13 denominations and 4 suits, so the total number of cards in a standard deck is $13 \cdot 4 = 52$ cards.

A *riffle shuffle*, or simply, a *shuffle*, is the result of splitting a deck cards into two piles and taking cards randomly from each pile to form a new pile. As an example, if we have a deck of 5 cards $\{A, B, C, D, E\}$ split into piles ABC and DE, there are 10 possible shuffles of these two piles, namely

$$ABCDE, ABDCE, ADBCE, DABCE, ABDEC,$$
$$ADBEC, DABEC, ADEBC, DAEBC, DEABC.$$

In mathematics, the sum of all possible ways to shuffle two words is called the *shuffle product*.

Definition 1.1. *For two words $x = x_1 \cdots x_m = x'x_m$ and $y = y_1 \cdot y_n = y'y_n$ the shuffle product of x and y is*

$$x \sqcup y = (x' \sqcup y)x_m + (x \sqcup y')y_n.$$

1.2. **Gilbreath's First Principle.** We are now ready to perform our first magic trick.

Magic Trick 1.2 (Pairs of Cards)**.** The magician shows a standard deck of 52 cards to the audience. Audience members are allowed to *cut* the deck, that is, removing some cards from the top of the deck and replacing them on the bottom of the deck in the same order. The magician cuts the deck of cards and then *shuffles* the deck of cards once. The magician announces,

Date: April 4, 2025.

"I am going to show a pair of cards. What is the probability that the two cards have different colors?"

The audience realizes since the number of black cards in the deck equals the number of red cards in the deck, there are four possibilities, namely two red cards, two black cards, a red card and black card, and finally, a black card and a red card. Label these as $\{RR, BB, RB, BR\}$. Thus the probability that a pair of cards has different colors is 2 out of 4, which equals 1/2, and the same color is 2 out of 4, again equaling 1/2[1].

The magician shows a pair of cards. They are two different colors. The magician shows the next pair of cards. Again, they have different colors. Notice since there are still the same number of red and black cards in the deck, the probability of having a pair with different colors is still 1/2. The magician continues to handout to show pairs of cards. Each pair of cards has different colors, despite the fact one quarter of the time pairs should be two red cards and 1/4 of the time the pairs should be two black cards.

This magic trick is known as *Gilbreath's First Principle*. It is due to Norman Gilbreath in 1958 [6].

In order to perform this trick, which is really mathematics in disguise, you need to prepare the deck.

Preparation

(1) Arrange the deck so that the cards are alternating red and black.
(2) When you cut the deck in preparation to perform one riffle shuffle, the bottom cards of the two piles must be different.

In Exercise 1.1 you will study the case when the bottom cards of the two piles to be shuffled are the same.

Some observations:

Lemma 1.3. *Given a deck of $2n$ cards which has the alternating property, the following holds:*

(1) *Cutting the deck has no effect on the alternating property.*
(2) *In order to guarantee Magic Trick 1.16 to work, the bottom cards of the two piles to be shuffled must have different colors.*

[1] Here we are actually lying. The probability that a pair has different colors is $P(RR \text{ or } BB) = 2 \cdot \frac{26 \cdot 25}{52 \cdot 51} = \frac{25}{51} \approx 0.4902$ and $P(RB \text{ or } BR) = 2 \cdot \frac{26 \cdot 26}{52 \cdot 51} = \frac{26}{51} \approx 0.5098$ So there is a slightly large chance of selecting a pair of cards having two different colors than a pair having the same color.

Proof. Suppose the deck is arranged from top to bottom as

$$B_1, R_1, B_2, R_2, \cdots, B_n, R_n.$$

Cutting the deck so that the top pile of the cut has an even number $2j$ of cards and replacing these $2j$ cards at the bottom results in the deck

$$B_{2j+1}, R_{2j+1}, \ldots, B_n, R_n, B_1, R_1, \cdots, B_j, R_j.$$

Note the result is still a deck which has the alternating property. Similarly, cutting the deck so that the top pile has an odd number $2j+1$ of cards means that the bottom pile of the cut also has an odd number of cards. Hence the top card of the bottom pile of the cut has its bottom card having a different color than the top card of the original deck. Moving the top pile of the cut beneath the bottom pile thus preserves the alternating property.

Part (2) of the lemma is left as an exercise. □

In mathematical terms, cutting the deck is known as a *cyclic permutation.* We will soon discuss what this means.

In Exercise 1.1 you will study the case when the bottom cards of the two piles to be shuffled are the same.

Theorem 1.4 (Gilbreath's First Principle)**.** *Suppose you have a deck of* $2n$ *cards consisting of* n *black cards and* n *red cards. The deck is arranged so that the cards are alternating by color. Then*

- *i Cutting the deck several times does not change the alternating property.*
- *ii Cutting the deck in two piles where the bottom cards have different colors and then shuffling the cards once results in a deck where every two cards dealt from the top of the deck results in a pair of cards having two different colors.*

1.3. **Permutations, cards and shuffles.** Many card tricks involve the use of group theory, so it will be useful to introduce the concept of a permutation. A *permutation* π of the elements $\{1, \ldots, n\}$ is a bijection from the elements 1 through n to itself. In *two-line notation*, one can write a permutation π as

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix},$$

where $\pi_i = \pi(i)$ for $i = 1, \ldots, n$. Since the top line of two-line notation is always the same, it is customary to omit this line and simply write $\pi = \pi_1\pi_2 \cdots \pi_n$, known as *one line notation.* A third way to represent a permutation is to use a directed graph on the elements $\{1, 2, \ldots, n\}$, where we draw an edge from the element i to π_i. See Figure 1 for an example.

Example 1.5. There are $3! = 6$ permutations of the three element set $\{1, 2, 3\}$, namely $123, 132, 213, 231, 312, 321$. One can verify the following result.

Figure 1. Directed graph representation of the permutation $\pi = 526143$.

Lemma 1.6. *There are $n!$ permutations of the n element set $\{1, 2, \ldots, n\}$.*

Proof. Begin by simply noting that there are n ways to select the image of 1 under π, that is, the value of $\pi(1)$. Next, there are $n-1$ ways to select the image of 2 under π as we already used up one of the n original values. In general, once the values of $\pi_1, \ldots, \pi_k$ are selected from $\{1, \ldots, n\}$, there are $n-k$ elements remaining from the set $\{1, \ldots, n\}$ from which we can choose the value π_{k+1}, namely, $\{1, \ldots, n\} - \{\pi_1, \ldots, \pi_k\}$. Overall, this gives $n \cdot (n-1) \cdots 2 \cdot 1 = n!$ permutations. $\square$

Notice that for 10 cards there are $10! = 3,628,800$ permutations of this deck, whereas for 52 cards there are

$$52! = 80658175170943878571660636856403766975289505440883277824000000000000$$

permutations.

Just as we can compose two functions, we can compose two permutations. Let $\pi = \pi_1 \cdots \pi_n$ and $\sigma = \sigma_1 \cdots \sigma_n$ be two permutations of an n element set. Then $(\pi \circ \sigma)(i) = \pi_{\sigma_i}$. See Figure **??** for an example. The set of all permutations on an n elements set with the operation of composition is known as the *symmetric group on n elements*. It is denoted by $\mathfrak{S}_n$.

Now that we have the language of permutations, we can easily describe other types of shuffles.

Given a deck of $2n$ cards labeled from top to bottom with 0 through $2n-1$, that is, card i appears in position i for $i = 0, \ldots, 2n-1$, a *perfect outshuffle* is one where the deck is split precisely in half, where the left-hand deck consists of cards labeled 0 through $n-1$ and the right-hand deck labeled n through $2n-1$. The two decks are shuffled in an alternating fashion, with the card labeled x (and thus originally in position x), where $0 \leq x \leq n-1$ (the top half of the deck) goes to position $2x$, and the cards in the bottom half of the deck, that is, labeled x for $n \leq x \leq 2n-1$ goes to position $2x - (2n-1)$.

Example 1.7. A perfect outshuffle of a deck of 8 cards is as follows:

$$
\begin{array}{ccccccccc}
0 & & & & & 0 & & & 0 \\
1 & & & & & & 4 & & 4 \\
2 & & 0 & 4 & & 1 & & & 1 \\
3 & \longrightarrow & 1 & 5 & \longrightarrow & & 5 & \longrightarrow & 5 \\
4 & & 2 & 6 & & 2 & & & 2 \\
5 & & 3 & 7 & & & 6 & & 6 \\
6 & & & & & 4 & & & 3 \\
7 & & & & & & 7 & & 7
\end{array}
$$

Example 1.8. Suppose you have a deck of n cards labeled $\{1, \ldots, n\}$ where the top card is labeled 1, the second card from the top is labeled 2, continuing in this fashion until the bottom card is labeled n. Fix j where $1 \leq j \leq n$. Deal the top j cards face down into a second pile. This gives two piles which look like this:

$$
\begin{array}{cc}
j+1 & \\
\vdots & j \\
 & \vdots \\
n-1 & 2 \\
n & 1
\end{array}
$$

Use these two decks to perform one riffle shuffle. This is called a *Gilbreath shuffle*. We will return to this in Section 1.5.

At this point, one would like to know how many perfect outshuffles does it take until the deck is returned to its original order? Again, suppose we have a deck with $2n$ cards. Again, we label the cards and positions of the cards as 0 through $2n-1$. Cut the deck in half and perform exactly one perfect outshuffle. In general, note that a card appearing in the top half of the deck in position x, where $0 \leq x \leq n-1$, goes to position $2x$. For a card from the bottom half of the deck in position x, where $n \leq x \leq 2n-1$, goes to position $2x - (2n-1)$.

We would like to encode this rule using modular arithmetic. In order to do this, for the moment, please forget about the bottom card (in position $2n-1$) since it never moves under the operation of a perfect outshuffle. Combining the two rules we see that a card in position $x \neq 2n-1$ goes to position $2x$ modulo $2n-1$, that is,

$$x \longmapsto 2x \mod 2n-1.$$

Notice we can think of this map as combining two physical phenomena: multiplying each card location spreads the cards out, and then performing modulus $2n-1$ interleaves the top half of the cards with the bottom half.

The problem of determining the number of perfect outshuffles that brings the deck back to its original order reduces to finding the smallest exponent

$k \geq 1$ such that

$$2^k \equiv 1 \mod 2n-1.$$

In order to determine k, we need some number theory. For a positive integer n, *Euler's totient function*, or simply the *Euler phi function*, denoted $\varphi(n)$, counts the number of integers in the set $\{1, \ldots, n\}$ that are relatively prime to n. Recall two integers a and b are relatively prime if their greatest common divisor is 1, that is, $\gcd(a, b) = 1$. Notice that if p is a prime number then $\varphi(p) = p - 1$.

We now establish some properties of the Euler phi function.

Lemma 1.9. *The Euler phi function satisfies the following properties:*

i. It is a multiplicative function, that is, if $\gcd(a, b) = 1$ *then*

$$\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b).$$

ii. For p a prime and $k \geq 1$

$$\varphi(p^k) = p^{k-1} \cdot (p-1)$$

iii.

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product is over all primes p that divide n.

Proof. Rather than showing part i directly, we will illustrate it in the special case of $n = 15$. Arrange the integers 1 through 15 in a table, where the rows indicate the congruence classes modulo 3 and the columns modulo 5.

	0	1	2	3	4
0	15	6	12	3	9
1	10	1	7	13	4
2	5	11	2	8	14

Notice the integers rows 1 and column 1 of this chart, corresponding to being equivalent to 0 mod 3 and 0 mod 5, are not relatively prime to 15. In contrast, the remaining integers occurring in the 2×4 submatrix are relatively prime to 15. Hence $\varphi(15) = 2 \cdot 4 = \varphi(3)\varphi(5)$.

To show ii, we instead show $\varphi(p^k) = p^k - p^{k-1}$. Arrange the integers 1 through p^k as follows:

$1, 2, 3, \ldots, p-1, pp+1, p+2, p+3, \ldots, 2p-1, 2p \ldots, \ldots, p^{k-1} \cdot p - 1, p^{k-1} \cdot p$

Notice the integers in the right-hand column are $\{p, 2p, \ldots, p^{l-1} \cdot p\}$, which are each divisible by p. There are p^{k-1} such integers. The $p^k - p^{k-1}$ remaining integers are not divisible by p, that is, they are relatively prime to p. Hence $varphi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1)$, as claimed.

n	$\varphi(n)$
1	1
2	1
3	2
4	2
5	4
6	2
7	6
8	4
9	6
10	4
11	10
12	4
13	12
14	6
15	8
16	8
17	16
18	6
19	18
20	8

TABLE 1. Values of the Euler phi function $\varphi(n)$ for $1 \leq n \leq 20$.

To show *iii*, let $n = p_1^{e_1} \cdots p_k^{e_k}$ be a prime factorization of n, where $p_1, \dots, p_k$ are distinct primes, and the exponents satisfy $e_i \geq 1$ for $i = 1, \dots, k$, Since the integers $p_1^{e_1}, \dots, p_k^{e_k}$ are relatively prime, we have

$$\begin{aligned}
\varphi(n) &= \varphi(p_1^{e_1}) \cdot \varphi(p_2^{e_2}) \cdots \varphi(p_k^{e_k}) \\
&= \prod_{i=1}^{k} p_i^{e_i - 1}(p_i - 1) \\
&= \prod_{i=1}^{k} p_i^{e_i - 1} \cdot p_i \left(1 - \frac{1}{p_i}\right) \\
&= n \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) \\
&= n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right)
\end{aligned}$$

Here we have used the fact the integers $p_1^{e_1}, \dots, p_k^{e_k}$ are relatively prime and applied part *i* and then part *ii*. $\square$

We recall Euler's theorem

n	$\varphi(n)$	n	$\varphi(n)$
1	1	11	10
2	1	12	4
3	2	13	12
4	2	14	6
5	4	15	8
6	2	16	8
7	6	17	16
8	4	18	6
9	6	19	18
10	4	20	8

TABLE 2. Values of the Euler phi function $\varphi(n)$ for $1 \leq n \leq 20$.

Theorem 1.10 (Euler)**.** *Assume a and m are relatively prime integers. Then*

$$a^{\varphi(m)} \equiv 1 \mod m,$$

where $\varphi(\cdot)$ is the Euler phi-function.

Corollary 1.11. *The smallest power of k where $2^k \equiv 1 \mod 2n-1$ is a divisor of $\varphi(2n-1)$.*

Example 1.12. We consider the case where we have a deck of $2n = 52$ cards and determine how many perfect outshuffles to return the deck to the original order. We have

$$\varphi(2n-1) = \varphi(51) = \varphi(3 \cdot 17) = 2 \cdot 16 = 32.$$

The divisors of 32 are 1, 2, 4, 8, 16 and 32. But $2^4 = 16 \not\equiv 1 \mod 51$. However,

$$2^8 = 256 \equiv 5 \cdot 51 + 1 \equiv 1 \mod 51.$$

Hence eight perfect outshuffles suffices to return a deck of 52 cards back to its original state.

Let us repeat this with a smaller deck.

Example 1.13. We next consider the case of a deck of $2n = 26$. We have

$$\varphi(2n-1) = \varphi(25) = \varphi(r^2) = 4 \cdot 5 = 20.$$

The divisors of 20 are 1, 2, 4, 5, 10 and 20. Note $2^{10} = 1024 \equiv 1025 - 1 \equiv -1 \mod 25$. But Euler's theorem implies $2^{20} \equiv 1 \mod 25$. Hence with a deck of 26 cards one has to perform 20 perfect outshuffles to return it to the original deck order.

1.4. **More shuffles.**

Example 1.14. Backwards perfect outshuffle

Example 1.15. Perfect inshuffle.

1.5. **Gilbreath's Second Principle.**

Magic Trick 1.16 (Types of Cards)**.** The magician shows a standard deck of 52 cards to the audience. Audience members are allowed to *cut* the deck, that is, removing some cards from the top of the deck and replacing them on the bottom of the deck in the same order. The magician then asks for a number x between 1 and 52. The magician deals the top x cards off the top of the deck into a second pile, and then performs one riffle shuffle with the two piles of cards. The magician repeatedly deals off four cards from the top of the deck and shows them to the audience. Every time there are four cards, each of a different suit.

To analyze this card trick, let us first see what dealing cards off the top of the deck does. Let us try this with cards labeled from top to bottom with 1 through 10. This gives

$$\begin{array}{ccc} 1 & & 10 \\ 2 & & 9 \\ 3 & & 8 \\ \vdots & \longrightarrow & \vdots \\ 8 & & 3 \\ 9 & & 2 \\ 10 & & 1 \end{array}$$

Observe this reverses the order of the cards. Let us next perform a Gilbreath shuffle with a deck of n cards labeled 1 through n. Fix j with $1 \leq j \leq n$. Deal the top j cards face down onto a second deck. This gives

$$\begin{array}{cc} j+1 & \\ j+2 & j \\ \vdots & \vdots \\ n-1 & 2 \\ n & 1 \end{array}$$

Notice that this preserves the order of the cards if you now view them as being arranged on an arc stretching from the right-hand side to the left-hand side.

We now explain the trick. First, pre-arrange the order of the deck by type, alternating club, diamond, heart, spade, club, diamond, heart, spade, etc. Notice that if we deal 52 or 0 cards the trick works trivially. Next deal j cards face down where $0 < j < 52$. Notice we can view the cards as being on an arc. Begin to shuffle the two decks. This corresponds to taking cards from either deck randomly. Convince yourself that the first four cards will have different type. Removing them is equivalent to removing an interval of length 4 from the original deck. Since any interval of length 4 from the deck has one of each type, we are left with a smaller deck where the trick holds by induction on the size of the deck. (Here we need the deck to have $4n$ cards.)

Example 1.17. We illustrate this with a deck of size 12 with 3 types of cards, say A, B and C. Dealing 5 cards gives

$$\begin{array}{cc}
C_2 & \\
A_3 & \\
B_3 & B_2 \\
C_3 & A_2 \\
A_4 & C_1 \\
B_4 & B_1 \\
C_4 & A_1
\end{array}$$

Turn the two decks sideways as

$$C_4B_4A_4C_3B_3A_3C_2 \mid B_2A_2C_1B_1A_1$$

Here the line indicates where the tops of the two decks were. Shuffling the cards so that C_2, A_3 and B_3 are the top three cards corresponds to the window

$$C_4B_4A_4C_3\mathbf{B_3A_3C_2} \mid B_2A_2C_1B_1A_1$$

Other windows:

$$C_4B_4A_4C_3B_3\mathbf{A_3C_2} \mid \mathbf{B_2}A_2C_1B_1A_1$$

$$C_4B_4A_4C_3B_3A_3\mathbf{C_2} \mid \mathbf{B_2A_2}C_1B_1A_1$$

$$C_4B_4A_4C_3B_3A_3C_2 \mid \mathbf{B_2A_2C_1}B_1A_1$$

Need to say something more about the window...

Theorem 1.18 (Gilbreath's Second Principle)**.** *Suppose you have a deck of $d = kn$ cards consisting of n cards of each of the k types. The deck is arranged so that the cards are alternating by type. Then*

i. Cutting the deck several times does not change the alternating property.
ii. Dealing j cards face down, where $0 \leq j \leq kn$ and then shuffling the cards once results in a deck where every k cards dealt from the top of the deck results in a set of k cards having exactly one of each of the k types.

In Exercise 1.6 you will be asked explain how Gilbreath's first principle is an instance of Gilbreath's second principle.

1.6. **Perfect inshuffles.** We would like to next study the case of how many perfect inshuffles are needed to return a deck to its original order.

Let us look at the case of 6 cards.

Example 1.19. For a deck of 6 cards, one perfect inshuffle looks as follows. Note we show splitting the deck into two equal halfs and then performing one riffle shuffle. The top card of the original deck now goes inside the deck, and as well the bottom card of the deck moves inside the deck.

$$
\begin{array}{ccccccccc}
1 & & & & & & 4 & & 4 \\
2 & & & & & 1 & & & 1 \\
3 & & 1 & 4 & & & 5 & & 5 \\
 & \longrightarrow & 2 & 5 & \longrightarrow & 2 & & \longrightarrow & 2 \\
4 & & 3 & 6 & & & 6 & & 6 \\
5 & & & & & 3 & & & 3 \\
6 & & & & & & & &
\end{array}
$$

To study this, let us add a card at the top of the deck labeled "0" and a card at the bottom labeled "$*$". With this new deck, perform one perfect *outshuffle*.

Example 1.20. For a deck of 6 cards, add a new top card labeled "0" and a new bottom card labeld "$*$". Next, perform a perfect outshuffle. We obtain

$$
\begin{array}{ccccccccc}
0 & & & & & 0 & & & 0 \\
1 & & & & & & 4 & & 4 \\
2 & & 0 & & & 1 & & & 1 \\
3 & & 1 & 4 & & & 5 & & 5 \\
 & \longrightarrow & 2 & 5 & \longrightarrow & 2 & & \longrightarrow & 2 \\
4 & & 3 & 6 & & & 6 & & 6 \\
5 & & & * & & 3 & & & 3 \\
6 & & & & & & * & & *
\end{array}
\; Noticethatifweignorethecardslabeled
$$

$0 and *, weexactlyobtaintheresultofperformingaperfectinshuffleontheoriginal 6 cards.$

Using this observation, we make the following claim.

Lemma 1.21. *Given a deck of an even number n cards, performing one perfect inner shuffle is the same as adding two extra cards 0 and $*$ to the original deck, with 0 at the top and $*$ at the bottom, performing one perfect outer shuffle, and then removing the two extra cards.*

Using Lemma 1.21, let us determine the smallest integer k such that k perfect inner shuffles returns a deck of n cards back to the original order. For a deck with $n = 6$ cards, adding two extra cards gives 8 cards. We have $\varphi(8-1) = \varphi(7) = 6$, so we need to check the divisors of 6, that is, 1, 2, 3 and 6. Easily

$$\begin{aligned} 2^1 = 2 \not\equiv 1 &\mod 7 \\ 2^2 = 4 \not\equiv 1 &\mod 7 \\ 2^3 = 8 \equiv 1 &\mod 7 \end{aligned}$$

One can easily verify this small case.

We next consider the case of a deck with 52 cards. Adding the two extra cards gives 54 cards, so we compute $\varphi(54-1) = \varphi(53) = 52$, as 53 is prime. The divisors of $52 = 2^2 \cdot 13$ are 1, 2, 4, 13, 26 and 52. We have

$$\begin{aligned} 2^1 = 2 \not\equiv 1 &\mod 53 \\ 2^2 = 4 \not\equiv 1 &\mod 53 \\ 2^4 = 16 \not\equiv 1 &\mod 53 \\ 2^{13} = 2 \cdot (2^6)^2 \equiv 2 \cdot (11)^2 \equiv 2 \cdot 15 \equiv 30 \not\equiv 1 &\mod 53 \\ 2^{26} = (2^{13})^2 \equiv (30)^2 \equiv 900 \equiv -1 &\mod 53 \\ 2^{52} = (2^{26})^2 \equiv (-1)^2 \equiv 1 &\mod 53 \end{aligned}$$

Unlike the case of perfect outer shuffles, for a deck with 52 cards it takes 52 perfect inner shuffles to return the deck back to the original order!

1.7. **A note on primes.** In this section we review a few facts about prime numbers.

Corollary 1.22. *In order to determine whether or not an integer n is prime, it is enough to test divisibility of n by all prime numbers p with $p \leq \sqrt{n}$.*

Lemma 1.23. *If the integer n can be factored as $n = a \cdot b$ with $a \leq b$ then $a \leq \sqrt{n}$.*

Proof. We provide a proof by contradiction. We are given that

$$n = a \cdot b,$$

where a and b are two integers and $a \leq b$. We claim that $a \leq \sqrt{n}$. If not, assume on the contrary that $a > \sqrt{n}$. Multiplying the inequality $a \leq b$ by a gives

$$a \cdot a \leq a \cdot b \tag{1.1}$$

But then the strict inequality $\sqrt{n} < a$ would imply

$$(\sqrt{n})^2 = n < a^2 \leq a \cdot b = n,$$

that is, $n < n$, which is impossible. Thus it must be that $a \leq \sqrt{n}$, as claimed. $\square$

Example 1.24. Let determine whether or not $n = 53$ is prime. It is enough to only check for divisors that are at most 7 since $49 < 53 < 64$ and taking the squareroot gives the bound

$$7 = \sqrt{49} < \sqrt{53} < \sqrt{64} = 8.$$

Clearly 2 is not a divisor as 53 is odd, and 3 is not a divisor as the sum of the digits of 53 are not divisible by 3. Also 53 is not divisible by 5 as its units place is not a 0 or 5. Finally, 7 does not divide 53, so 53 is a prime number.

Chapter 1 Exercises

Exercise 1.1. a. Explain why Trick **??** will not work if the two piles of cards to be shuffled have the same color bottom card.
b. What can you do to save the trick in part a?
c. Assuming you have an odd number of alternating cards, explain when the trick will work. (Hint: you need to be a very talented card shuffler.)

Exercise 1.2. Show that if $C = c_1 c_2 \cdots c_m$ is a deck consisting of m cards and $D = d_1 d_2 \cdots d_n$ is a deck consisting of n cards then there are $\binom{m+n}{m}$ ways to shuffle these two decks together.

Exercise 1.3. a. Show the shuffle product of two words is *commutative*, that is, if $x = x_1 \cdots x_m$ and $y = y_1 \cdots y_n$ then $x \sqcup\!\sqcup\, y = y \sqcup\!\sqcup\, x$.
b. Show the shuffle product is *associative*, that is, $x \sqcup\!\sqcup\, (y \sqcup\!\sqcup\, z) = (x \sqcup\!\sqcup\, y) \sqcup\!\sqcup\, z$.

Exercise 1.4. Compute the number of perfect outshuffles needed to return a deck of $2n$ cards to the original order for $n = 1, \ldots, 10$. Which sized deck would make a good trick? Which a bad trick?

Exercise 1.5. Devise a magic trick where you will ultimately show 12 cards at a time and guess the suit and denomination of the 13th card.

Exercise 1.6. Explain how Gilbreath's first principle is an instance of Gilbreath's second principle.

Exercise 1.7. Something about computing $\pi \circ \pi$ and $\sigma \circ \tau$ for a number of permutations.

Exercise 1.8. Exercise on basic divisibility facts, such as:
a. n is divisible by 2 if and only if its unit digit is an even number.
b. n is divisible by 3 if and only if the sum of its digits is divisible by 3.
c. n is divisible by 5 if and only if ... .
d. n is divisible by 11 if and only if the alternating sum of its digits equals 0.

2. Chapter II: Hummer

2.1. CATO moves.

2.2. Baby Hummer.

2.3. 10 card Hummer.

3. Chapter III: The Blind Bartender

(This feels like CATO)

Intro to group theory?

4. Chapter IV: Topology and Knots

4.1. **The Möbius strip.** Give history...

4.2. **Some knot theory tricks.**

4.3. **Introduction to unknot and trefoil knot.**

4.4. **Some serious knot stuff.** Reidemeister moves

Coloring argument that the unknot and trefoil knot are different.

4.5. **Hanging a picture with two nails.**

5. Chapter V: A more mathematical view of knots and tangles

5.1. Real def'n of a knot.

5.2. Alexander-Conway polynomial.

5.3. The Jones polynomial.

5.4. The braid group.

5.5. The Fundamental Group of the complement of a knot.

6. Chapter VI. Conway's Rational Tangles.

6.1. **The game.** To play this game, you will need 4 people and two ropes where each rope is 10 feet long. If possible, use two different colored ropes.

To begin the game, each person should hold the end of one rope. Do not let go of the rope! The starting configuration is two people next to a wall holding the first rope, and the other pair next to them (the audience) holding the second rope. The two ropes are parallel to each other.

There are two moves:

T = Twist: From the audience's perspective, the person on the right nearest the wall lifts their rope over the other person on the right furthest from the wall. The other two people do not move!

R = Rotate: Each student moves 1/4 turn clockwise.

We now need to discover the mathematics behind the twist and rotate moves. Let us assume that $T = 1$. (Try this with the rope.) Then $T^2 = TT = 1 + 1 = 2$. Start again at 0 and make a T twist. Can we apply just T and R moves to undo the twist T?

Since you found out that $TRT = 0$, you could guess that R sends a number to its negative. Is this true? Test this with $T^2 = 1 + 1 = 2$. What happens?

Now that you see that T does not send x to $-x$, let's look at RT. What happens? What about RT^2? What about RT^n?

It appears that R sends 0 to either $-\infty$ or $+\infty$, that is, R sends the number x to either $1/x$ or $-1/x$. Test this so see which it is.

Let's now try to undo the following rational tangles.

Find the moves to untangle each tangle and then try them out with the ropes.

Initial Tangle	Inverse Tangle
T	RT
$TT = T^2$	
$TTT = T^3$	
T^4	
T^n	

Fill in the blanks in this chart.

Rational Tangle	Rational Number	Inverse Tangle
$TTRTTTRT$		
T^5RT^2RTRT		
	$-1/2$	
	$-1/3$	
	$-1/4$	
	$-1/5$	
	$-3/2$	
	$-5/7$	

For a spectacular finish, find the tangle for $-17/43$ and its inverse. Twist the tangle for $-17/43$, then cover the tangle with a plastic bag to preserve the tangle. The twist the inverse tangle. Remove the bag and shake the ropes slightly. Were you correct?

6.2. **Euclid's Algorithm.** Recall that an integer q divides an integer a if $a = k \cdot q$ for some integer k. This is denoted by $q|a$. The *greatest common divisor* (gcd) of integers a and b, denoted $\gcd(a, b)$, is the largest integer that divides both a and b.

We recall two basic properties of the gcd.

Lemma 6.1. *1.* $\gcd(a, b) = \gcd(b, a)$.
2. $\gcd(a, b) = \gcd(a - b, b)$.

Proof. For (1), it is straightforward that the gcd is symmetric. For (2), suppose $d = \gcd(a, b)$. Since $d|a$ and $d|b$ we have $d|a - b$. Suppose on the contrary that d is not the largest common divisor of $a - b$ and b, and instead e is, with $d < e$. Then $e|a - b$ an $e|b$ implying e divides $(a - b) + b = a$. This means e is a larger integer that divides both a and b, contradicting the fact that $d = \gcd(a, b)$ $\square$

Euclid's algorithm consists of the following. Given integers a and b, write $\gcd(a, b) = \gcd(a - b, b)$. If $a - b = 0$, then the gcd is b. Otherwise, continue to iterate this step. If $a - b < 0$, then write $(a, b) = \gcd(b, -a)$.

Example 6.2. We use Euclid's algorithm to compute the gcd of 20 and 72.

$$\begin{aligned} \gcd(72, 20) &= \gcd(52, 20) = \gcd(32, 20) = \gcd(12, 20) = \gcd(-8, 20) \quad (6.1) \\ &= \gcd(20, 8) = \gcd(12, 8) = \gcd(4, 8) = \gcd(-4, 4) \quad (6.2) \\ &= \gcd(4, 4) = \gcd(0, 4) = 4. \quad (6.3) \end{aligned}$$

References

[1] Tom Davis, Conway's Rational Tangles, tomrdavis@earthlink.net, `http://www.geometer.org/mathcircles`

[2] John H. Conway, An enumeration of knots and links and some of their algebraic properties, in Computational Problems in Abstract Algebra, D. Welsh, Editor, pp. 329–358, Pergamon Press, New York, 1970.

Conway's original paper.

[3] Jay Goldman and Louis H. Kaufmann, Rational Tangles, Advances in Applied Mathematics, **18**, pp. 300–332, 1997.

The authors prove two rational tangles are topologically equivalent if and only if they represent the same rational number. An application to biology is included.

[4] David Austin, Untangling your square dance, AMS feature column http://www.ams.org/publicoutrea
column/fc-2017-08 *Even more moves...*

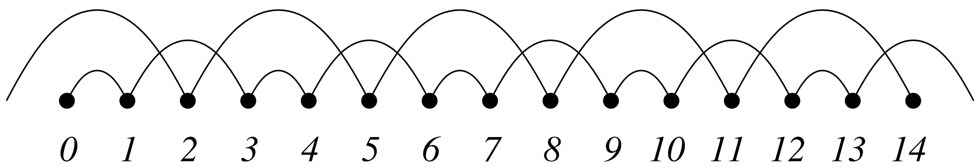

FIGURE 2. The juggling pattern $\mathbf{a} = (1, 2, 3)$ with period $d = 3$ and throw vector $\mathbf{x} = (0, 1, 2)$.

7. Chapter VI: q-counting

7.1. What is a q-analogue.

7.2. MacMahon.

7.3. Juggling.

A juggling sequence is a period pattern of throwing balls in the air and catching them. In order to talk about the mathematics of juggling, we need to simplify things somewhat. We will first assume (i) the juggler is one-handed and (ii) the juggler can catch and throw only one ball at the time. This is known as *simple juggling*.

As an example, consider the juggling pattern $\mathbf{a} = (1, 2, 3)$ in Figure 2. We think of the horizontal axis as time, where the first time point is 0. At time zero the juggler throws the ball so that it lands 1 time unit later, then at time 1 the juggler throws the ball so that it lands 2 time units later and at time 2 the juggler catches and throws a ball so that it lands three time units later. This pattern repeats itself every 3 time units, so we say this is a period $d = 3$ juggling sequence.

Next consider the sequences $(1, 3, 2)$. This is not a juggling sequence. The problem is that at time 4 the juggler would have to catch more than one ball, violating our requirement (i) for a simple juggling pattern the juggler can only catch and throw one ball at a time.

In the first example there are two balls being juggled. We can now state two results answers the questions: what is a legal simple juggling sequence and how many balls are being juggled?

Theorem 7.1. *1. Let* $\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$ *be a sequence of d positive integers. Then* $\mathbf{a}$ *is a simple juggling pattern of period d if*

$$\{a_i + i \mod d : \textit{for } i = 1, \ldots, d\} = \{0, 1, 2, \ldots, d-1\}.$$

2. Let $(a_1, a_2, \ldots, a_d)$ be a simple juggling pattern. Then the number n of balls is

$$n = \frac{a_1 + a_2 + \cdots + a_n}{d}$$

Let us now try to count all the simple juggling patterns having period d and at most n balls. Here is the theorem

Theorem 7.2. *[Buhler, Eisenbud, Graham, Wright] The number of simple juggling patterns having period d and at most n balls is*

$$n^d.$$

In Figure **??** we see there are $2^3 = 8$ such patterns. The proof of Theorem 7.2 is quite difficult. Instead of counting the juggling patterns, we will look at the number os crossings in a peeriod. For example, for $\mathbf{a} = (1, 2, 3)$ there are 2 crossings in a period. So we will weight this juggling pattern by q^2. In general, for a juggling pattern $\mathbf{a}$ with $\mathrm{cross}\mathbf{a}$ crossings, we will weight it by $q^{\mathrm{cross}\mathbf{a}}$.

In Figure **??** we have indicated the number of crossings in a period and the weight. Notice that if we add up all of the weightings of these 8 patterns we obtain the polynomial $1 + 3q + 3q^3 + q^3 = (1 + q)^3$. Notice this factors.

Theorem 7.3 (Ehrenborg-Readdy)**.** *The sum of the weights of simple juggling patterns having period d and at most n balls is*

$$[n]^d = (1 + q + \cdots + q^{n-1})^d.$$

Note that setting $q = 1$ in this theorem gives the n^d result in Theorem 7.2.

To prove this result, we are going to use juggling cards. One can take the picture of a juggling pattern and "push" over each crossing as far as possible to the right so that the ball being catched drops at the last moment. In Figure **??** we have the three juggling cards for at most 3 balls. Notice if we arrange three copies of the card C_o next to themselves, we obtain the sequence $\mathbf{a} = (1, 1, 1)$. Notice that two of the paths never fall down. In essence, two of the balls are floating in the air!

By arranging cards from this deck periodically, we will see we will obtain all possibly simple juggling sequences of period d and at most n balls. See Exercise? for determining the card sequences corresponding to the juggling patterns in Figure ??.

Proof. To construct a simple juggling pattern of period d, select a card from the deck $\{C_0, \ldots, C_{n-1}\}$ for the first throw. We have n possible cards to choose from, each having the weight $q^0 = 1$, q^1 through q^{n-1} respectively since card C_i has i crossings. The other throws have similar choices which are independent of previously made selections. Overall, the weights are

$$[n] \cdot [n] \cdots [n] = [n]^d.$$

□

References [9]

Chapter 6 Exercises

Exercise 6.1. a. Draw the cards C_0 and C_1 for simple juggling cards with at most 2 balls.
b. Determine the card sequences needed to generate the simple juggling sequences having period 3 and at most 2 balls.

Exercise 6.2. Let C_0, C_1, C_2 be the juggling cards for at most 3 balls. Determine the period 5 juggling sequence corresponding to the card sequence $C_1C_2C_2C_0C_1$.

7. Chapter VII: Origami

7.1. **Hills and Valleys.** We begin by learning how to fold a hill and valley. On a standard piece of paper, draw a point. Crease the paper from the point to the edge of the paper to create a hill. This creates a ray from the point to the edge of the paper. Continue to crease hills until you have created 3 to 5 hills. Next add a valley from the point to the edge of the paper. Continue adding enough valleys until your paper can be flattened.

In origami notation, hills are denoted by solid lines and valleys by dotted lines. See Figure **??** for an example.

Count the number of hills and valleys eminating from the point. Repeat the experiment again with a new sheet of paper using a different number of hills. What do you conjecture?

There is a basic relation between the number of hills and valleys in a flat foldable origami crease pattern.

Theorem 7.1 (Maekawa)**.** *For every flat foldable origami crease pattern, let $V(x)$ denote the number of valleys at a point x and $H(x)$ denote the number of hills at a point x. Then*

$$|V(x) - H(x)| = 2$$

Proof. The proof is to to walk around the point x while the crease pattern is folded flat. To travel completely around a given point, you (or an ant) must travel a total of 2π radians. When you walk over a hill, the change in angle is $+\pi$ radians, and when you walk over a valley, the change is π radians. Thus you will need 2 extra hills to complete the full circuit.

Observe that if you flip the paper over, this uniformly exchanges hills and valleys. Thus the difference in the number of valleys and hills around a point is, up to a sign, 2. □

Corollary 7.2. *For a flat foldable origami crease pattern with V valleys and H hills, turning the paper upside down yields a crease pattern with H valleys and V hills.*

Proof. Straightforward. □

Using the paper with valleys and hills eminating from a point, label the $2n$ angles

Theorem 7.3 (Kawasaki's Theorem)**.** *For every flat foldable origami crease pattern, let α_i for $1 \leq i \leq 2n$ denote the consecutive angles formed by the crease pattern around a fixed vertex. Then*

$$\alpha_1 - \alpha_2 + \alpha_3 - \cdots - \alpha_{2n} = 0.$$

Furthermore,

$$\sum_{i\ odd} \alpha_i = \sum_{i\ even} \alpha_i = \pi.$$

Proof. The result follows by traveling along a circle centered around the vertex x. When the paper is folded flat one sees all of the odd-indexed arcs have a positive direction and the even-indexed ones have the opposite direction. Once you circumnavigate along the entire circle and return to your initial starting point, the net sum of these signed arc lengths is zero, hence the same result holds for their respective angles.

The second result is a direct consequence of the first. □

Theorems ?? and ?? are two of the four axioms of origami. The other two are:

Theorem 7.4. *Every flat foldable origami crease pattern is* 2*-colorable.*

Theorem 7.5. *You cannot pass a fold through another.*

See [8].

*****

A walk through origami[2]

(1) **Maekawa's Theorem**
This exercise is to practice origami mountain and valley folds. All the folds we make will start at the center dot and end at the edge of the paper.
Begin by creasing the paper from center to an edge. Your paper should now have a small hill caused by the mountain fold. If you hold the paper by this fold, it already has a curved structure.
Make 2, 3 or 4 more mountain folds in different directions in the paper. We next wish to add valley folds from the center to an edge. Keep on adding valley folds until you can fold up your paper flat. You may need to add additional mountain folds to do this. (Ask an assistant for help, if needed.)
How many mountain and how many valley folds did you need to fold your paper flat? Compare your result with your neighbors.
The folds you made created sectors eminating from the center of the paper. Number these angles you created clockwise starting with 1, 2, etc. Look at the sum of the angles labeled with odd integers versus the sum of the angles labeled with even integers. Do you observe anything? This is known as **Kawasaki's Theorem**.

[2]This is an excerpt of the workshop "Origami structures for technological and design applications" held by origami scientist Robert Lang. http://langorigami.com/

(2) **Bird's foot**
Cut out the paper. Begin to fold the mountain and valley crease pattern in the center of the bird's foot. Straight lines are mountains and dotted lines are valleys. (There are 4 total folds.) To reinforce this structure, fold up each of the tab sides in half, then half again. Then turn the two corners over to lock. Repeat with the other four tabs. Now flex your bird's foot.
Fold a different bird's foot. Can you make any observations about the bellow you have made? Could you combine two similar structures and make them move simultaneously?

(3) **Helical Yoshimura pattern**
Fold mountains on the solid lines, valleys on the dotted lines. (Be patient!) Note how you are changing the nature of the paper. The paper will begin to form a familiar 3-dimensional structure.
Where could you use this origami?

(4) **Miura-ori**
Fold mountains on the solid lines, valleys on the dotted lines. (Be patient!) Eventually this will fold flat. Pull out from two sides to make it open. Note that unlike a rubberband which thins when you pull it, this object grows.
Where could you use this origami?

(5) **3D Miura-ori**
What do you get from this crease pattern?

(6) **7-fold twist**
What do you get from this crease pattern?

(7) **Cup à la Tomoko Fuse**
Try this!

(8) **TensionPot, Opus 695**
This is a Robert Lang design. Note the curved folds.

Reference:

1. Robert J. Lang, Twists, Tilings, and Tessellations: Mathematical Methods for Geometric Origami, CRC Press.

8. Chapter X: Miscellaneous

9. Preliminaries

The magic in this book requires very few materials. You will need a standard deck of cards, that is, a 52 card deck. For the rope tricks, magic rope is the best. A less expensive alternative is to use cotton clothesline. It is important to use cotton as otherwise some of the knots will slip.

References

[1] Arthur T. Benjamin, An amazing mathematical card trick, *The Mathematical Intelligencer* **2010** (32), 37–40.

[2] Joe Buhler, David Eisenbud, Ron Graham and Colin Wright, Juggling drops and descents, *Amer. Math. Monthly* **101** (1994), 507–519.

[3] Persi Diaconis and Ron Graham, *Magical Mathematics. The mathematical ideas that animate great magic tricks,* Princeton University Press, Princeton, NJ.2012

[4] Richard Ehrenborg and Margaret Readdy, Juggling and applications to q-analogues, *Discrete Math.* **157** (1996), 107–125.

[5] Gardner, *Martin Gardner,* Mathematics Magic and Mystery, Dover Publications. New York, NY 1956

[6] Norman Gilbreath, Magnetric Colors Principle, *The Linking Ring* **July 1958** (38), No. 5, 60–61.

[7] Bob Hummer, Face-up/Face-down Mysteries, *privately printed manuscript*, 1942.

[8] Robert J. Lang, *Twists, Tilings, and Tessellations. Mathematical Methods for Geometric Origami,* CRC Press, Boca Raton, FL. 2018

[9] Burkard Polster, *The Mathematics of Juggling,* Springer, New York. 2003