Name: Prof. Readdy Student ID: Answer Key

Section: +∞

Do not remove this answer page. You will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or communication capabilities is permitted. You may not use a phone or other communication device during the exam.

The exam consists of 12 multiple choice questions that are worth 5 points each and 4 free response questions that are worth 10 points each. You should work the multiple choice questions on the question page. After you have checked your work carefully, record your answers by completely filling in the circle below that corresponds to your answer. If you must change your answer, make a note on the front of the exam. Be sure to check carefully when you transfer your answers to the cover sheet.

Show all work to receive full credit on the free response problems. You do not need to compute a decimal approximation to your answer. For example, the answer 4π is preferred to 12.57.

Multiple Choice Questions

1	A	B	C	D	E	**7**	A	B	C	D	E
2	A	B	C	D	E	**8**	A	B	C	D	E
3	A	B	C	D	E	**9**	A	B	C	D	E
4	A	B	C	D	E	**10**	A	B	C	D	E
5	A	B	C	D	E	**11**	A	B	C	D	E
6	A	B	C	D	E	**12**	A	B	C	D	E

SCORE

Multiple Choice	13	14	15	16	Total Score
60	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Give the domain of the function $f(x) = \sqrt{4+2x}$.

   A. $[-10, 10]$
   B. $(-\infty, \infty)$
   C. $(-10, \infty)$
   D. $[-2, \infty)$
   E. $(-\infty, 2]$

or ≥ 0

Need $4+2x \geq 0$

$2x \geq -4$

$x \geq -2$.

(Note $f(-2) = \sqrt{4-4} = \sqrt{0}$)

2. (5 points) Let $f(x) = \dfrac{2x+4}{x-5}$. Find the inverse function $f^{-1}(x)$ and give its domain.

   A. $f^{-1}(x) = \dfrac{2x+4}{x-5}$ with the domain $(-\infty, 5) \cup (5, \infty)$
   B. $f^{-1}(x) = \dfrac{x+2}{x-5}$ with the domain $(-\infty, 5) \cup (5, \infty)$
   C. $f^{-1}(x) = \dfrac{5x+4}{x-2}$ with the domain $(-\infty, 2) \cup (2, \infty)$
   D. $f^{-1}(x) = \dfrac{5x+4}{x-2}$ with the domain $(-\infty, -4/5) \cup (-4/5, \infty)$
   E. $f^{-1}(x) = \dfrac{x+2}{x-1}$ with the domain $(-\infty, 1) \cup (1, \infty)$

$y = \dfrac{2x+4}{x-5}$

Reflect over line $y = x$

$x = \dfrac{2y+4}{y-5}$

$x(y-5) = 2y+4$.

$xy - 5x = 2y + 4$

$xy - 2y = 4 + 5x$

$y(x-2) = 4+5x$

$f^{-1}(x) = y = \dfrac{4+5x}{x-2}$.

Domain: Need $x - 2 \neq 0$

$x \neq 2$.

$(-\infty, +2) \cup (+2, +\infty)$

3. (5 points) Assume that the position (measured in meters) of an object at time t (measured in seconds) is given by $s(t) = 2t^2 + 1$. Find the average velocity of the object on the interval $[2, 4]$.

   A. 33 meters/second
   B. 24 meters/second
   C. 12 meters/second
   D. 5 meters/second
   E. -12 meters/second

Average velocity on $[2,4]$ $= \frac{s(4) - s(2)}{4-2}$

$= \frac{2(4)^2+1-(2(2)^2+1)}{2}$

$= \frac{32-8}{2} = \frac{24}{2} =$ 12 m/sec.

4. (5 points) Consider the function f defined by

$$f(x) = \begin{cases} x - 5, & x < 2 \\ x^2 + 5, & 2 \leq x \end{cases}.$$

Find the **one-sided** limit $\lim_{x \to 2^-} f(x)$.

   A. -3
   B. 0
   C. 9
   D. 2
   E. The limit does not exist

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x - 5$

$= 2 - 5$

$= -3$

5. (5 points) Find $\lim_{x\to 2} \dfrac{x^2-4}{x}$.

   A. $-1/2$

   B. The limit does not exist

   C. $1/2$

   D. -3

   E. 0

$$\lim_{x\to 2} \frac{x^2-4}{x} = \frac{\lim_{x\to 2} x^2-4}{\lim_{x\to 2} x} = \frac{2^2-4}{2} = \frac{0}{2} = 0$$

6. (5 points) A function f satisfies $2-2x^2 \le f(x) \le -x^2+2x+3$ for all real numbers x. There is exactly one number c where we may use the Squeeze Theorem to compute the limit $\lim_{x\to c} f(x) = L$. Find c and L.

   A. $c = 0$ and $L = -1$.

   B. $c = 1$ and $L = 0$.

   C. $c = -1$ and $L = 1$.

   D. $c = 0$ and $L = 1$.

   E. $c = -1$ and $L = 0$.

$$\lim_{x\to c} 2-2x^2 \le \lim_{x\to c} f(x) \le \lim_{x\to c} -x^2+2x+3$$

$$2-2c^2 \le \lim_{x\to c} f(x) \le -c^2+2c+3$$

Need equal.

$$2-2c^2 = -c^2+2c+3$$
$$0 = 2c^2-c^2+2c+3-2$$
$$= c^2+2c+1$$
$$= (c+1)^2$$
$$c+1=0 \text{ or } c=-1$$

$$L = 2-2(-1)^2 = 2-2 = 0$$

(RHS: $-(-1)^2+2(-1)+3 = -1-2+3 = 0$.)

7. (5 points) Consider the function f defined by

$$f(x) = \begin{cases} 2x+4, & x < 3 \\ a, & x = 3 \\ x^2+1, & x > 3 \end{cases}$$

For which value of a is the function f continuous on $(-\infty, \infty)$?

A. $a = 3$

B. $a = 0$

C. $a = 10$

D. $a = -1$

E. This function is not continuous for any value of a.

$\lim_{x\to a} f(x) = f(a)$.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 2x+4 = 2(3)+4 = 6+4 = 10$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2+1 = (3)^2+1 = 9+1 = 10$

$\Rightarrow$ Need $f(3) = 10$

8. (5 points) Select the statement that best describes the behavior of $f(x) = \dfrac{-4}{x^2}$ near $x = 0$.

A. $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to 0^-} f(x) = +\infty$

B. $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to 0^-} f(x) = -\infty$

C. $\lim_{x\to 0^+} f(x) = +\infty$ and $\lim_{x\to 0^-} f(x) = -\infty$

D. $\lim_{x\to 0^+} f(x) = +\infty$ and $\lim_{x\to 0^-} f(x) = +\infty$

E. $\lim_{x\to 0} f(x) = 0$

$\lim_{x\to 0^+} \frac{-4}{x^2} \to \frac{-4}{(\text{small positive})^2} = -\infty$

$\lim_{x\to 0^-} \frac{-4}{x^2} \to \frac{-4}{(\text{small negative})^2} = \frac{-4}{\text{smaller positive}} = -\infty$

9. (5 points) Find the limit $\lim_{x\to\infty} \frac{-x^3+x^2+72}{5x^3+32x^2+1}$.

A. $-1/5$

B. $+\infty$

C. $-\infty$

D. 0

E. 72

Guess.

$\lim_{x\to+\infty} \frac{-x^3}{5x^3} = \lim_{x\to-\infty} \frac{-1}{5} = -\frac{1}{5}$.

Show:

$$\lim_{x\to\infty} \frac{x^3\left(-1+\frac{1}{x}+\frac{72}{x^3}\right)}{x^3\left(5+\frac{32}{x}+\frac{1}{x^3}\right)}$$

$$= \lim_{x\to+\infty} \frac{(-1+1/x+72/x^3)}{(5+32/x+1/x^3)} = \frac{-1+0+0}{5+0+0} = -1/5$$

10. (5 points) Find the equation of the tangent line to the curve $y = x^2 + 4x$ at the point $(1, 5)$.

A. $y = -2x + 3$

B. $y = -6x + 11$

C. $y = 2x + 3$

D. $y = 2x - 2$

E. $y = 6x - 1$

$y - y_0 = m(x - x_0)$

$y - 5 = 6(x - 1)$

$y = 5 + 6x - 6$

$= 6x - 1$

Need slope: $y'(1)$

$y = x^2 + 4x$

$y' = 2x + 4$

$y'(1) = 2(1) + 4 = 6$

Need point: $(1, 5)$

9: what if $\lim_{x\to-\infty}$?

Guess $\lim_{x\to-\infty} \frac{-x^3}{5x^3}$

$= -\frac{(-\infty)^3}{5(-\infty)^3}$

$\to \frac{+\infty \cdot 1}{5(-\infty)}$

$= 1/5$

11. (5 points) Consider the function $f(x) = x^2 + 3$. If you evaluate and simplify the expression

$$\frac{f(1+h) - f(1)}{h}$$

you obtain

A. $2h + h^2$

B. $2 + h$

C. $2h + h^2 + 3$

D. 0

E. $1/h$

$$= \frac{(1+h)^2+3-(1^2+3)}{h}$$

$$= \frac{h^2+2h+1+3-1-3}{h}$$

$$= \frac{h^2+2h}{h}$$

$$= h + 2$$

12. (5 points) The graph of f is shown below. Find all the values of x for which the derivative $f'(x)$ does not exist.

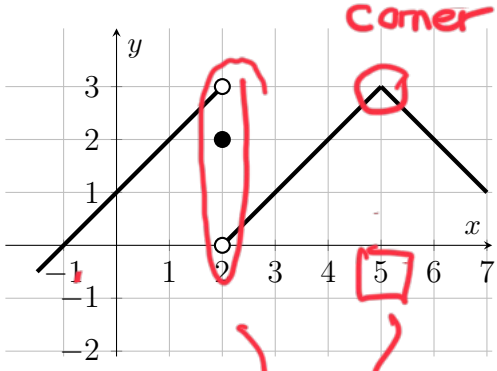
A. $x = 0$ and $x = 5$

B. $x = -1$ and $x = 0$

C. $x = 2$ and $x = 3$

D. $x = 2$ and $x = 5$

E. $x = 3$ and $x = 5$



Corner

No derivative

Recall:

Differentiability implies continuity.

(D ⇒ C)

Not C ⇒ Not D.

Not continuous ⇒ Not diffble

contn ⇏ diffblty. ✶ $y = |x|$

Free response questions, show all work, and clearly label your answers

13. (10 points) Let $f(x) = x^2 + 4$ with the domain $[2, \infty)$.
    (a) Find the formula for the inverse function f^{-1}.
    (b) Give the range of f.
    (c) Use the relation between the domain of f^{-1} and the range of f to give the domain and range of f^{-1}.

$y = x^2 + 4$

Reflect over $y = x$

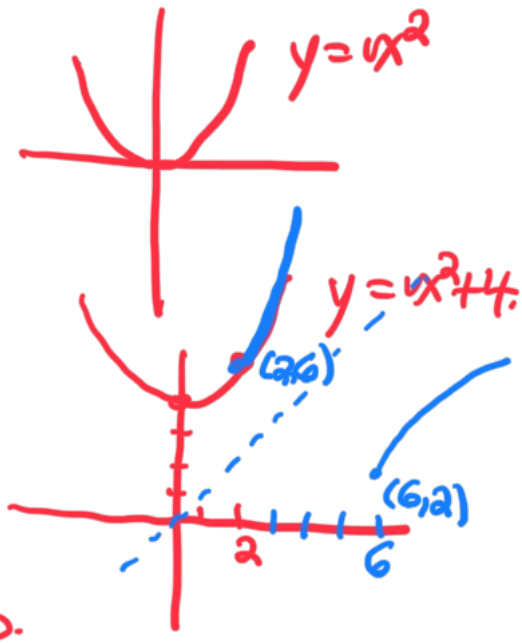
$x = y^2 + 4$.

$y^2 = x - 4$

$y = \pm\sqrt{x-4}$.

Choose branch

a. $y = +\sqrt{x-4}$



since range of $f(x)$ b.
on $[2,+\infty)$ is $[6,+\infty)$. (positive values).

c. domain f is $[2,+\infty)$
range f is $[6,+\infty)$.
So inverse function f^{-1} swaps these
domain f^{-1} is $[6,+\infty)$
range f^{-1} is $[2,+\infty)$.

14. (10 points) For each limit, find the limit or state that it does not exist. Show steps clearly using proper notation.

(a) $\lim_{x\to 0} \dfrac{2x^2-2}{x^2-2x+1}$

(b) $\lim_{x\to 1} \dfrac{2x^2-2}{x^2-2x+1}$

(c) $\lim_{x\to -\infty} \dfrac{2x^2-2}{x^2-2x+1}$

Guess

$\lim_{x\to 0} x^2-2x+1 = 0^2-2(0)+1 = 1 \neq 0$

(Note limit of quotient is quot. of limits if lim of denom $\neq 0$)

a. $\lim_{x\to 0} \dfrac{2x^2-2}{x^2-2x+1} = \dfrac{\lim_{x\to 0} 2x^2-2}{\lim_{x\to 0} x^2-2x+1}$

poly. is continuous everywhere so can substitute.

$\dfrac{2\cdot 0^2-2}{0^2-2(0)+1} = \dfrac{0-2}{0+1} = -2.$

b. (Guess $\dfrac{2\cdot 1^2-2}{1^2-2\cdot 1+1} = \dfrac{0}{0}$ ... No Need more work)

$\lim_{x\to 1} \dfrac{2x^2-2}{x^2-2x+1} = \lim_{x\to 1} \dfrac{2(x^2-1)}{(x-1)(x-1)}$

$= \lim_{x\to 1} \dfrac{2(x-1)(x+1)}{(x-1)(x-1)}$

$= \lim_{x\to 1} \dfrac{2(x+1)}{x-1} = \dfrac{2(1+1)}{1-1} = \dfrac{4}{0}$ DNE.

(In fact $\lim_{x\to 1^+} \dfrac{2(x+1)}{x-1} = \dfrac{4}{\text{small pos \#}} \to +\infty$.

$\lim_{x\to 1^-} \dfrac{2(x+1)}{x-1} = \dfrac{4}{\text{small neg \#}} \to -\infty$).

c. $\lim_{x\to -\infty} \dfrac{2x^2-2}{x^2-2x+1} = \lim_{x\to -\infty} \dfrac{x^2(2-\frac{2}{x^2})}{x^2(1-\frac{2}{x}+\frac{1}{x^2})}$

$= \lim_{x\to -\infty} \dfrac{2-2/x^2}{1-2/x+1/x^2} = \dfrac{2-0}{1+0+0} = 2/1 = 2$

15. (10 points) (a) State the Intermediate Value Theorem.



(b) Use the Intermediate Value Theorem to show that the equation

$$x^4 - 3x^2 + 5x - 4 = 0$$

has a solution. Be sure to give the interval on which you are applying the intermediate value theorem.

a. Let f be a function that is continuous on $[a,b]$. Let k be a value where $f(a) < k < f(b)$. Then there is a value c with $a \le c \le b$ where $f(c) = k$.

b. Let $f(x) = x^4 - 3x^2 + 5x - 4$.

Interval: $[0, 10]$. $k = 0$.

$f(0) = -4$

$f(1) = 1 - 3 + 5 - 4 = 6 - 7 = -1$

$f(2) = 2^4 - 3(2^2) + 5(2) - 4 = 16 - 12 + 10 - 4 = 26 - 31 = -5$

$f(10) = 10^4 - 3(10^2) + 5(10) - 4 = 10000 - 300 + 50 - 4 = 9700 + 46 = 9746$

$f(0) = -4 < 0 < 9746 = f(10)$

Since f is a polynomial it is continuous everywhere.

By IVT, have a c with $0 \le c \le 10$ and $f(c) = 0$.

(Note: $f'(x) = -2x$)

Free response questions, show all work, and clearly label your answers

16. (10 points) Let $f(x) = -x^2 + 5$. Use the **limit definition** of the derivative to find the derivative $f'(2)$.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{-(2+h)^2+5-(-2^2+5)}{h}$$

$$= \lim_{h\to 0} \frac{-(h^2+4h+4)+5-(-4+5)}{h}$$

$$= \lim_{h\to 0} \frac{-h^2-4h-4+5+4-5}{h}$$

$$= \lim_{h\to 0} \frac{-h^2-4h}{h}$$

$$= \lim_{h\to 0} -h-4 = -0-4 = -4$$

(Check: shortcut $f'(x) = -2x$

$f'(2) = -2(2)$

$= -4$).

COWABUNGA!