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Lattice minors and Eulerian posets

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Lattice minors and Eulerian posets

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
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ABSTRACT OF DISSERTATION

Lattice minors and Eulerian posets

We study a partial ordering on pairings called the uncrossing poset which first appeared in the literature in connection with a certain stratified space of planar electrical networks. We begin by examining some of the relationships between the uncrossing poset and Catalan combinatorics, and then proceed to study the structure of lower intervals. We characterize the lower intervals in the uncrossing poset that are isomorphic to the face lattice of an n-dimensional cube. Moving up in complexity certain lower intervals are isomorphic to the poset of simple vertex labeled minors of an associated graph.

Inspired by this structure, we define a notion of minors for lattices enriched with a generating set. This notion abstracts the notion of simple vertex labeled minors of a graph. We can associate a generator-enriched lattice to any polymatroid, a far reaching generalization of graphs, and show that conversely any generator-enriched lattice has an associated polymatroid. The generator-enriched lattice encodes the simple information of the closure operator of the polymatroid analagous to how a geometric lattice encodes the simple information of a matroid. For a generator-enriched lattice associated to a graph, we show the minors of the generator-enriched lattice are in bijection with the simple vertex labeled minors of the graph. This bijection is generalized to any generator-enriched lattice and its associated polymatroid.

We proceed to study a partial order structure on the minors of a given generator-enriched lattice called the minor poset whose relations correspond to performing deletions and contractions. A construction for minor posets in terms of the zipping operation introduced by Reading is given. This construction implies any minor poset is isomorphic to the face poset of a regular CW sphere, and in particular, implies minor posets are Eulerian. This construction also yields **cd**-index inequalities for minor posets. We characterize the generator-enriched lattices whose minor poset is itself a lattice as meet-distributive lattices avoiding a single forbidden minor. As a special case, minor posets of distributive lattices avoiding this forbidden minor are isomorphic to the face lattice of the order polytope of the dual of the poset of join-irreducibles.

The deletion and contraction operations of generator-enriched lattices do not commute. We introduce a modified version of contractions, called weak contractions, that do commute with deletions. From this operation we define weak minor posets whose relations correspond to performing deletions and weak contractions. The theory of weak minor posets closely parallels that of minor posets. Most notably, weak minor posets are shown to be complemented lattices and strong maps between generator-enriched lattices induce meet-preserving maps between the weak minor posets in analogy with the zipping construction for minor posets. We characterize graded weak minor posets and give EL-labelings for weak minor posets of generator-enriched lattices whose minors each have an EL-labeling. In particular, we find any graded weak minor poset is also shellable.

KEYWORDS: uncrossing poset, minors, deletion and contraction, polymatroids, lattices, Eulerian posets

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April 27, 2023

Lattice minors and Eulerian posets

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To my wife, for endlessly encouraging me to better myself.

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Chapter 1 Introduction

1.1 Posets

In this section we briefly define some basic notions of posets and lattices. The reader may refer to [46, Chapter 3] or [10] for a more thorough introduction.

A *poset*, or partially ordered set, is a set P together with a binary relation $\leq$ such that for all $x, y, z \in P$

(i) $x \leq x$,

(ii) if $x \leq y$ and $y \leq x$ then $x = y$,

(iii) if $x \leq y$ and $y \leq z$ then $x \leq z$.

Let $[n]$ denote the set $\{1, \ldots, n\}$. The *Boolean algebra* of rank n, denoted B_n, is the poset consisting of all subsets of $[n]$ ordered by inclusion. The *length n chain* is the poset $C_n = \{0 < 1 < 2 < \cdots < n\}$. The *face lattice of the n-dimensional cube*, denoted Q_n, has underlying set $\{0, 1, *\}^n \cup \{\widehat{0}\}$. The order relation of Q_n is the componentwise ordering induced by the partial order $0 < * > 1$ among vectors, and $\widehat{0} \leq x$ for all $x \in Q_n$.

If $x \leq y$ and for any z with $x \leq z \leq y$ either $z = x$ or $z = y$ holds then we say x *is covered by* y or y *covers* x and denote this by $x \prec y$. Posets are depicted by Hasse diagrams. The *Hasse diagram* of a poset P is a directed graph with vertex set P and edges directed from x to y whenever $x \prec y$. Hasse diagrams will be depicted so that all edges are directed upwards. See Figure 1.1 for examples.

Given two posets P and Q, a map $f : P \to Q$ is said to be *order-preserving* if $p_1 \leq p_1$ in P implies that $f(p_1) \leq f(p_2)$ in Q. A map is an *isomorphism* of posets when it is an order-preserving bijection and the inverse map is order-preserving as well.

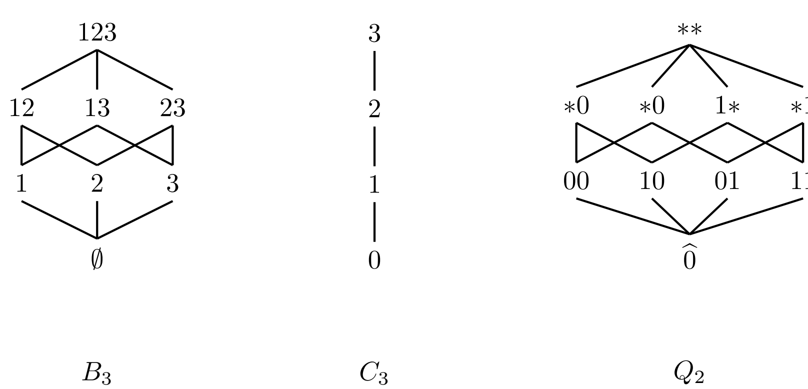


Figure 1.1: Examples of Hasse diagrams

A subposet of P is a subset $Q \subseteq P$ equipped with the same order relation. A *(closed) interval* of P is a subposet of the form $[x, y] = \{z \in P : x \leq z \leq y\}$ where $x \leq y$. An *open interval* is a subposet of the form $(x, y) = [x, y] \setminus \{x, y\}$. The *dual poset* of P is the poset P^* with underlying set P and order relation defined by $x \leq_{P^*} y$ when $x \geq_P y$. A poset P is said to be *ranked* when the lengths of all inclusionwise maximal chains in P are the same. If P is ranked the *rank* of an element x, denoted $\mathrm{rk}(x)$, is the length of any maximal chain in the subposet $\{y \in P : y \leq x\}$.

An element x is *maximal* if $x \leq y$ only holds for $x = y$ and dually x is *minimal* if $x \geq y$ only holds for $x = y$. When P has a unique maximal element it will be denoted $\widehat{1}$ and when P has a unique minimal element it will be denoted $\widehat{0}$. A poset is said to be *graded* when it is ranked and has a unique minmal element and a unique maximal element. When P has a $\widehat{0}$ the elements which cover $\widehat{0}$ are referred to as *atoms*.

Let P be a poset and let $x, y \in P$. The *join* of x and y is an element $z \geq x, y$ such that whenever $w \geq x, y$ we have $z \leq w$. Dually the *meet* of x and y is an element $z \leq x, y$ such that whenever $w \leq x, y$ we have $z \geq w$. We view taking the join of two elements x and y as a binary operation denoted by $x \vee y$, and likewise for the meet which is denoted as $x \wedge y$. Both of these operations are commutative and associative and are idempotent in the sense that $x \vee x = x$ and $x \wedge x = x$ for any element x.

A *lattice* is a poset in which the join and meet of any two elements exist. Boolean algebras are the prototypical examples of lattices. In a Boolean algebra the join of two elements is their union and the meet of two elements is their intersection. A chain of length n and the face lattice Q_n of an n-dimensional cube are examples of lattices as well. An element x is said to be *join irreducible* if whenever $x = y \vee z$ then either $y = x$ or $z = x$. Given a lattice L we denote the set of join irreducibles by $\mathrm{irr}(L)$. It is a basic fact of lattice theory that any finite poset that has a $\widehat{0}$ and in which any two elements have a join is a lattice. Dually any finite poset that has a $\widehat{1}$ and in which any two elements have a meet is a lattice. See [46, Proposition 3.3.1]

1.2 Overview of the uncrossing poset

In this subsection we give an overview of the uncrossing poset, which we study in Chapter 2. References for this material are [28], [36, Section 4.5] and [35].

The uncrossing poset UC_n of order n consists of all pairings on $[2n]$ adjoined with an element $\widehat{0}$. The order relations correspond to resolving crossings in an associated diagram for the pairing. We view pairings on $[2n]$ as fixed point free involutions. The *medial diagram* of a pairing τ consists of a circle with the points $1, \ldots, 2n$ placed in clockwise order on the boundary of the circle and arcs drawn within the circle between the points. If every pair of arcs in a medial diagram cross at most once then an arc from i to j indicates that i is paired with j. If two arcs cross more than once then locally resolving one of the intersections so that the resulting arcs have one less intersection does not change the pairing represented. See Figure 1.2 for an example of removing a double-crossing in this way.

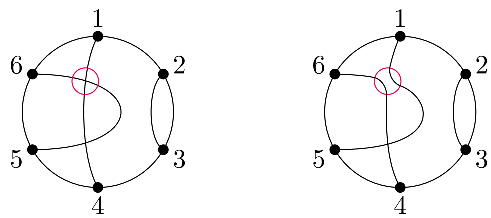


Figure 1.2: Locally resolving a double intersection. Both diagrams represent the same pairing (15)(23)(46).

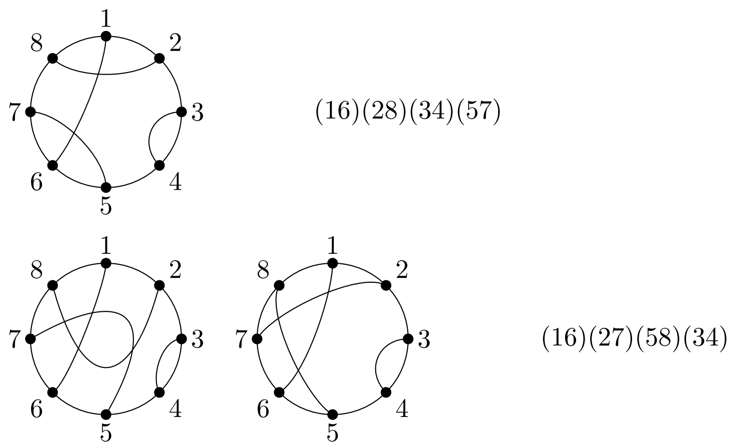


Figure 1.3: Pairings and examples of medial diagrams.

Given a medial diagram M let $\tau(M)$ denote the pairing represented by M. By definition the pairing $\tau(M)$ pairs i and j when there is an arc from i to j in the medial diagram obtained from M by removing all double crossings. Figure 1.3 depicts examples of medial diagrams for pairings.

Definition 1.2.1. *The uncrossing poset* UC_n *is the poset consisting of all pairings of* $[2n]$ *along with a minimal element* $\widehat{0}$. *The order relation is defined as* $\sigma \leq \tau$ *when there is a medial diagram* M *for* τ *and a sequence of local crossing resolutions that when applied to* M *result in a medial diagram for* σ.

Figure 1.4 depicts the Hasse diagram of UC_3 and Figure A11 in Appendix A depicts the Hasse diagram of UC_4.

Basic enumerative information of the uncrossing poset is of combinatorial interest, for instance the rank function, number of atoms and number of elements. Let $<_i$ be the ordering on $[2n]$ defined by

$$i <_i i+1 <_i \cdots <_i 2n <_i 1 <_i \cdots <_i i-1.$$

Given a pairing τ define the *crossing number* $\mathrm{cross}(\tau)$ to be the number of pairs of arcs $(i, \tau(i))$ and $(j, \tau(j))$ such that $i <_i j <_i \tau(i) <_i \tau(j)$. Given a medial diagram

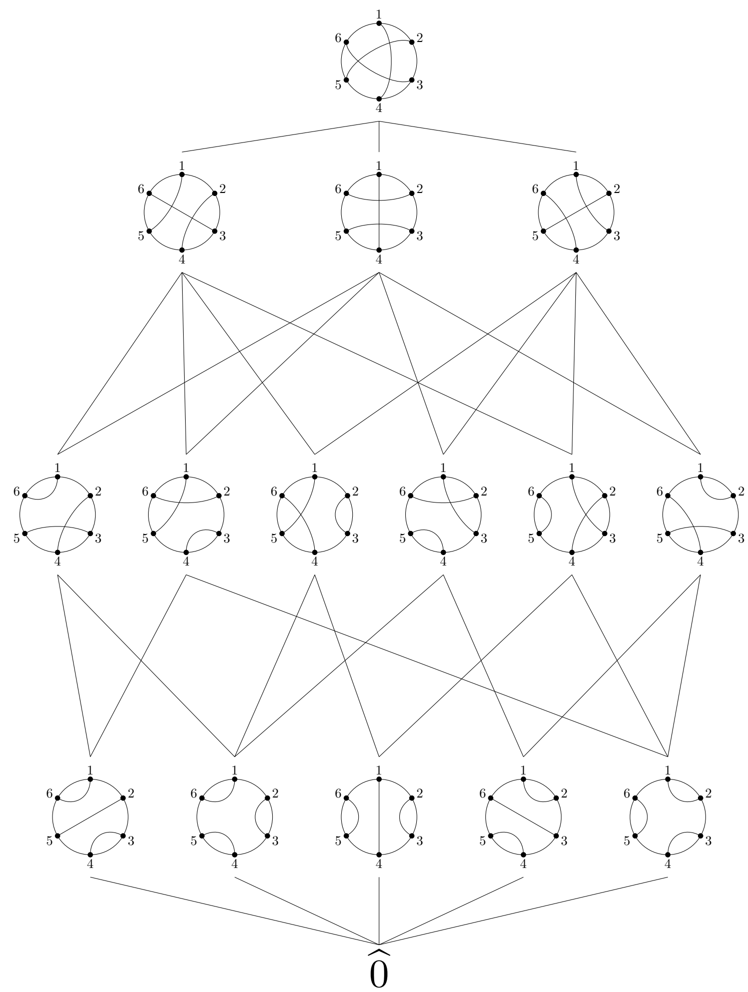


Figure 1.4: The uncrossing poset UC_3.

for τ in which no two arcs cross more than once, a pair of arcs crosses if and only if the pair contributes to $\mathrm{cross}(\tau)$. The uncrossing poset is graded: given a pairing $\tau \in \mathrm{UC}_n$ its rank is given by its crossing number $\mathrm{rk}(\tau) = \mathrm{cross}(\tau) + 1$. The atoms of UC_n are thus the pairings with no crossings.

Noncrossing chord diagrams with n chords are a classical object counted by the *Catalan numbers*, $\dfrac{1}{n+1}\dbinom{2n}{n}$, which is a well known sequence that enumerates many well studied combinatorial objects. [45, Exercise 6.19] and [47] give an extensive listing of combinatorial objects enumerated by the Catalan numbers.

To count the number of pairings on $[2n]$ one can construct a pairing by choosing what is paired to 1, for which there are $2n-1$ options, then choosing what is paired to the smallest as of yet unpaired point, for which there are $2n-3$ options, continuing in this manner we see the number of pairings is

$$(2n-1)(2n-3)\cdots 1 = (2n-1)!!.$$

The maximal element of UC_n is the pairing $(1, n+1)(2, n+1)\cdots(n, 2n)$ in which every pair of arcs forms a crossing. This pairing has $\binom{n}{2}$ crossings so the uncrossing poset UC_n is rank $\binom{n}{2}+1$.

Lam showed that the uncrossing poset UC_n is Eulerian in [35, Theorem 1]. This means that every nontrivial interval in UC_n has an equal number of odd and even rank elements. The proof relied on an embedding of the uncrossing poset into the dual of an affine Bruhat order. Lam leveraged this map to give a direct counting proof that the uncrossing poset is Eulerian.

Theorem 1.2.2 (Lam [35, Theorem 1])**.** *The uncrossing poset* UC_n *is Eulerian.*

A poset can be viewed topologically via its order complex. The *order complex* of a poset P is the simplicial complex $\Delta(P)$ consisting of all chains of P. When P has a $\widehat{0}$ and $\widehat{1}$ we are usually interested in the order complex of the *proper part* of P defined as the subposet $P \setminus \{\widehat{0}, \widehat{1}\}$. A shelling of a simplicial complex is an ordering $F_1, \dots, F_k$ of the facets such that for each facet F_i the intersection $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ is pure of dimension $\dim(F_i) - 1$. A poset is shellable if its order complex is shellable.

A lexicographic shelling of a poset is one induced by certain types of edge labelings of the Hasse diagram of the poset. The simplest type is an EL-labeling. An *EL-labeling* of a poset P is a labeling λ of the covers of the poset with an ordering of the labels given such that each interval $[x, y]$ of P has a unique maximal chain whose sequence of labels is weakly increasing and furthermore this chain's sequence of labels is lexicographically least among all maximal chains of $[x, y]$. There are other techniques to produce lexicographic shellings in which the notion of a descent in a label sequence is weakened or in which the labeling λ is allowed to depend upon a choice of a maximal chain.

Lam conjectured that UC_n is lexicographically shellable in [35, Conjecture 1], and Hersh and Kenyon proved this fact in [28, Theorem 3.18]. Hersh and Kenyon gave an EC-labeling of the dual poset UC_n^*. This induces a shelling for the uncrossing poset UC_n as a chain of UC_n is a chain of UC_n^* and vice versa hence the order complexes $\Delta(\mathrm{UC}_n)$ and $\Delta(\mathrm{UC}_n^*)$ coincide.

Theorem 1.2.3 (Hersh–Kenyon [28, Theorem 3.18])**.** *The poset* UC_n^* *has an EC-labeling.*

The EC-labeling that Hersh and Kenyon constructed depended in part upon a classical EL-labeling for the Bruhat order on the symmetric group. Certain intervals from this Bruhat order appear as intervals in the dual poset UC_n^*.

Given a permutation π of $[n]$ define its *inversion number* to be

$$\mathrm{inv}(\pi) = |\{i < j : \pi(i) > \pi(j)\}|.$$

The *Bruhat order* on the symmetric group $\mathfrak{S}_n$ has cover relations defined by $\pi_1 \prec \pi_2$ when $\mathrm{inv}(\pi_2) = \mathrm{inv}(\pi_1) + 1$ and $\pi_2 = (i, j)\pi_1$ for some transposition (i, j). Figure A1 in Appendix A depicts the Hasse diagram of the Bruhat order on $\mathfrak{S}_3$ and Figure A5 depicts the Hasse diagram of the Bruhat order on $\mathfrak{S}_4$. This partial order can be visualized by placing points $1, \dots, n$ along the top and bottom of a rectangle and connecting a point i along the bottom the point $\pi(i)$ along the top with a strand. If

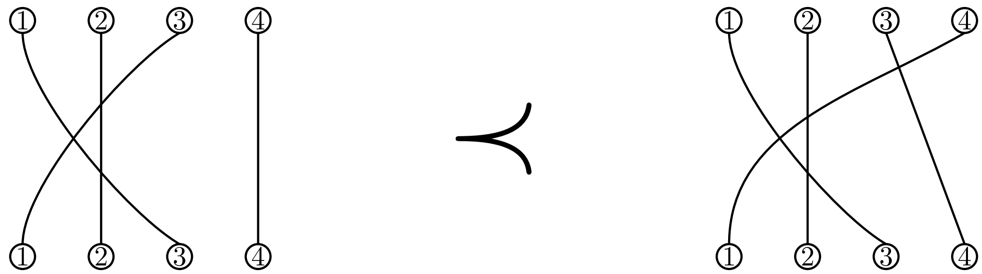


Figure 1.5: Diagrams associated to the cover relation $3214 \prec 4213$.

this diagram is drawn such that no two strands cross more than once, and no strand crosses itself then the inversion number of the permutation is the number of crossings between the strands. Multiplying the permutation on the left by a transposition (i, j) corresponds to swapping the ends of the strands at i and j along the top of the diagram. See Figure 1.5.

Hersh and Kenyon showed if $\sigma < \tau$ are pairings such that the sets

$$\{i \in [2n] : i < \sigma(i)\} \text{ and } \{i \in [2n] : i < \tau(i)\}$$

coincide then the interval $[\sigma, \tau]$ in UC_n is dual to an interval of the Bruhat order. We give a more detailed statement and a proof in Proposition 2.2.4.

We recall the definition of a regular CW complex.

Definition 1.2.4 ([12, Definition 4.7.4])**.** *A regular CW complex is a topological space* Γ *presented as a union* $\Gamma = \bigcup_\alpha \gamma_\alpha$ *of closed cells* γ_α *satisfying the following properties.*

(i) Each cell γ_α *is homeomorphic to a closed ball of some dimension.*

(ii) For any two distinct cells γ_α *and* γ_β *the interiors* $\gamma_\alpha \setminus \partial\gamma_\alpha$ *and* $\gamma_\beta \setminus \partial\gamma_\beta$ *are disjoint.*

(iii) For any cell γ_α *the boundary* $\partial\gamma_\alpha$ *is a union of closed cells.*

The dimension of a regular CW complex Γ is the maximum dimension of a cell of Γ. The *face poset* of a regular CW complex Γ consists of all closed cells of Γ along with a $\widehat{0}$ and $\widehat{1}$, with the cells ordered by inclusion. A regular CW complex which is homeomorphic to a sphere is said to be a *regular CW sphere*. Figure 1.6 depicts a regular CW sphere whose face poset is isomorphic to the uncrossing poset UC_3. For any finite regular CW complex the face poset determines the CW complex up to homeomorphism since the order complex of the proper part of the face poset is homeomorphic to the CW complex [37, Theorem 1.7].

For a thorough treatment of CW complexes in full generality see [37], or for a shorter introduction see the appendix in [27]. Section 4.7 (b) in [12] contains an introduction to CW complexes.

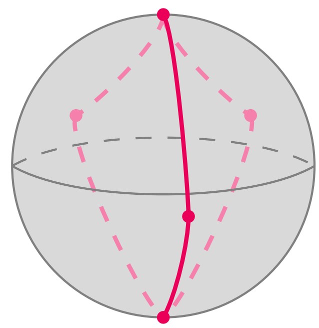


Figure 1.6: A regular CW sphere of dimension 2.

Hersh and Kenyon also used their shelling result to show that the uncrossing poset is isomorphic to the face poset of a regular CW complex.

Corollary 1.2.5 (Hersh-Kenyon [28, Corollary 3.19])**.** *The uncrossing poset* UC_n *is isomorphic to the face poset of a regular CW complex.*

Since the uncrossing poset UC_n is isomorphic to the lower interval

$$[\widehat{0}, (1, n+1) \cdots (n-1, 2n)(2n+1, 2n+2)]$$

of UC_{n+1} this corollary implies the CW complex whose face poset is UC_n is a sphere.

1.2.1 Electrical Networks

The uncrossing poset originates from the theory of planar electrical networks. Medial diagrams are essentially in bijection with certain graphs, and local crossing resolutions in medial diagrams corresponds to deleting and contracting edges of these graphs. Viewing the uncrossing poset in terms of these graphs will be useful in Chapter 2. The paper [36] by Lam is a reference for the material in this subsection.

A *circular planar embedding* of a graph is an embedding into the closed disk of the plane with all edges embedded within the interior of the disk. The vertices may lie on the boundary. A *planar electrical network* is a finite graph together with a circular planar embedding and real valued edge weights which are thought of as modeling conductance. A planar electrical network has an associated linear transformation called the *response matrix* which maps a vector of voltages input to the boundary vertices to a vector of currents flowing in to the network at each boundary vertex. Two planar electrical networks are said to be *electrically equivalent* when they have the same response matrix. A partition of the boundary vertices of a planar graph is said to be *noncrossing* when the convex hulls of the vertices in each block of the partition are disjoint.

For each noncrossing partition of the vertices there is an associated *grove measurement* for the network, these grove measurements determine the response matrix and vice versa. In [36] Lam uses these grove measurements as a coordinatization of the space of planar electrical networks by the real projective space indexed by noncrossing partitions. The space of projective coordinates of planar electrical networks

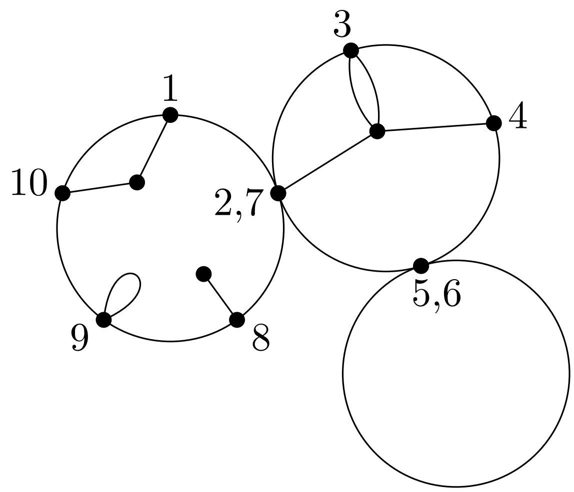


Figure 1.7: A cactus graph.

is stratified with each stratum consisting of the coordinates for networks with a fixed set of vanishing coordinates.

To every planar electrical network G there is an associated medial diagram $M(G)$. Any medial diagram for the pairing represented by $M(G)$ is associated to a graph which is electrically equivalent to G. Thus certain pairings index the equivalence classes of electrical networks. In [36] Lam introduced a compactification of the space of electrical networks, consisting of networks called cactus networks, in which the electrical equivalence classes are indexed by all pairings. The poset UC_n is the face poset of the stratification of the space of projective coordinates of cactus networks ([36, Theorem 5.8]).

A *cactus network* with n boundary vertices is a finite graph together with real valued edge weights, a planar embedding into the disc with edges in the interior, n vertices on the boundary, and a boundary partition identifying the boundary vertices that is a noncrossing partition. A *cactus graph* is the underlying embedded graph of a cactus network, that is, a cactus network without any edge weights. We depict cactus graphs embedded into the union of discs, each attached to another at a boundary vertex, that is obtained by pinching the disc so that any two boundary vertices identified by the boundary partition come together. Figure 1.7 depicts a cactus graph.

There is a correspondence between cactus graphs and medial diagrams. Let G be a cactus graph with n boundary vertices $v_1, \ldots, v_n$. To construct the medial diagram $M(G)$, on each disk for each boundary vertex v place two new points v^+ and v^- just to the left and right of v. We construct the arcs of $M(G)$ piecewise. For each newly added vertex u placed left or right of a boundary vertex v, add an arc segment between u and the edge e that is incident to v and that is closest to u near v. If no such edge exists connect v^+ and v^- by an arc. For each pair (v, F) where v is an internal vertex of the cactus graph G and F is a face of G incident to v, add an arc segment between the two edges contained in F incident to v. If there is only

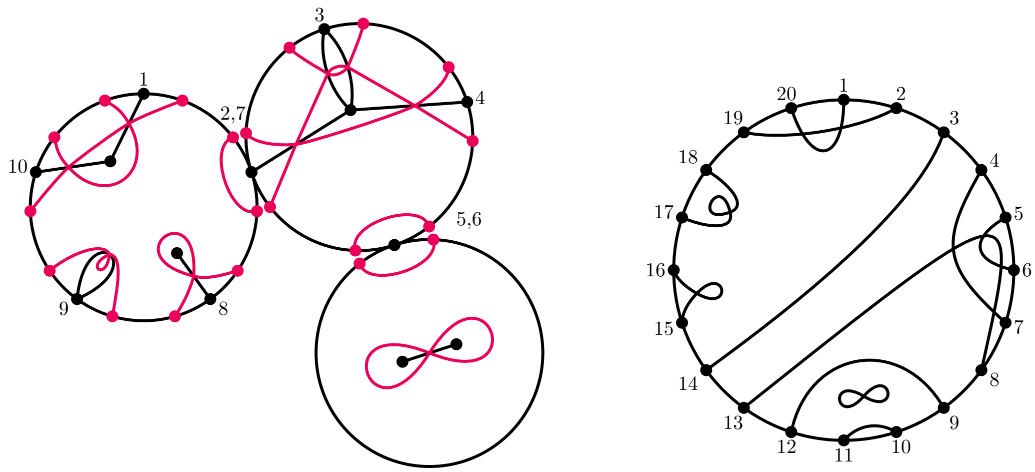


Figure 1.8: The construction of a medial diagram from a cactus graph.

one such edge, add an arc segment which is a closed loop attached to this edge and wrapping around v. If v is an isolated internal vertex, add an arc that is a closed loop around v. The endpoints of any arc segment are either a boundary vertex or the endpoint of four arc segments. Where four arc segments meet is interpreted as a transversal intersection of arcs, loosely speaking the two arcs pass straight through the intersection.

Closed loop arcs in medial diagrams were not discussed previously, to clarify adding any such arcs does not change what pairing is represented, and neither will resolving any crossings involving such an arc.

The correspondence between cactus graphs and medial diagrams is essentially a bijection. Figure 1.8 shows a cactus graph and its associated medial diagram. Given a medial diagram M with $2n$ points, the cactus graph G is constructed as follows. For $i = 1, \ldots, n$, in between the points $2i - 1$ and $2i + 1$ place a boundary vertex v_i. Color the regions of the medial diagram black and white so that no two regions sharing an edge are the same color. The regions containing the boundary vertices v_i are all the same color, say white. For each white internal region of the medial diagram place an internal vertex. Then for each pair of white regions sharing an intersection in the medial diagram draw an edge between the associated internal vertices. Lastly the boundary partition of the cactus graph identifies any two boundary vertices that are in the same region of the medial graph.

Given a medial diagram M let $G(M)$ denote the associated cactus graph, and given a cactus graph G let $M(G)$ denote the associated medial diagram. Any cactus graph has an associated pairing $\tau(M(G))$ which we denote simply as $\tau(G)$. A cactus graph G is said to be *critical* when the number of edges of G equals the crossing number $\mathrm{cross}(\tau(G))$. Each pairing τ corresponds to a stratum in the stratification of the space of cactus networks and this stratum is parameterized by varying edge weights on a critical cactus graph representation of τ.

Observe that when given a medial graph M along with its corresponding cactus graph $G = G(M)$ that the crossings in M correspond to the edges of G. Further-

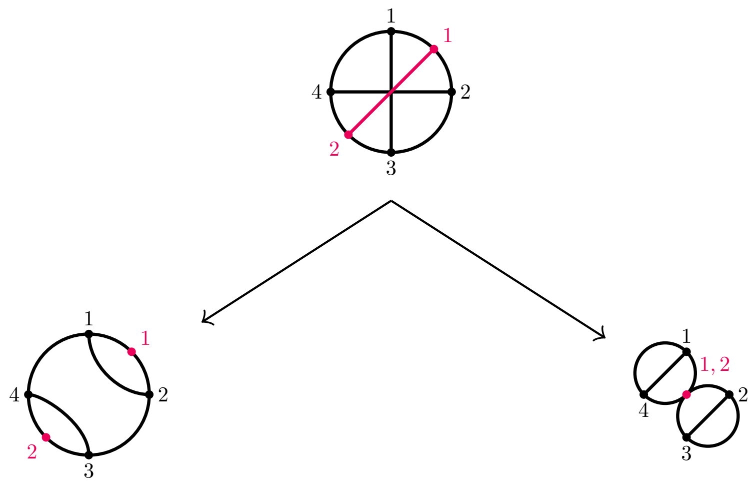


Figure 1.9: Resolving a crossing in a medial diagram along with the associated cactus graphs.

more resolving a crossing in the medial diagram M corresponds to either deleting or contracting the corresponding edge in the cactus graph G. See Figure 1.9. Note that a cactus graph has no edge labels or labelings of internal vertices, but the boundary vertices are labeled. If a contraction identifies two boundary vertices, that is, the contraction merges two blocks in the boundary partition, then the labels of these two boundary vertices are merged. Given two pairings $\tau_1, \tau_2 \in \mathrm{UC}_n$ we have $\tau_1 \leq \tau_2$ if and only if there is a cactus graph G_2 with $\tau(G_2) = \tau_2$ and a sequence of deletions and contractions which when applied to G_2 result in a cactus graph G_1 with $\tau(G_1) = \tau_1$.

1.2.2 Noncrossing partition lattice

We have already seen one relationship between noncrossing partitions and the uncrossing poset, namely, every cactus graph has a noncrossing boundary partition. In this subsection we describe the lattice structure of noncrossing partitions. For details of this structure and some of its varied connections, the reader can refer to [38].

Given a partition p of $[n]$ and $i \in [n]$ let $p(i)$ denote the block of p that contains i. A partition p of $[n]$ is said to be *noncrossing* when for any indices $i <_i j <_i k <_i \ell$ if $p(i) = p(k)$ and $p(j) = p(\ell)$ then $p(i) = p(j)$, that is, all four indices i, j, k, ℓ are in the same block. Noncrossing partitions are typically drawn via a circular diagram where the points $1, \ldots, n$ are placed clockwise along the boundary of a circle and the blocks are depicted as the convex hull of their points. A partition is noncrossing if and only if no two blocks in this circular diagram intersect.

The *noncrossing partition lattice* $\mathcal{NC}_n$ is the set of noncrossing partitions of $[n]$ partially ordered by reverse refinement, that is, $p_1 \leq p_2$ when $p_1(i) \subseteq p_2(i)$ for all i. This does indeed define a lattice, the maximal element has a single block $[n]$ and the meet $p = p_1 \wedge p_2$ of two partitions is defined by $p(i) = p_1(i) \cap p_2(i)$ for $i \in [n]$. Figure 1.10 depicts the noncrossing partition lattice $\mathcal{NC}_4$.

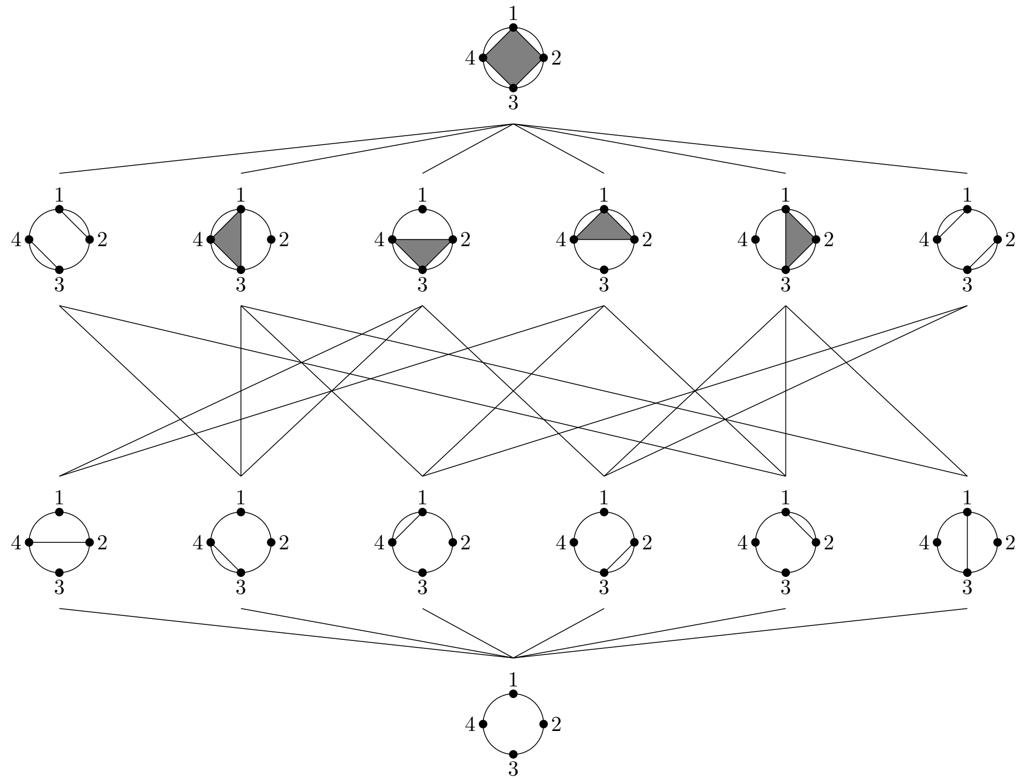


Figure 1.10: The noncrossing partition lattice $\mathcal{NC}_4$.

The number of noncrossing partitions of $[n]$ is given by the nth Catalan number thus noncrossing partitions of $[n]$ are in bijection with pairings on $[2n]$ with no crossings. The construction of a cactus graph gives such a bijection. Given a pairing τ with no crossings, draw a medial diagram M for τ with no crossings. The cactus graph $G(M)$ has no edges and is thus essentially the same as its boundary partition which is a noncrossing partition of $[n]$. This bijection also shows up in the computation of a certain map on noncrossing partitions called the Kreweras complementation, first introduced by Kreweras in [33]. Given a noncrossing partition p, to construct the *Kreweras complement* $\kappa(p)$ first add $2n$ pairing points on either side of the partition points $1, \ldots, n$. Each pairing point lies in a region bordered by the boundary of the circle and the blocks of the partition. Draw an arc from each pairing point to the first pairing point in the same region when proceeding clockwise or counterclockwise around the region in the direction towards the associated partition point. Then add n new complement partition points. The ith complement partition point is placed in between the pairing points $2i-1$ and $2i-2$ for $i = 2, \ldots, n-1$ and the complement partition point 1 is placed in between the pairing points $2n$ and 1. The complement partition block $\kappa(p)(i)$ consists of all complement partition points in the same region as i cut out by the arcs of the pairing.

Figure 1.11 depicts the Kreweras complements of all elements of the noncrossing partition lattice $\mathcal{NC}_4$. The noncrossing partition lattice $\mathcal{NC}_n$ is self dual and the

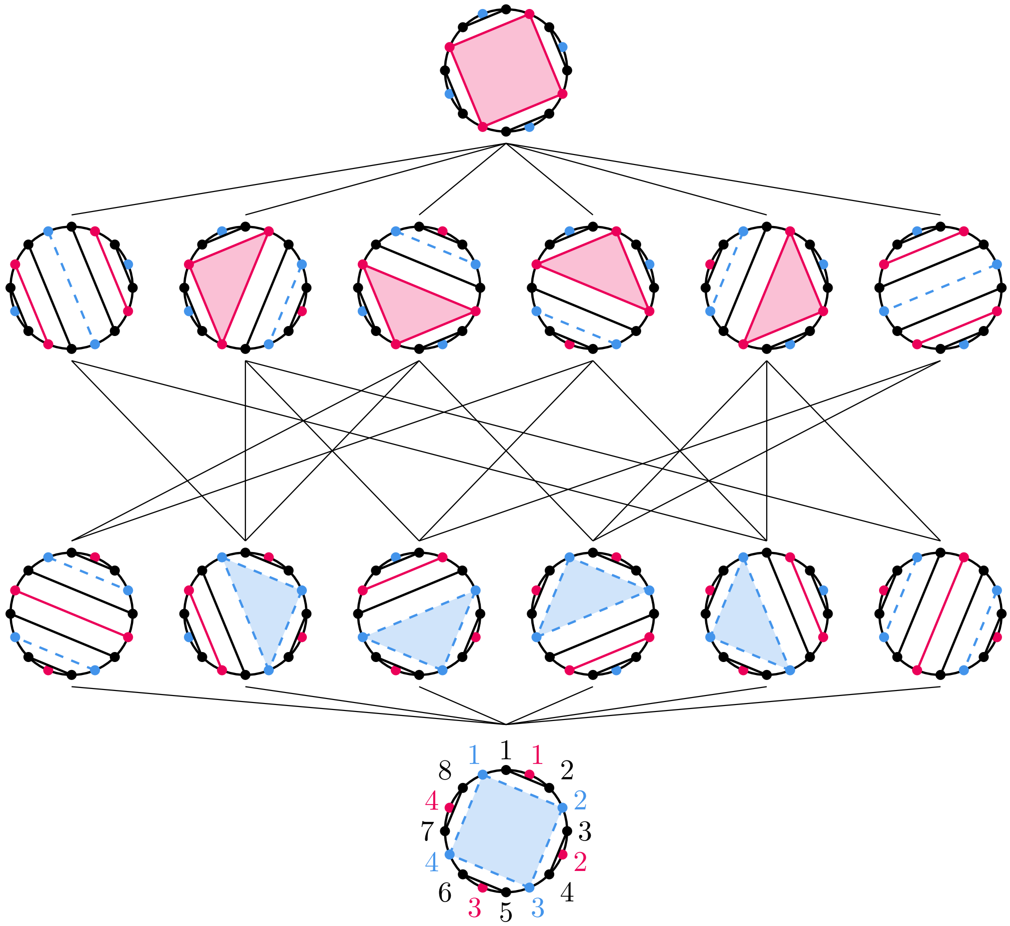


Figure 1.11: Construction of the Kreweras complements for elements of the noncrossing partition lattice $\mathcal{NC}_4$. The partitions are depicted in red with their Kreweras complements in blue with dashed lines.

Kreweras complementation is an isomorphism between $\mathcal{NC}_n$ and its dual $\mathcal{NC}_n^*$.

Biane determined in [6, Theorem 2] the *skew automorphism group* of the noncrossing partition lattice. The skew automorphism group is the group of all bijections which either are order-preserving and inverse order-preserving or are order-reversing and inverse order-reversing.

Theorem 1.2.6 (Biane)**.** *For $n \geq 3$ the skew automorphism group of the noncrossing partition lattice $\mathcal{NC}_n$ is isomorphic to the dihedral group D_{4n} of order $4n$.*

The dihedral group D_{4n} is identified with the skew automorphism group of the lattice $\mathcal{NC}_n$ via the usual action of D_{4n} applied to the partition points and the complement partition points arranged as depicted in Figure 1.11. The Kreweras complementation corresponds to the rotation by $2\pi/2n$ radians.

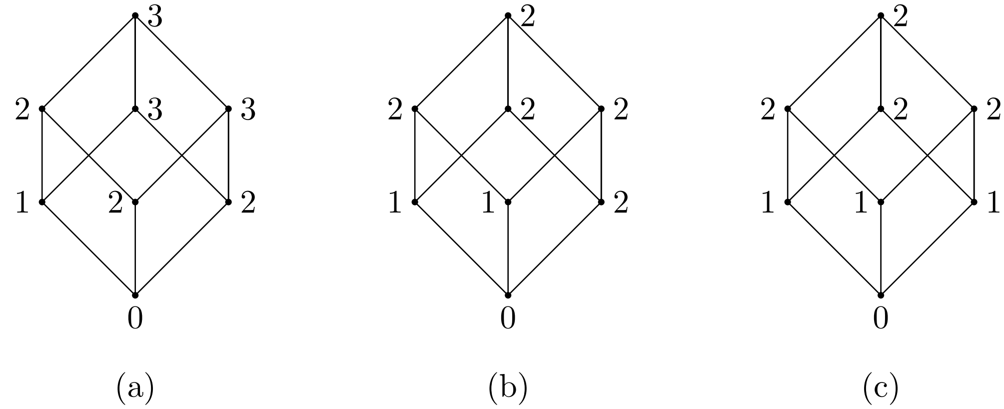


Figure 1.12: Examples of polymatroids with a ground set of size 3.

1.3 Polymatroids

In Chapter 3 we introduce the notion of a generator-enriched lattice which will be the basis for the minor posets studied in Chapter 4. Generator-enriched lattices are intimately related with polymatroids, which were introduced by Edmonds in [21] in connection with optimization theory. Edmonds introduced polymatroids as polytopes which generalize the concept of a matroid by replacing independent subsets with independent weightings by nonnegative real numbers (subsets corresponding to 0,1 weightings). Edmonds gave an equivalent definition in terms of a rank function which we will prefer for our purposes.

Given a set E let B_E denote the Boolean algebra of all subsets of E.

Definition 1.3.1. *A* polymatroid *over the ground set* E *is a function* $r : B_E \to \mathbb{R}_{\geq 0}$ *such that the following three conditions hold.*

(i) $r(\emptyset) = 0$.

(ii) The function r *is order-preserving.*

(iii) For any subsets $X, Y \subseteq E$ *we have* $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Figure 1.12 depicts several examples of polymatroids. For a subset X of the ground set the image $r(X)$ is referred to as the rank of X (with respect to r). Condition (iii) in Definition 1.3.1 is referred to as *submodularity.* When a polymatroid $r : B_E \to \mathbb{R}_{\geq 0}$ is integer valued and also satisfies $r(X) \leq |X|$ for all $X \subseteq E$, it is a *matroid.* Figure 1.12 (c) is a matroid while Figure 1.12 (a) and (b) are not.

Matroids are rich combinatorial objects that abstract notions of independence such as acylic subsets of graphs and linear independence in vector spaces. The book [39] is a standard reference for matroid theory. Two examples of matroids motivate much of the theory and its terminology. Given a graph G there is an associated matroid whose ground set is the set of edges of G. The rank of set X of edges is the size of a minimal spanning subgraph of X. The other motivating example comes from vector configurations. Given a set E of vectors in a vector space V there is a matroid whose ground set is the set E and the rank of a subset is the dimension of the linear span in V. If E is a set of any subspaces of V, not all 1-dimensional, defining the rank of a subet X of E to be the dimension of the smallest subspace of V including X does not define a matroid, but this gives an example of a polymatroid.

Let r be a polymatroid with ground set E. An element $e \in E$ is said to be a *loop* when $r(\{e\}) = 0$. Two elements $e, f \in E$ are said to be *parallel* when $r(\{e\}) = r(\{e, f\}) = r(\{f\})$. The *parallel class of* e is the collection of elements parallel to e. A polymatroid with no loops or distinct parallel elements is said to be *simple*.

The operations of deletion and contraction generalize to (poly)matroids. The *deletion* of r by $X \subseteq E$ is the polymatroid $r \setminus X = r|_{B_{E \setminus X}}$, that is, the usual function restriction of r to $B_{E\setminus X}$. The *contraction* of r by X is the polymatroid r/X defined on $E \setminus X$ by $(r/X)(Y) = r(Y \cup X) - r(Y)$. These operations are essentially the restriction of r to a lower, respectively, an upper interval of the Boolean algebra B_E, but with a reindexing and a shift in the latter case.

The *simplification* of a polymatroid is any polymatroid obtained by deleting all loops and deleting all but one element in each parallel class. Any two simplifications of a polymatroid are isomorphic in that there is a bijection between the ground sets that preserves the rank function.

A matroid r on the ground set E has an associated closure operator

$$\overline{\cdot} :\to B_E \to B_E$$

defined by $\overline{X} = \{e \in E : r(X \cup \{e\}) = r(X)\}$. Note that the submodularity of r implies that $r(\overline{X}) = r(X)$. The closure operator determines the matroid, and this can be used to give an equivalent definition for matroids. The closed sets $\overline{X}$ of the matroid r are referred to as the *flats* of r. The flats have the structure of a lattice when order by inclusion, the meet is the intersection and the join is the closure of the union.

A lattice L is said to be *submodular* if whenever x and y in L cover the meet $x \wedge y$ then the join $x \vee y$ covers both x and y. Equivalently a lattice is submodular if it is graded and the rank function is submodular in that for all x and y we have $\text{rk}(x \vee y) + \text{rk}(x \wedge y) \leq \text{rk}(x) + \text{rk}(y)$; see [46, Proposition 3.3.2]. A lattice is said to be *atomic* when every element can be expressed as a join or atoms, in other words, only the atoms are join irreducible. A lattice that is submodular and atomic is said to be *geometric*. Birkhoff characterized the lattices that are isomorphic to the lattice of flats of a matroid as finite geometric lattices in [9]. The lattice of flats when considered only up to isomorphism encodes only the simple information of the matroid in that the simplification of the matroid can be recovered.

The notion of the closure operator of a polymatroid makes sense via the same definition as for matroids, but this does not uniquely determine the polymatroid. Similarly, the flats of a polymatroid can be defined in the same way and the flats form a lattice, but this carries less information about the polymatroid than the lattice of flats carries of a matroid. In Theorem 3.3.7 we show that generator-enriched lattices encode the simple information of polymatroid closure operators in a way analogous to geometric lattices and matroids. We describe in Section 3.4 the operations of deletion and contraction, which are well-defined on closure operators of polymatroids, in terms of the associated generator-enriched lattice. These operations are further studied in Chapter 4 in regard to a partial ordering.

1.4 PL-spheres, CW posets and the zipping operation

In Chapter 4 we introduce minor posets of generator-enriched lattices. The order relations of this poset correspond to performing deletions and contractions on a generator-enriched lattice. The main result of this chapter is a construction for minor posets in terms of Reading's zipping operation. This operation was introduced in [41, Section 4] where it was used to study intervals in Bruhat orders and their **cd**-indices. We use this construction to show that the minor poset of any generator-enriched lattice is isomorphic to the face poset of a regular CW sphere and that the order complex of the proper part of any minor poset is a PL-sphere. A generalization of the dual of the zipping operation to quasi-graded posets that geometrically corresponds to merging strata in Whitney stratifications was introduced in [23]. We introduce the zipping operation and related background needed for this construction and its corollaries here.

Definition 1.4.1. *Let P be a poset. A triple $x, y, z \in P$ is said to be a* zipper *when the following three conditions hold:*

(i) The element z covers x and y and no other element in P.

(ii) $\{p \in P : p < x\} = \{p \in P : p < y\}$.

(iii) The element z is the join of x and y.

We denote a zipper as $x, y \prec z$. Note that in a zipper $x, y \prec z$ the element z uniquely determines the zipper since z only covers x and y.

Definition 1.4.2. *Let P be a poset and let $x, y \prec z$ be a zipper. The* zipped poset, *denoted* $\mathrm{zip}(P, z)$*, is obtained from P by identifying the elements x, y, z as a new element w with order relation defined by the following three conditions holding for all $p, q \in P \setminus \{x, y, z\}$:*

1. $p \leq q$ in $\mathrm{zip}(P, z)$ if and only if the same holds in P.

2. $p \leq w$ in $\mathrm{zip}(P, z)$ if and only if $p \leq x$, hence $p \leq y$ and $p \leq z$, in P.

3. $p \geq w$ in $\mathrm{zip}(P, z)$ if and only if $p \geq x$ or $P \geq y$ in P.

When the poset P is the face poset of a regular CW complex, a zipper $x, y \prec z$ consists of an open cell z such that

$$\partial z = x \cup y \cup \partial x$$

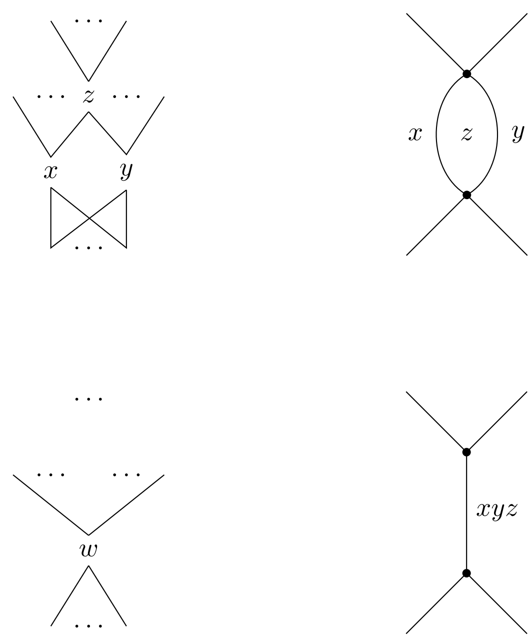


Figure 1.13: A schematic depiction of a zipping operation in the Hasse diagram on the left and in a CW complex on the right.

where x and y are open cells such that $\partial x = \partial y$ and such that item (iii) in Definition 1.4.1 holds. Zipping the zipper contracts the cell z while identifying the two cells x and y. Figure 1.13 illustrates this.

A graded poset is said to be *thin* when every rank 2 interval forms a diamond.

Remark 1.4.3. *If P is a thin poset then a triple $x, y \prec z$ forms a zipper whenever conditions (i) and (iii) of Definition 1.4.1 are satisfied. Condition (ii) follows from thinness and condition (i).*

Proposition 1.4.4 (Reading [41, Proposition 4.4])**.** *If P is a graded and thin poset and $x, y \prec z$ form a zipper then the poset* $\mathrm{zip}(P, z)$ *is graded and thin as well.*

The zipping operation is well-behaved topologically in that it preserves PL-sphericity. Recall a simplicial complex Δ is said to be a *PL-sphere* when there is a piecewise linear homeomorphism from Δ to the boundary of a simplex. Equivalently Δ is a PL-sphere when there is a simplicial subdivision of Δ that is combinatorially isomorphic to a simplicial subdivision of the boundary of a simplex. A poset P is said to be a PL-sphere when the order complex $\Delta(P)$ is a PL-sphere. Background on PL-spheres can be found in [12, 4.7 (d)] and [30]. Proper parts of face lattices of polytopes, in particular of the face lattice of a cube, are examples of posets that are PL-spheres.

In order to derive the **cd**-index inequalities for minor posets in Section 4.3, it will be important to observe that when a poset P is a PL-sphere every open interval of P is a PL-sphere as well. To see this, given a simplicial complex Δ and a face X, define the *link of* X to be the subcomplex

$$\text{link}_\Delta(X) = \{Y \in \Delta : X \cap Y = \emptyset,\ X \cup Y \in \Delta\}.$$

Let P be a poset with $\widehat{0}$ and $\widehat{1}$ and let $x < y$ in P. The order complex of the open interval (x, y) appears as a link in the order complex $\Delta(P \setminus \{\widehat{0}, \widehat{1}\})$, namely, as the link of any chain $C = C_x \cup C_y$, where C_x is a maximal chain of $(\widehat{0}, x]$ and C_y is a maximal chain of $[y, \widehat{1})$. Links of PL-spheres are themselves PL-spheres.

Lemma 1.4.5 (Hudson [30, Corollary 1.16])**.** *Given a simplicial complex* Δ *that is a PL-sphere, for any face* X *of* Δ *the link* $\text{link}_\Delta(X)$ *is a PL-sphere.*

Theorem 1.4.6 (Reading [41, Theorem 4.7])**.** *Let* P *be a poset such that* $x, y \prec z$ *form a zipper. If the proper part of* P *is a PL-sphere then so is the proper part of* $\text{zip}(P, z)$*.*

Reading's zipping operation also behaves nicely with respect to the **cd**-index. The **cd**-index and the effect of zipping operations on the **cd**-index is discussed at the end of the chapter in Section 1.5.

We also use the following characterization, due to Björner, in Chapter 4 to establish that the minor poset of any generator-enriched lattices is isomorphic to the face poset of a regular CW sphere.

Definition 1.4.7 (Björner [11, Definition 2.1])**.** *A poset* P *is a CW poset if the following three conditions hold:*

1. P *has a unique minimal element* $\widehat{0}$ *and a unique maximal element* $\widehat{1}$*.*
2. $|P| \geq 3$*.*
3. $\Delta((\widehat{0}, p))$ *is homeomorphic to a sphere for all elements* p *in the open interval* $(\widehat{0}, \widehat{1})$*.*

Definition 1.4.7 differs slightly from the one given in [11] since in the present context face posets are adjoined with a maximal element which does not correspond to a cell.

Theorem 1.4.8 (Björner [11, Proposition 3.1])**.** *A poset is a CW poset if and only if it is isomorphic to the face poset of a regular CW complex.*

1.5 The cd-index

In this section we give an overview of the **cd**-index. For a more complete, discussion see [3]. The **cd**-index of a poset is a polynomial in noncommutative variables **c** and **d** which enumerates chains in a very compact way, abstracting the problem of counting flags in polytopes. The **cd**-index was introduced by Bayer and Klapper in [5].

Let P be a graded rank $n + 1$ poset with $\widehat{0}$ and $\widehat{1}$. For $S \subseteq [n]$ let P_S denote the *rank selected poset*, the subposet of P consisting of elements whose rank is an element of S adjoined with $\widehat{0}$ and $\widehat{1}$. Define f_S to be the number of maximal chains of P_S. The collection $(f_S)_{S\subseteq[n]}$ is called the *flag f-vector of P*. Define h_S by

$$h_S = \sum_{T\subseteq S} (-1)^{|S\setminus T|} f_T.$$

The collection $(h_S)_{S\subseteq[n]}$ is called the *flag h-vector of P*. Note that the flag h-vector is the image of the flag f-vector under a linear transformation. Furthermore this transformation is invertible. Using Möbius inversion the above relation is equivalent to

$$f_S = \sum_{T\subseteq S} h_T.$$

Let $\mathbf{a}$ and $\mathbf{b}$ be noncommutative variables each of degree 1. For $S \subseteq [n]$ we encode the set S as a monomial $u_S = u_1 \cdots u_n$ defined by

$$u_i = \begin{cases} \mathbf{a} & i \notin S, \\ \mathbf{b} & i \in S. \end{cases}$$

Define the **ab**-*index* of P to be the polynomial

$$\Psi(P) = \sum_{S\subseteq[n]} h_S u_S.$$

The **ab**-index is a generating function for the flag h-vector of P and thus contains all the information of the flag f-vector as well. Note that the **ab**-index is homogeneous of degree n.

Define noncommutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. We consider $\mathbf{c}$ to be degree 1 and $\mathbf{d}$ to be degree 2. If the **ab**-index $\Psi(P)$ can be expressed as a polynomial in the variables $\mathbf{c}$ and $\mathbf{d}$ then this polynomial is the **cd**-*index* of P. We denote the **cd**-index of P by $\Psi(P)$ as well. This is unlikely to cause confusion, we will discuss the **ab**-index only sparingly.

The **cd**-index is a very compact way to enumerate chains in P in which all linear redundancies are removed. This feature is reflected in a result due to Bayer and Billera [4, Proposition 2.2], namely the existence of the **cd**-index implies all homogeneous linear relations between entries of the flag f-vector that hold for any Eulerian poset of rank $n+1$. The additional fact that the coefficient of $\mathbf{c}^n$ in the **cd**-index is 1 implies all such linear relations, homogeneous and nonhomogeneous.

Example 1.5.1. *Consider the Boolean algebra B_3 depicted in Figure 1.1. Since B_3 is rank 3 here $n = 2$. The entries of the flag f-vector and flag h-vector are:*

S	f_S	h_S	u_S
$\emptyset$	*1*	*1*	**aa**
1	*3*	*2*	**ba**
2	*3*	*2*	**ab**
12	*6*	*1*	**bb**

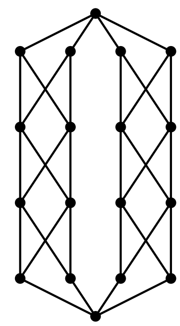


Figure 1.14: A poset whose **cd**-index has a negative coefficient.

Summing the terms $h_S u_S$ we obtain

$$\begin{aligned}\Psi(B_3) &= \mathbf{a}^2 + 2\mathbf{ab} + 2\mathbf{ba} + \mathbf{b}^2 \\ &= \mathbf{c}^2 + \mathbf{d}.\end{aligned}$$

The coefficients of the **cd**-index can be negative. For example, the poset depicted in Figure 1.14 has **cd**-index $\mathbf{c}^4 + 2\mathbf{c}^2\mathbf{d} + 2\mathbf{dc}^2 - 4\mathbf{d}^2$. See [26] for a discussion of this poset and similar posets in regard to the existence of an R-labeling.

In small examples such as the computation of $\Psi(B_3)$ shown above, the translation from the **ab**-index to the **cd**-index can be done without too much difficulty in an ad hoc manner. For larger examples and for computer calculations this is not really appropriate and an algorithm is desirable.

The following process, which is described in the discussion after Lemma 1.1 in [44], gives an algorithmic way to compute the **cd**-index from the **ab**-index. This is implemented in the computer program described in Section 6.3. First, we define a new variable $\mathbf{e} = \mathbf{a} - \mathbf{b}$. Observe $2\mathbf{a} = \mathbf{c} + \mathbf{e}$ and $2\mathbf{b} = \mathbf{c} - \mathbf{e}$, so we may express the **ab**-index in terms of **e** and **c** simply by substituting $(\mathbf{c} + \mathbf{e})/2$ for **a** and substituting $(\mathbf{c} - \mathbf{e})/2$ for **b**. Call the result of this substitution the **ce**-index. Since $\mathbf{e}^2 = \mathbf{c}^2 - 2\mathbf{d}$ any even power of **e** can be expressed as a polynomial in **c** and **d**. No odd power of **e** is a polynomial in **c** and **d** since both **c** and **d** are symmetric in **a** and **b** but **e** is not. Thus, the **cd**-index exists if and only if the **ce**-index has only even powers of **e**, and when this is the case the **cd**-index can then be computed by substituting $\mathbf{c}^2 - 2\mathbf{d}$ for $\mathbf{e}^2$.

The **ab**-index can also be computed more directly by assigning a certain weight to chains and summing over all chains of P that contain $\widehat{0}$ and $\widehat{1}$. This viewpoint is often more useful in arguments that require directly counting chains in posets. The weight of a chain C is $w(C) = w_1 \cdots w_n$ defined by

$$w_i = \begin{cases} \mathbf{b} & \text{if there is a rank } i \text{ element of } C, \\ \mathbf{a} - \mathbf{b} & \text{otherwise.} \end{cases}$$

The **ab**-index is $\Psi(P) = \sum_C w(C)$ where the sum is over all chains C in P such that $\widehat{0}, \widehat{1} \in C$.

There has been a lot of interest in establishing nonnegativity of the **cd**-index of geometrically motivated classes of Eulerian posets. It turns out a sufficient condition for nonnegativity is the Gorenstein* condition. Without going into detail, a poset is said to be Gorenstein* when the order complex of its proper part and every link in this complex have the homology of a sphere of the same dimension. See [24, Section 2.1] for a precise definition of the Gorenstein* condition. Any poset that is shellable is Gorenstein*, in particular face lattices of polytopes are Gorenstein*. Additionally, face posets of regular CW spheres are Gorenstein* as are posets with a $\widehat{0}$ and $\widehat{1}$ whose proper part is a PL-sphere.

Many results lead up to the nonnegativity result for Gorenstein* posets. Purtill showed that the coefficients of the **cd**-index are nonnegative for face lattices of polytopes of dimension at most 5 [40, Proposition 7.11], and polytopes with simplicial facets [40, Corollary 7.8]. More generally Purtill showed that any CL-shellable Eulerian poset whose proper upper intervals are all Boolean algebras has a **cd**-index with nonnegative coefficients in [40, Corollary 7.4]. Stanley showed that the **cd**-index of the face poset of any S-shellable (spherically shellable) regular CW sphere, a class which includes all polytopes, has nonnegative coefficients in [44, Theorem 2.2], and also that the **cd**-index of any Gorenstein* poset such that all lower intervals are Boolean algebras has nonnegative coefficients in [44, Corollary 3.1]. Reading established the nonnegativity of certain coefficients of **cd**-indices of all Gorenstein* posets in [42, Theorem 3]. The most general case below was conjectured by Stanley in [44, Conjecture 2.1] and proved by Karu.

Theorem 1.5.2 (Karu [31, Theorem 1.3])**.** *If P is a Gorenstein* poset then the coefficients of the* **cd***-index $\Psi(P)$ are nonnegative.*

There has also been interest in inequalities relating **cd**-indices of different posets. Billera, Ehrenborg and Readdy showed that the set of **cd**-indices of lattices of regions of oriented matroids is minimized by the **cd**-index of the cross polytope [8, Corollary 7.5], and in particular that the set of **cd**-indices of zonotopes is minimized by the **cd**-index of the cube [8, Corollary 7.6]. Billera and Ehrenborg gave certain inequalities between the **cd**-index of a polytope and the **cd**-index of a face [7, Theorem 5.1], a particular case being $\Psi(P) \geq \Psi(\mathrm{Pyr}(F))$ where F is a facet of the polytope P and $\mathrm{Pyr}(F) = F \times B_1$ [7, Corollary 5.2]. This result was then used to show that the set of **cd**-indices of polytopes is minimized by the **cd**-index of the simplex, that is, the **cd**-index of the Boolean algebra [7, Theorem 5.3]. Billera and Ehrenborg also showed that the **cd**-indices of the cyclic polytopes coefficientwise maximize **cd**-indices of polytopes in [7, Theorem 6.5]. Generalizing the lower bound for polytopes, Ehrenborg and Karu showed in [24, Corollary 1.3] that the **cd**-index of the Boolean algebra is the minimum among all Gorenstein* lattices of a given rank. In Chapter 4 we prove some inequalities for minor posets (Corollaries 4.3.10, 4.3.11 and 4.3.14).

Reading's zipping operation (Definition 1.4.2) behaves nicely with respect to the **cd**-index. This is the main tool we use to prove inequalities in Chapter 4.

Theorem 1.5.3. *If P is an Eulerian poset and $x, y \prec z$ form a zipper, then the poset $\mathrm{zip}(P, z)$ is Eulerian as well.*

(a) (Reading [41, Proposition 4.5, Theorem 4.6]) If $z \neq \widehat{1}_P$ *then*

$$\Psi(\mathrm{zip}(P,z)) = \Psi(P) - \Psi([\widehat{0},x]_P) \cdot \mathbf{d} \cdot \Psi([z,\widehat{1}]_P).$$

(b) (Stanley [44, Lemma 1.1]) If $z = \widehat{1}_P$ *then*

$$\Psi(\mathrm{zip}(P,z)) = \Psi(P) \cdot \mathbf{c}.$$

Chapter 2 The uncrossing poset

2.1 Introduction

In this chapter we study the uncrossing poset from a combinatorial and structural view. In Section 2.2 we encode pairings as certain pairs consisting of a Dyck path and a permutation. This encoding leads to a decomposition of the uncrossing poset into lower intervals of the Bruhat order; see Proposition 2.2.6. Furthermore, when examining the atoms of the uncrossing poset the encoding gives a bijection between Dyck paths and 312-avoiding permutations. We show this bijection is an isomorphism between the Bruhat order, respectively weak order, resricted to 312-avoiding permutations and dominance order on Dyck paths, respectively the Tamari lattice.

In Section 2.3 we show in Proposition 2.3.2 the CW complex whose face poset is isomorphic to the uncrossing poset has a 1-skeleton isomorphic to the Hasse diagram of the noncrossing partition lattice. We then use this result to show the automorphism group of the uncrossing poset is the dihedral group.

Finally, in Section 2.4 we study the structure of the simpler lower intervals in the uncrossing poset. In Theorem 2.4.3 we characterize the lower intervals that are isomorphic to the face lattice of a cube. In Proposition 2.4.6 we describe a zipping construction to produce the next simplest lower intervals in the uncrossing poset from the face lattice of a cube. To end the chapter we discuss some of the difficulties preventing us from extending this construction to more complicated cases which motivates us to study generator-enriched lattices in the following chapters.

2.2 312-avoiding permutations and Dyck paths

In this section we discuss a bijection between pairings and certain pairs of permutations and Dyck paths. By considering the pairs that correspond to a pairing with no crossings this leads to a bijection between Dyck paths and 312-avoiding permutations which is well structured. This bijection also leads to a decomposition of the uncrossing poset into lower intervals from the Bruhat order on the symmetric group generated by 312-avoiding permutations.

We first give an alternate definition of the order relation of the uncrossing poset which is more intuitive than local crossing resolutions and does not depend on the choice of a representative for pairings. This characterization is stated in terms of the action of $\mathfrak{S}_n$ on pairings, viewed as fixed point free involutions, by conjugation. In other words for $\pi \in \mathfrak{S}_n$ and $\tau \in \mathrm{UC}_n \setminus \{\widehat{0}\}$, we define $\pi(\tau) = \pi\tau\pi^{-1}$. Acting with a transposition (i, j) on a pairing τ corresponds to swapping i and j in the pairing or by commuting the transposition on both sides past τ swapping $\tau(i)$ and $\tau(j)$.

Lemma 2.2.1. *We have $\sigma \leq \tau$ in $\mathrm{UC}_n \setminus \{\widehat{0}\}$ if and only if there is a sequence of transpositions $t_1, \ldots, t_k$ such that $\sigma = t_k \cdots t_1(\tau)$ and letting t_0 be the identity permutation* *for* $i = 1, \ldots, k$ *we have*

$$\operatorname{cross}(t_i \cdots t_0(\tau)) - \operatorname{cross}(t_{i-1} \cdots t_0(\tau)) = 1.$$

Proof. Let σ and τ be pairings on $[2n]$. It will suffice to show that $\sigma \prec \tau$ if and only if there exists indices i and j such that $\sigma = (i,j)\tau(i,j)$ and $\operatorname{cross}(\tau) - \operatorname{cross}(\sigma) = 1$. Let M be a reduced medial diagram representing τ. We have $\sigma \prec \tau$ if and only if there is a crossing in M which when locally resolved results in a reduced medial diagram M' representing σ. Let i and j be points such that there is an arc from i to $\tau(j)$ in the medial diagram M' for σ. The local resolution creates a double intersection for indices k that satisfy $i <_i k <_i j$ and $\tau(i) <_i \tau(k) <_i \tau(j)$. On the other hand, applying the transposition (i,j) removes an extra crossing for each such index k. Note the effects of the local resolution and of the transposition agree if and only if the local resolution creates no pair of distinct arcs that cross an even number of times. Furthermore, since the medial diagram M is reduced and a single resolution was performed to obtain M', two arcs in the medial diagram M' cross no more than twice. Thus applying the transposition (i,j) to the pairing τ has the same effect as applying the local resolution if and only if $\operatorname{cross}(\tau) - \operatorname{cross}((i,j)\tau(i,j)) = 1$. Therefore $(i,j)\tau(i,j) \prec \tau$ if and only if $\operatorname{cross}(\tau) - \operatorname{cross}((i,j)\tau(i,j)) = 1$. □

Recall a *Dyck path* is a lattice path with steps $(1,0)$ (or an E step) and $(0,1)$ (or an N step) from the origin $(0,0)$ to the point (n,n) that lies weakly above the line $y = x$. We will typically encode Dyck paths as an NE-word with $2n$ letters. An NE-word defines a Dyck path if and only if there are n N steps and n E steps and the ith E step has at least i N steps preceding it. Given a Dyck path we use the notation N_i to refer to the ith N step (when not indexing the E steps), and similarly E_i refers to the ith E step of the given path.

Given a Dyck path P define the *height sequence* $h(P) = (h_1, \ldots, h_n)$ by letting h_i be the number of N steps preceding the ith E step. Equivalently h_i is the height of the path at the ith E step when drawn in the plane. Observe that the map $P \mapsto h(P)$ is a bijection onto the set of sequences with n terms $h_1, \ldots, h_n$ such that $h_i \leq h_{i+1}$ and $i \leq h_i \leq n$.

Let $\tau \in \mathrm{UC}_n$ be a pairing. We associate a Dyck path $P(\tau) = P_1, \ldots, P_{2n}$ by setting $p_i = N$ if $i < \tau(i)$ and otherwise setting $p_i = E$. Note that $P(\tau)$ is indeed a Dyck path: the ith E step is the larger point in the ith pair and the corresponding N step is preceded by $i-1$ other N steps, those which correspond to the smaller point in the first $i-1$ pairs.

We also associate to each pairing τ on $[2n]$ a permutation $\pi(\tau)$ on $[n]$. Let $P = p_1, \ldots, p_{2n} = P(\tau)$. The permutation $\pi = \pi(\tau)$ is defined by the condition $\pi(i) = j$ if the ith E step is paired with the jth N step by τ. It will be convenient to view the pair (P, π) as the Dyck path P with each step E_i labeled by $\pi(i)$. We will denote labeled Dyck paths with the labels as superscripts on the E steps. See Figure 2.1. The mapping $\tau \mapsto (P(\tau), \pi(\tau))$ is a bijection onto pairs (P, π) such that $\pi(i) \leq h(P)_i$. The index of the step E_i in P is paired to the index of the step $N_{\pi(i)}$ by τ.

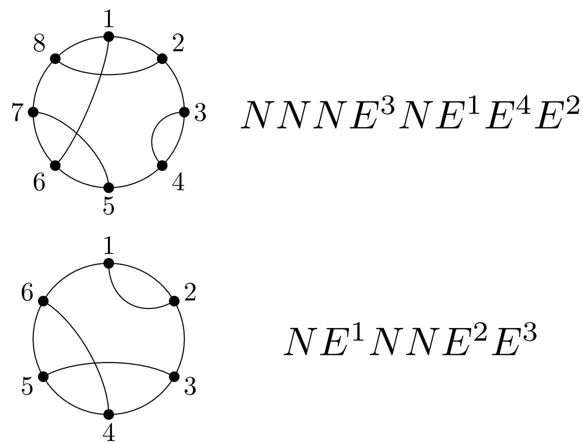


Figure 2.1: Examples of pairings and the associated labeled Dyck path.

This bijection is not in essence new. Hersh and Kenyon considered certain words in bijection with pairings, obtained from the labeled Dyck path by replacing N_i with i and E_i with $\tau(i)$ (see [28, Definition 3.2]). In [36, Section 4.7] Lam discussed Catalan subsets which encode the Dyck path as the set of indices of the N steps, and Catalan necklaces which are in bijection with pairings. These Catalan subsets also appeared in [15].

Observe that applying a transposition to a pairing corresponds to applying the same transposition to the associated labeled Dyck path as a word, though with one complication. If the N steps of the path are reordered, say by a permutation σ, then the labels of the E steps are permuted by σ^{-1} corresponding to resorting the N steps. Note that the same effect of swapping two N steps can be achieved by swapping two E steps. This corresponds to changing which ends of the strands on the medial diagram are swapped. From this perspective there are two kinds of swaps we can apply to a given pairing: those which swap two E steps, and those which swap an N step and an E step. We show in Proposition 2.2.4 that the E, E swaps correspond to relations in the Bruhat order, a result originally due to Hersh and Kenyon [28, Theorem 3.8].

The number of crossings of a pairing can be expressed in terms of the associated Dyck path and permutation. Recall that $\mathrm{inv}(\pi)$ is the number of pairs $i < j$ such that $\pi(i) > \pi(j)$. For a Dyck path $P = P_1, \dots, P_{2n}$ define $\mathrm{inv}(P)$ to be the number of pairs $i < j$ such that $P_i = E$ and $P_j = N$.

Lemma 2.2.2. *For any pairing τ we have*

$$\mathrm{cross}(\tau) = \binom{n}{2} - \mathrm{inv}(\pi(\tau)) - \mathrm{inv}(P(\tau)).$$

Proof. We show that $\binom{n}{2} = \mathrm{cross}(\tau) + \mathrm{inv}(\pi(\tau)) + \mathrm{inv}(P(\tau))$ by partitioning the set of unordered tuples from $[n]$ into three classes. Let $i < j$ be indices. Consider the labeled Dyck path corresponding to τ and the subword consisting of the four steps corresponding to the four points incident to the ith and the jth arcs of τ. We have the following three possibilities:

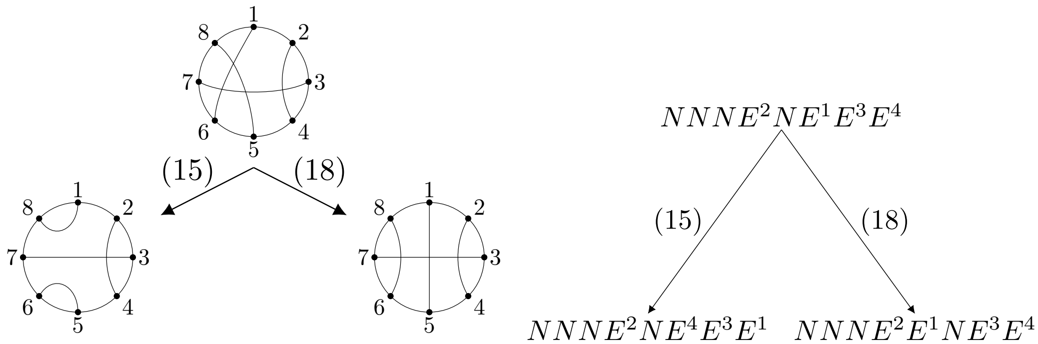


Figure 2.2: Two transpositions, (15) and (28), applied to a pairing, viewed on the medial graph and in terms of the labeled Dyck paths.

$$N_iN_jE^iE^j, \tag{2.1}$$

$$N_iN_jE^jE^i, \tag{2.2}$$

$$N_iE^iN_jE^j. \tag{2.3}$$

Observe that Case 2.1 is the only case corresponding to a crossing of τ, that Case 2.2 is the only case where E_j precedes E_i hence corresponding to an inversion of $\pi(\tau)$ and that Case 2.3 is the only case contributing to $\mathrm{inv}(P(\tau))$. □

Dominance order on Dyck paths is the poset Dom_n consisting of all Dyck paths with $2n$ steps with the ordering $P_1 \leq P_2$ if when drawn in the plane the path P_2 always lies weakly above the path P_1. Equivalently, we have $P_1 \leq P_2$ when $h(P_1)_i \leq h(P_2)_i$ for $i = 1, \ldots, n$. Figures A2 and A6 in Appendix A depict the Hasse diagrams of Dom_3 and Dom_4. This poset was studied in [2]. This is in fact a distributive lattice. Define a poset on the transpositions of $[n]$ by setting $(i, j) \leq (k, \ell)$ if $i \leq k \leq \ell \leq j$. Mapping a Dyck path P to the set of transpositions (i, j) such that the unit square whose bottom left corner is at the coordinate (i, j) lies below P gives an isomorphism between Dom_n and the lattice of lower order ideals. This is a well known bijection.

The following is a stronger version of [28, Proposition 3.9].

Lemma 2.2.3. *If $\sigma < \tau$ in UC_n then $P(\sigma) \leq P(\tau)$ in Dom_n.*

Proof. It suffices to consider the case where $\sigma \prec \tau$, that is, where $\sigma = (i, j)\tau$ and $\mathrm{cross}(\tau) - \mathrm{cross}(\sigma) = 1$. If the transposition (i, j) swaps two E steps in the labeled Dyck path of τ then the underlying path $P(\tau)$ is unchanged. The only other possibility, by Lemma 2.2.2 is that an E step and an N step are swapped with the N step preceding the E step in $P(\tau)$. This implies that $P(\sigma) < P(\tau)$ since $h(P(\sigma))$ and $h(P(\tau))$ agree except between the two steps that were swapped where $h(P(\tau))$ exceeds $h(P(\sigma))$. □

Proposition 2.2.4 (Hersh–Kenyon [28, Theorem 3.8])**.** *Given pairings $\sigma < \tau$ in UC_n with $P(\sigma) = P(\tau)$ the interval $[\sigma, \tau]$ is anti-isomorphic to the interval $[\pi(\tau), \pi(\sigma)]$ of $\mathfrak{S}_n$.*

Proof. We show that the covers of the two intervals are in bijection with the order reversed. Consider a cover relation $\eta_1 \prec \eta_2$ in the interval $[\sigma, \tau]$. By Lemma 2.2.1 $\text{cross}(\eta_2) - \text{cross}(\eta_1) = 1$ and $\eta_1 = (i, j)\eta_2(i, j)$ for some $i, j \in [2n]$. Since $P(\sigma) = P(\tau)$ by Lemma 2.2.3 we have that $P(\tau) = P(\eta_1) = P(\eta_2)$. Thus the transposition (i, j) must exchange either two N steps or two E steps. We may assume (i, j) exchanges two E steps by replacing this transposition with $(\eta_2(i), \eta_2(j))$ if necessary. The effect on the permutation $\pi(\eta_2)$ is multiplying on the left by a transposition, namely (i', j') assuming it was the i'th and j'th E steps exchanged. Thus, $\pi(\eta_1)$ is obtained from $\pi(\eta_2)$ via a single transposition and Lemma 2.2.2 shows that this transposition creates one inversion. Thus, $\pi(\eta_1) \succ \pi(\eta_2)$ in Bruhat order.

Consider applying a transposition (i, j) to the permutation $\pi = \pi(\sigma)$ such that $\text{inv}(\pi) - \text{inv}((i, j)\pi) = 1$. Since the transposition (i, j) removes an inversion it necessarily moves the larger of the two between i and j rightward in π. Since the larger element is moved rightward we have that $(i, j)\pi(i) \leq h(P)_i$. Thus the labeled Dyck path $(P, (i, j)\pi)$ defines a pairing. By continuing in this manner every permutation π' such that $\sigma \leq \pi' \leq \pi$ defines a pairing from the labeled Dyck path (P, π'). Furthermore by the same reasoning as above applying a transposition to a permutation π' corresponds to exchanging two E steps in the labeled Dyck path (P, π') and removing an inversion of π' creates a crossing in the pairing. Therefore, the inverse map is cover reversing as well. □

We now proceed to examine this bijection with labeled Dyck paths on the atoms in UC_n, that is, the pairings on $[2n]$ with no crossings. Noncrossing chord diagrams are a classical Catalan object. When restricted to pairings with no crossings the map $\tau \mapsto P(\tau)$ is a bijection onto the set of Dyck paths with $2n$ steps. To verify this one can check that the inverse map is as follows. Given a Dyck path $P = P_1, \ldots, P_{2n}$ construct a pairing τ on the steps of P by pairing each E step from left to right to the first N step preceding it which is not already paired. This clearly has no crossings: if $i < j < \tau(i)$ then $i < \tau(j) < \tau(i)$ as if P_j is an E step then it is paired to an element greater than i since i is paired to a step after P_j, and if P_j is an N step then $\tau(j) < \tau(i)$ as otherwise $\tau(i)$ would be paired to j.

Since the map $\tau \mapsto (P(\tau), \pi(\tau))$ is a bijection and when considering atoms the map $\tau \mapsto P(\tau)$ is a bijection, we have that the map $\tau \mapsto \pi(\tau)$ is a bijection between pairings with no crossings and some subset of permutations. This gives a bijection between Dyck paths and this subset of permutations by defining $\pi(P)$ to be the permutation $\pi(\tau)$ where τ is the pairing with no crossings satisfying $P(\tau) = P$. Lemma 2.2.2 tells us that the permutation $\pi(P)$ has the most inversions among permutations σ satisfying $\sigma(i) \leq h(P)_i$ for $i = 1, \ldots, n$. Thus we can explicitly construct $\pi = \pi(P)$ by defining

$$\pi(i) = \max\left(\{x \in [n] \setminus \{\pi(1), \ldots, \pi(i-1)\} : x \leq h(P)_i\}\right).$$

We use this description below to show that this map $P \mapsto \pi(P)$ is a bijection between Dyck paths and 312-avoiding permutations. A permutation π is said to have an *occurrence of 312* when there are indices $i < j < k$ such that $\pi(j) < \pi(k) < \pi(i)$. A permutation is said to be *312-avoiding* when it has no occurrences of the pattern 312.

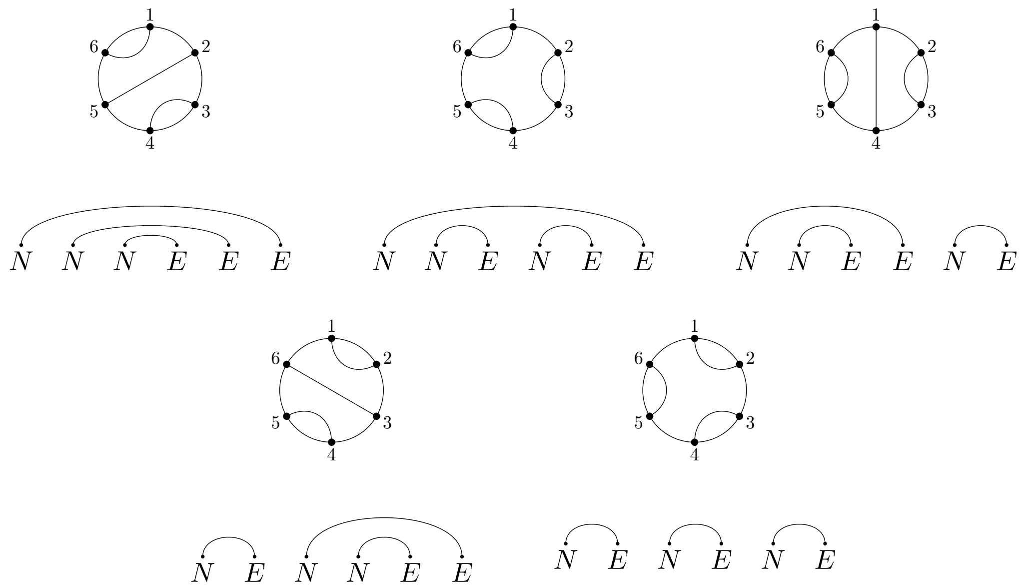


Figure 2.3: The bijection between atoms of the uncrossing poset UC_3 and the five Dyck paths with 6 steps.

The class of 312-avoiding permutations, and more generally σ-avoiding permutations for any $\sigma \in \mathfrak{S}_3$, are a classical Catalan object [32].

Lemma 2.2.5. *The map $P \mapsto \pi(P)$ defined above is a bijection between Dyck paths with $2n$ steps and 312-avoiding permutations of $[n]$.*

Proof. For any permutation $\pi \in \mathfrak{S}_n$ define $h(\pi)$ to be the sequence defined by $h(\pi)_i = \max\{\pi(1), \ldots, \pi(i)\}$. Note that $h(\pi)$ is a nondecreasing sequence and $h(\pi)_i \geq i$, hence it is the height sequence of some Dyck path. Define $P(\pi)$ to be the Dyck path P with $h(P) = h(\pi)$.

Let $\pi = \pi(P)$ for some Dyck path P. As discussed above we have

$$\pi(i) = \max\left(\{x \in [n] \setminus \{\pi(1), \ldots, \pi(i-1)\} : x \leq h(P)_i\}\right).$$

For each index i where P achieves a new height, that is, $h(P)_i > h(P)_{i-1}$, we must have $\pi(i) = h(P)_i$ since $\pi(j) < h(P)_j < h(P)_i$ for $j < i$. Hence the successive maximums of π, that is, the terms of the sequence $h(\pi)_i$, are exactly the heights $h(P)_i$ of the path P. Therefore $P(\pi(P)) = P$.

Let π be a 312-avoiding permutation. To show that $\pi(P(\pi)) = \pi$ we must show that, setting $h = h(\pi)$, we have

$$\pi(i) = \max\left(\{x \in [n] \setminus \{\pi(1), \ldots, \pi(i-1)\} : x \leq h_i\}\right).$$

Suppose for some index j that $\pi(j)$ is not this maximum, that is, there exists an index $k > j$ such that $\pi(k) > \pi(j)$ and $\pi(k) \leq h_j$. Clearly we cannot have $\pi(j) = h_j$, hence there exists an index $i < j$ where $\pi(i) = h_j$. Observe that $\pi(i) > \pi(k) > \pi(j)$ hence we have an occurrence of 312 in π. Since π is 312 avoiding we then must have that $\pi(j)$ is this maximum hence $\pi(P(\pi)) = \pi$. □

This bijection allows us to establish the following decomposition for the uncrossing poset. To state this decomposition, given a Dyck path P define $\mathrm{UC}_n(P)$ to be the subposet of the uncrossing poset UC_n consisting of all pairings τ with $P(\tau) = P$.

Proposition 2.2.6. *The uncrossing poset* UC_n *decomposes as a disjoint union consisting of the minimal element and the subposets* $\mathrm{UC}_n(P)$*, that is,*

$$\mathrm{UC}_n = \{\widehat{0}\} \cup \dot{\bigcup}_P \mathrm{UC}_n(P).$$

Furthermore, for all Dyck paths P the subposet $\mathrm{UC}_n(P)$ *is an interval of* UC_n *and is anti-isomorphic to the lower interval of Bruhat order on the symmetric group generated by the 312-avoiding permutation* $\pi(P)$.

Proof. The fact that the subposets $\mathrm{UC}_n(P)$ are disjoint and their union is UC_n is evident. Let τ be the atom corresponding to the Dyck path P. The identity permutation id certainly satisfies $\mathrm{id}(i) \leq h(P)_i$, so there is a pairing σ whose labeled Dyck path is (P, id). The interval $[\tau, \sigma]$ is included in $\mathrm{UC}_n(P)$. Conversely given any pairing η with $P(\eta) = P$ one can resolve all crossings with transpositions that exchange two E steps in the labeled Dyck path. This results in a pairing with no crossings whose associated Dyck path is P, hence $\eta \geq \tau$. We also have $\eta \leq \sigma$ since applying transpositions to $\pi(\eta)$ to remove all inversions results in the identity permutation, and these transpositions correspond to applying E, E swaps to η resulting in the pairing σ. Therefore $\mathrm{UC}_n = [\tau, \sigma]$ as claimed. This interval is anti-isomorphic to the interval $[\mathrm{id}, \pi(\tau)]$ by Proposition 2.2.4. Lemma 2.2.5 shows that the permutation $\pi(\tau)$ is 312-avoiding. Given any 312-avoiding permutation π Proposition 2.2.4 and Lemma 2.2.5 show that the interval $[\mathrm{id}, \pi]$ is anti-isomorphic to the interval $[\tau, \sigma]$ in UC_n where τ has labeled Dyck path $(P(\pi), \pi)$ and σ has labeled Dyck path $(P(\pi), \mathrm{id})$. □

Figure 2.4 depicts the decomposition of the uncrossing poset UC_3 and Figure A14 in Appendix A depicts the decomposition of the uncrossing poset UC_4.

It can be seen from Lemma 2.2.2 that the bijection between Dyck paths and 312-avoiding permutations sends the area statistic of a Dyck path

$$\sum_i h(P)_i - i = \binom{n}{2} - \mathrm{inv}(P)$$

to the inversion number of the associated permutation. This area statistic is the rank of the path in dominance order, so the bijection sends the rank function of dominance order to the rank function of Bruhat order. We show below that in fact this bijection is a poset isomorphism between dominance order on Dyck paths and Bruhat order on 312-avoiding permutations. The result that dominance order on Dyck paths is isomorphic to Bruhat order on 312-avoiding permutations was first shown in [2, Theorem 5.1] using a map passing through noncrossing partitions. We first need the following lemma

Before this we discuss the map $\pi \mapsto h(\pi)$ defined in the proof above. It gives the nearest 312-avoiding permutation above π.

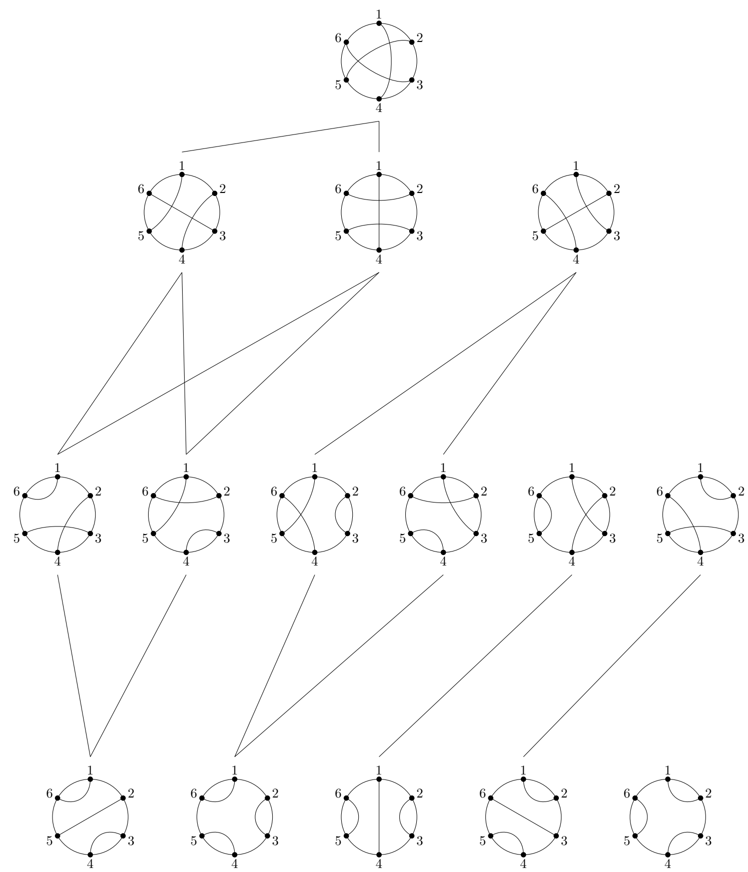


Figure 2.4: The decomposition of UC_3 into 312-avoiding permutation lower intervals.

Proposition 2.2.7. *Let $\pi, \sigma \in \mathfrak{S}_n$ be permutations with π 312-avoiding. We have $\sigma \leq \pi$ if and only if $h(\sigma)_i \leq h(\pi)_i$ for $i = 1, \ldots, n$.*

Proof. Set $P = P(\pi)$. Let τ be the pairing whose labeled Dyck path is (P, π). Let η be the pairing with labeled Dyck path (P, id). There exists a pairing with labeled Dyck path (P, σ) if and only if $h(P)_i = h(\pi)_i \geq \sigma(i)$. By Proposition 2.2.6 we have $\sigma \leq \tau$ if and only if there exists a pairing with labeled Dyck path (P, σ). □

We note as a curiosity that the above result may be restated as that the maps $\pi \mapsto h(\pi)$ and $P \mapsto \pi(P)$ form a Galois correspondence between $\mathfrak{S}_n$ and dom_n.

Corollary 2.2.8. *The poset of 312-avoiding permutations under Bruhat order is isomorphic to dominance order on Dyck paths via the map $\pi \mapsto P(\pi)$.*

Proof. Applying Proposition 2.2.7 with both σ and π as 312-avoiding permutations shows that $\sigma \leq \pi$ if and only if $P(\sigma) \leq P(\pi)$. □

Surprisingly our bijection between 312-avoiding permutations and Dyck paths is also an isomorphism between two other classical partial orderings of permutations

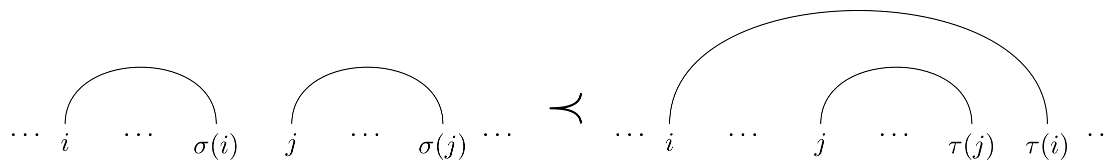


Figure 2.5: A schematic depiction of a cover relation of the poset AN_n.

and Catalan objects, namely the weak order on $\mathfrak{S}_n$ restricted to 312-avoiding permutations and the Tamari lattice of binary parenthesizations of $n+1$ symbols. The Hasse diagrams of the weak order on $\mathfrak{S}_3$ and $\mathfrak{S}_4$ are depicted in Figures A3 and A7 in Appendix A. The fact that these posets are isomorphic was shown by Björner and Wachs in [14, Theorem 9.6].

The Tamari lattice T_n consists of all ways to parenthesize a product of $n+1$ symbols, with the entire product enclosed in one pair of parentheses. Cover relations correspond to applying the associative law moving a pair of parentheses rightwards. Schematically, covers have the following form.

$$\cdots\big((\ldots)\cdot(\ldots)\big)\cdot(\ldots)\cdots\prec\cdots(\ldots)\cdot\big((\ldots)\cdot(\ldots)\big)\cdots$$

The Hasse diagrams of the Tamari lattices T_3 and T_4 are depicted in Figures A4 and A8 in Appendix A. Recall binary parenthesizations on $n+1$ symbols are in bijection with Dyck paths with $2n$ steps. Given a binary parenthesization, the associated Dyck path has NE word obtained from the parenthesization by replacing multiplication symbols $\cdot$ with N and end parentheses) with E and removing the remaining symbols.

To show that the map $\pi \mapsto P(\pi)$ is an isomorphism between weak order on 312-avoiding permutations and the Tamari lattice, we use the following intermediate poset defined on pairings. Given a pairing τ two arcs $i, \tau(i)$ and $j, \tau(j)$ are said to be *aligned* if $i < \tau(i) < j < \tau(j)$ and said to be *nested* if $i < j < \tau(j) < \tau(i)$. The *align-nest order* AN_n is the poset consisting of the pairings on $[2n]$ that have no crossings. We have a cover relation $\sigma \prec \tau$ in AN_n when there are aligned arcs $i, \sigma(i)$ and $j, \sigma(j)$ with $j = \sigma(i)+1$ and τ is obtained from σ by moving $\sigma(i)$ to the position immediately following $\sigma(j)$. See Figure 2.5.

Theorem 2.2.9. *The map $\tau \mapsto P(\tau)$ is an isomorphism between the poset AN_n and the Tamari lattice T_n, and the map $\tau \mapsto \pi(\tau)$ is an isomorphism between the poset AN_n and the weak order on 312-avoiding permutations in $\mathfrak{S}_n$.*

Proof. The pairing with no crossings corresponding to a Dyck path is obtained by preceding left to right and pairing each E step to the first N step to its right which is as of yet unpaired. Observe that in terms of the associated parenthesization the associated pairing is the one that pairs each right parenthesis with the corresponding multiplication symbol.

A cover in the Tamari lattice corresponds to moving rightwards a right parenthesis which is immediately followed by the symbol $\cdot$ to the position immediately after the right parenthesis corresponding to this multiplication symbol. This exactly corresponds to a cover in the poset AN_n. Let τ be the pairing corresponding to the lower parenthesization in the cover from the Tamari lattice. Let i be the index such that $\tau(i)$ is the index of the moved right parenthesis, when not indexing occurrences of left parantheses, and let $j = \tau(i) + 1$. Then the arcs $i, \tau(i)$ and $j, \tau(j)$ are aligned, and moving the end parenthesis rightwards corresponds to moving $\tau(i)$ to the position immediately following $\tau(j)$. Thus the associated pairings form a cover in AN_n. Conversely the conditions for a cover in AN_n translate to the conditions for a cover in the Tamari lattice formed by the associated parenthesizations. Since the cover relations are in bijection we have that AN_n is isomorphic to the Tamari lattice of order n.

We have that $\pi_1 \prec \pi_2$ in the weak order on 312-avoiding permutations if there is a sequence $t_1, \ldots, t_k$ of simple transpositions such that $\pi_2 = \pi_1 t_1 \cdots t_k$, no permutation $\pi_1 t_j \cdots t_1$ is 312-avoiding and setting $t_0 = \mathrm{id}$ we have $\mathrm{inv}(\pi_1 t_0 \cdots t_j) < \mathrm{inv}(\pi_2)$ for $i = 1, \ldots, k$. Consider a permutation π that has a 312-pattern, and a pairing τ such that $\pi(\tau) = \pi$. There are indices $i < j < k$ such that $k < \pi(k) < \pi(i) < \pi(j)$. Observe that the arcs $i, \tau(i)$ and $j, \tau(j)$ cross, and each nests over the arc $k, \tau(k)$. Conversely given a pairing with three arcs, two of which cross each other and nest over the third, there is a 312-pattern in the associated permutation. Applying the transpositions t_r to the permutation corresponds to moving $\tau(i)$ in the pairing to the position immediately following the next point corresponding to an E step of the Dyck path. Note that if moving the point $\tau(i)$ in this way creates a crossing that the crossing arcs necessarily nest over a third arc. This sequence of transpositions corresponds to a cover in the weak order precisely when every intermediate permutation has a 312-pattern, hence precisely when every intermediate pairing obtained by the corresponding swaps has a crossing. The intermediate pairings all have crossings when the sequence corresponds to a cover in the poset AN_n. Therefore the covers of the weak order on 312-avoiding permutations are in bijection with the covers of the poset AN_n, hence these posets are isomorphic. $\square$

2.3 Noncrossing Partitions

Noncrossing partitions are a classical object enumerated by the Catalan numbers and thus the atoms of UC_n are in bijection with the noncrossing partitions of $[n]$. A bijection between these two objects arises quite naturally when viewing the uncrossing poset in terms of electrical networks as discussed in Section 1.2.1. To every noncrossing partition we have an associated cactus graph with no edges. This graph is formed by identifying the elements of each block of the partition. Conversely, every cactus graph with no edges is associated to a pairing with no crossings. This bijection for $n = 3$ can be seen from the atoms in the poset depicted in Figure 2.8. This bijection can also be viewed as taking a noncrossing partition to the pairing represented by the medial diagram used in the construction of the Kreweras complement; see Figure 1.11.

In this section we show that this bijection injects the Hasse diagram into the regular CW complex $\Gamma(\mathrm{UC}_n)$ that has face poset isomorphic to the uncrossing poset UC_n. We then use this result to show that the automorphism group of the uncrossing poset is the dihedral group.

We will need the following observation.

Lemma 2.3.1. *Every pairing $\tau \in \mathrm{UC}_n$ lies above a unique set of atoms.*

Proof. This follows from results in Lam's paper [36]. Lam associated to each cactus network a set of noncrossing partitions called an electroid which only depends on the electrical equivalence class. In [36, Theorem 5.28] Lam showed the electroid associated to a pairing τ is the set of noncrossing partitions which correspond to an atom in $a \in \mathrm{UC}_n$ with $a \leq \tau$. Corollary 5.33 in [36] says that we have $\tau_1 \leq \tau_2$ in UC_n if and only if the associated electroids $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy $\mathcal{E}_1 \subseteq \mathcal{E}_2$. Therefore no two distinct elements have the same electroid. □

Recall the 1-skeleton of a regular CW complex is the subcomplex consisting of the cells of dimension at most 1, that is, the vertices and the edges.

Proposition 2.3.2. *For $n \geq 3$ the 1-skeleton of the CW complex $\Gamma(\mathrm{UC}_n)$ and the Hasse diagram of the lattice $\mathcal{NC}_n$ are isomorphic as graphs.*

Proof. Given a pairing τ with no crossings let $p(\tau)$ be the associated noncrossing partition obtained as the cactus graph from any reduced medial graph of τ. We extend this bijection to a bijection between edges of the Hasse diagram of $\mathcal{NC}_n$ and rank 2 elements of the uncrossing poset UC_n. Observe that each pairing with one crossing has one associated critical cactus graph. This critical cactus graph has a single edge, and this edge has two distinct vertices lying on the boundary. The edge corresponds to the crossing of τ. Given a pairing τ with one crossing we associate two partitions $p(\tau)$ and $q(\tau)$. Let G be the critical cactus graph associated to τ. The partition $p(\tau)$ is the cactus graph obtained by deleting the edge of G, and the partition $q(\tau)$ is the partition obtained by contracting the edge of G. Observe that $q(\tau)$ is obtained by merging two blocks of $p(\tau)$, namely the vertices of G connected by an edge, hence $p(\tau) \prec q(\tau)$ in $\mathcal{NC}_n$. Conversely, given a cover relation $p \prec q$ in $\mathcal{NC}_n$, form a cactus graph G whose boundary partition is p and which has an edge between the two vertices corresponding to the two blocks of p merged to form q. This procedue is a bijection between cover relations of $\mathcal{NC}_n$ and critical cactus graphs with a single edge, hence between cover relations and pairings with a single crossing. Furthermore the two elements of the cover relation are taken to the two atoms in UC_n below the corresponding pairing with a single crossing. □

Figure 2.6 shows the noncrossing partition lattice $\mathcal{NC}_4$ with elements labeled by edgeless cactus graphs and the Hasse diagram edges labeled by cactus graphs with a single edge.

The above result can be used to describe the automorphism group of the uncrossing poset. Recall that the group of skew automorphisms of the noncrossing partition

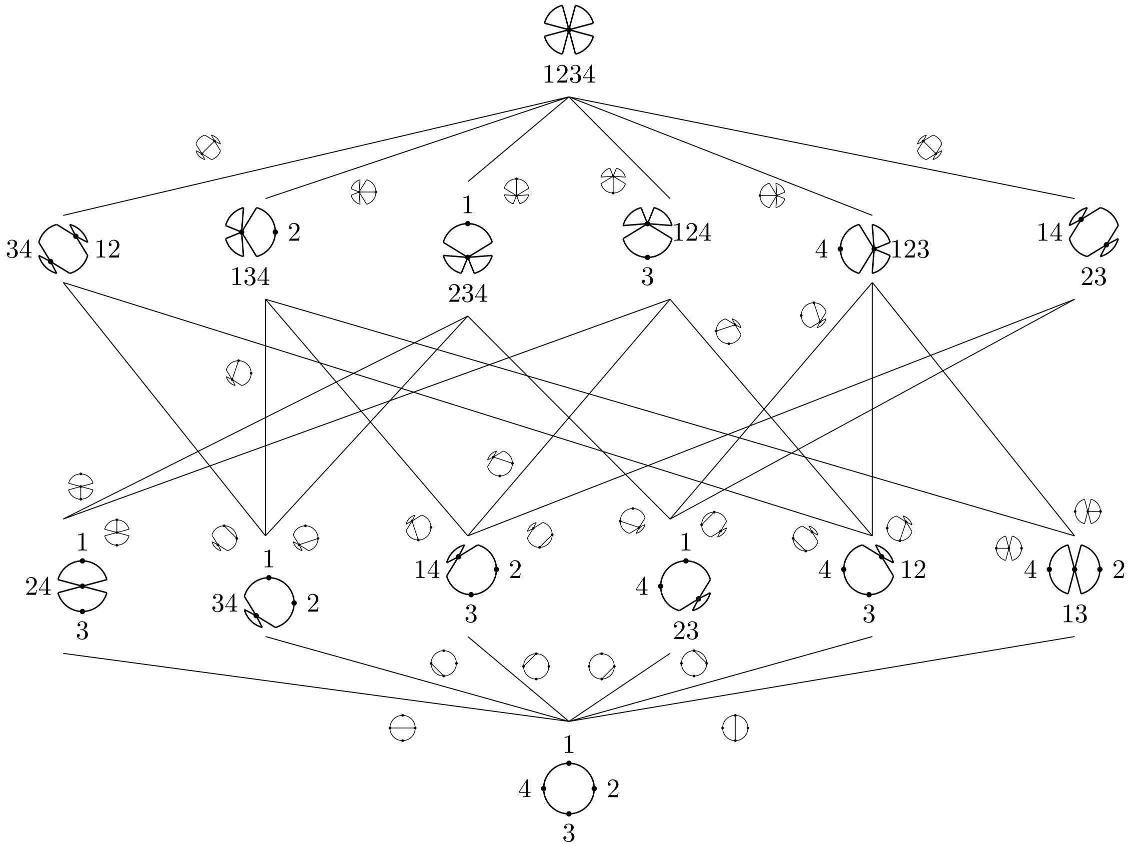


Figure 2.6: The noncrossing partition lattice $\mathcal{NC}_4$ labeled with cactus graphs. The Hasse diagram is isomorphic to the 1-skeleton of $\Gamma(\mathrm{UC}_4)$.

lattice $\mathcal{NC}_n$ is isomorphic to the dihedral group D_{4n} via the action of D_{4n} on the $2n$ points forming the partition and its Kreweras complementation as in Figure 1.11.

Proposition 2.3.3. *For $n \geq 3$ the automorphism group of the uncrossing poset UC_n is isomorphic to the dihedral group D_{4n} via the usual action on the $2n$ points of medial diagrams.*

Proof. The action of D_{4n} on medial diagrams is clearly an automorphism of UC_n for all group elements. It remains to be seen that any automorphism of UC_n may be realized as an element of D_{4n}.

Let ϕ be an automorphism of UC_n. The map ϕ induces a graph automorphism of the 1-skeleton of the CW complex $\Gamma(\mathrm{UC}_n)$, hence by Proposition 2.3.2 a graph automorphism of the Hasse diagram of $\mathcal{NC}_n$.

We claim that this graph automorphism of the Hasse diagram of $\mathcal{NC}_n$ is in fact a skew automorphism of $\mathcal{NC}_n$. We denote the map on $\mathcal{NC}_n$ induced by ϕ by ϕ as well. To prove the claim it will suffice to show that the image under ϕ of the finest partition $1/2/\cdots/n$ is either itself or the coarsest partition $12\cdots n$. Once this is established, it follows that ϕ either preserves or reverses all cover relations: any cover relation lies in some maximal chain which under ϕ maps to a path between the minimum and maximum elements of $\mathcal{NC}_n$ of the same length hence the path must be either strictly ascending or descending since $\mathcal{NC}_n$ is graded.

Recall the ordering $<_i$ on $[n]$ is defined by $i <_i i+1 <_i \cdots <_i i-1$. To prove the claim, first consider a pairing τ with no crossings with indices i, j, k such that $j <_i \tau(i) <_i \tau(j) <_i k <_i \tau(k)$. The arcs $(i, \tau(i))$ and $(k, \tau(k))$ cannot be crossed without also introducing a crossing between the arcs $(j, \tau(j))$ and $(k, \tau(k))$. Thus no such pairing can be covered by $\binom{n}{2}$ elements. A pairing without such a triple of indices must either have no nestings or one arc that nests over all the other arcs with no nestings inside the outermost arc. These two situations correspond to the pairings $(1,2)\cdots(2n-2, 2n-1)$ and $(2n-1, 1)\cdots(2n-3, 2n-2)$. Since the pairing $(1,2)\cdots(2n-1, 2n)$ has no nested arcs any two arcs can be crossed without creating a second crossing, this pairing is covered $\binom{n}{2}$ elements. Since the pairing $(2n-1, 1)\cdots(2n-3, 2n-2)$ is a rotation of the pairing $(1,2)\cdots(2n-2, 2n-1)$, it is also covered by $\binom{n}{2}$ elements.

Having proven the claim we have established that the graph automorphism ϕ acting on the Hasse diagram of $\mathcal{NC}_n$ is in fact a skew automorphism of the lattice $\mathcal{NC}_n$. Thus we have a map from the automorphism group of UC_n to the dihedral group D_{4n}. This map is in fact an injection, the image of any element of UC_n under an automorphism is determined by the images of the atoms by Lemma 2.3.1, hence by the corresponding skew automorphism of $\mathcal{NC}_n$. On the other hand the action of D_{4n} on UC_n injects D_{4n} into the automorphism group of UC_n and hence by finiteness these injections are in fact bijections. □

2.4 Lower interval structure of the uncrossing poset

In this section we study certain lower intervals of the uncrossing poset. We classify the cubic lower intervals of UC_n and describe a construction for some slightly more complicated lower intervals in terms of Reading's zipping operation. Our investigations here motivate the content of Chapters 3 and 4.

Let $\tau \in \mathrm{UC}_n$ be a pairing and choose a critical cactus graph G corresponding to τ. Recall that resolving a crossing locally in the medial diagram associated to G corresponds to deleting and contracting edges of G. We use the graph G to define a map from the face lattice Q_k of the k-dimensional cube where k is the number of edges of G onto the lower interval $[\widehat{0}, \tau]$. The face lattice Q_k is isomorphic the poset $\{0, 1, *\}^{E(G)} \cup \{\widehat{0}\}$ with componentwise order induced by the order $0 < * > 1$. We define $\phi_G : Q_k \to [\widehat{0}, \tau]$ on faces $F \in \{0, 1, *\}^{E(G)}$ by letting $\phi_G(F)$ be the pairing associated to the cactus graph obtained by contracting all edges of G which index a 1 in F and deleting the edges which index a 0 in F. We also define $\phi_G(\widehat{0}) = \widehat{0}$.

Lemma 2.4.1. *Let $\tau \in \mathrm{UC}_n$ be a pairing with k crossings and let G be a critical cactus graph representing τ. The map $\phi_G : Q_k \to [\widehat{0}, \tau]$ defined above is an order-preserving surjection. Furthermore we have $\sigma_1 < \sigma_2 < \tau$ in UC_n if and only if for all $F \in Q_k$ with $\phi(F) = \sigma_2$ there exists some $F_1 < F$ with $\phi(F_1) = \sigma_1$.*

Proof. Let M be the medial graph associated to G. Since resolving crossings in a medial diagram corresponds to deleting and contracting edges in the corresponding cactus graph, the image under ϕ_G of any face is a pairing in the interval $[\widehat{0}, \tau]$. Furthermore ϕ_G is order-preserving. To show that ϕ_G is surjective, let $\sigma < \tau$ in UC_n. Choose a saturated chain $\sigma = \sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_r = \tau$. For $i = 1, \ldots, r$ there is a transposition (x_i, y_i) such that $\sigma_{i-1} = (x_i, y_i)\sigma_i(x_i, y_i)$. Since the transposition (x_r, y_r) resolves exactly one crossing of τ, locally resolving the crossing in the corresponding way in the medial graph M of τ does not create any pair of arcs crossing more than once. This resolution does not create any loops either since the medial graph M has no pair of arcs that cross more than once. The same argument now applies to σ_{r-1}, so continuing in this manner we have a sequence of crossing resolutions which take the medial diagram M for τ to a medial diagram for σ. These crossing resolutions correspond to deletion and contraction of edges of the graph G hence to a chain in Q_k. Thus σ is in the image of ϕ_G.

Let $\sigma_1 < \sigma_2$ in $[\widehat{0}, \tau]$, and let $\phi_G(F) = \sigma_2$. Consider the cactus graph H which is obtained from G by performing the deletions and contractions as indicated by F. If H is critical then the map ϕ_H is an order-preserving surjection onto $[\widehat{0}, \sigma_2]$. Let F_1 be a face of the cross(σ_2)-dimensional cube such that $\phi_H(F_1) = \sigma_1$. Now observe that the entries of F_1 correspond to the $*$ entries of F and that the image $\phi_H(F_1)$ is the image under ϕ_G of the face obtained from F by setting the $*$ entries to either $0, 1$ or $*$ as indicated by F_1.

To finish the proof we claim that every cactus graph has a minor which is critical and has the same associated pairing. If deleting and contracting any edge of a cactus graph H results in a cactus graph H' with $\tau(H') \neq \tau(H)$ then every edge of H corresponds to a crossing of $\tau(H)$ hence H is critical. Otherwise, there is a minor H' of H with $\tau(H') = \tau(H)$. The minor H' has fewer edges than H so the claim follows by induction on the number of edges. □

Lemma 2.4.2. *Any pairing has at most one associated cactus graph which is critical and has no internal vertices.*

Proof. Let τ be a pairing and let G be an associated critical cactus graph with no internal vertices. Let $\mathcal{E}$ be the set of partitions that correspond to an atom $a < \tau$. We show that the set $\mathcal{E}$ determines the edges of G. Each partition in $\mathcal{E}$ is obtained as the boundary partition of a contraction by some set of edges of G. Hence this set has a unique minimal element, namely the boundary partition of G. Let p be this minimal element. For every edge e of G we have a partition p_e which is the boundary partition of G/e. Note that $p_e \succ p$ in $\mathcal{NC}_n$ since p_e is obtained by merging in p the two blocks corresponding to the vertices of e. On the other hand if we have any partition $q \succ p$ in $\mathcal{E}$ then there is some set of edges of G which when contracted results in a cactus graph with boundary partition q. Since $q \succ p$ only two vertices were identified in this contraction so the set of edges consists of a single edge. Therefore, the partitions $q \succ p$ in $\mathcal{E}$ are in bijection with the edges of G. Of course the edges of a simple graph determine the graph, so there can only be one critical cactus graph for τ with no internal vertices. □

Theorem 2.4.3. *Given a cactus graph G, the map ϕ_G is injective if and only if G is acyclic and has no internal vertices.*

Proof. First suppose that G has a cycle C. Let H be the result of deleting all edges of G not contained in the cycle C and contracting all but 2 edges of C. The edges of H form a 2-cycle and thus deleting either edge results in a cactus graph with the same associated pairing as H. Choosing faces on the cube which correspond to these sequences of deletions and contractions we see that ϕ_G is not injective. Now suppose G has an internal vertex. Since G is a cactus graph there is an edge e incident to this internal vertex. Consider the subgraph H of G whose only edge is e. The pairing associated to H and the pairing associated to the result of deleting all edges of G are the same. Thus, we see ϕ_G is not injective.

Every pairing $\sigma < \tau$ has some associated cactus graph which is a minor of G. Every minor of G is acyclic and has no internal vertices. This observation implies the minors are critical as well. By Lemma 2.4.2 it remains to be seen that for every minor H of G there is only one representation as an unordered sequence of deletions and contractions applied to G. This follows from the fact that G is acyclic. For every pair of vertices v and w of G which are identified in H there is exactly one path from v to w in G and this path must be contracted to form H. Every contraction of G is acylic, hence in particular has no 2-cycles or loops. This implies that the choice of edges to delete in order to form H is unique. □

Corollary 2.4.4. *Given a pairing $\tau \in \mathrm{UC}_n$ the lower interval $[\widehat{0}, \tau]$ is isomorphic to the face lattice of a cube if and only if in a reduced medial diagram for τ every region borders the boundary of the disc.*

Proof. It will suffice to show there exists a cactus graph G for τ such that ϕ_G is injective. Lemma 2.4.1 then implies that the inverse map is order-preserving. We show that there exists a reduced medial diagram of τ with every region bordering the boundary of the disc precisely when there exists a cactus graph for τ which is acyclic and has no internal vertices.

Consider the construction of the cactus graph from the medial diagram. The regions in the medial graph are two colored black and white and every black region contains a vertex of the cactus graph. A region R that does not border the disc corresponds to a sequence $i_1, i_2, \ldots, i_k, i_1$ of crossings in the medial diagram. Every intersection of arcs in the medial diagram lies in four regions, two are colored black and two are colored white. The sequence of intersections has the property that i_j and i_{j+1} are both contained in a region R_j of the opposite color of the region R for $j = 1, \ldots, k$ with the indices taken modulo k. If the regions R_j are colored black then they each have a vertex, and since R_j and R_{j+1} share a crossing they are connected by an edge in the cactus graph. Thus, in this case we have a cycle in the cactus graph. If the regions R_j are colored white then the region R is colored black and thus the cactus graph has an interior vertex.

Conversely, if the cactus graph has an interval vertex then this corresponds to a region of the medial diagram. Furthermore this region does not border the boundary simply because otherwise the corresponding vertex would be a boundary vertex. Now suppose there is a cycle in the cactus graph. There exists a cyclic sequence $R_1, \ldots, R_k, R_1$ of regions where R_j and R_{j+1} share a crossing for $j = 1, \ldots, k$.

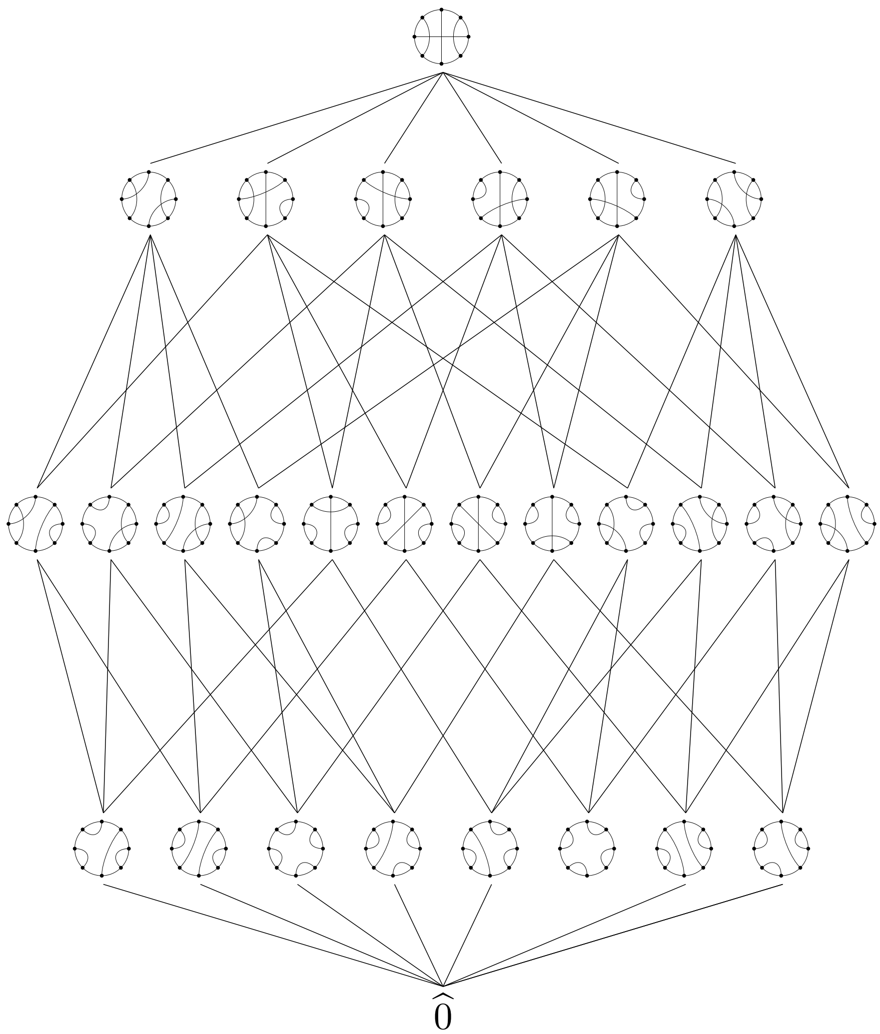


Figure 2.7: A lower interval of the uncrossing poset UC_4 that is isomorphic to the face lattice of the 3-dimensional cube.

There is a portion of an arc from the medial diagram which connects the two crossings of each region R_j which are shared with R_{j-1} and R_{j+1}. Connecting these arc portions together the medial diagram contains a closed loop consisting of arc portions. This closed loop may not be the boundary of a region in the medial diagram, but it must enclose one, hence the medial diagram contains a region which does not border the boundary. □

Figure 2.7 shows a lower interval of the uncrossing poset UC_4 that is isomorphic to the poset Q_3.

Expanding our scope to somewhat more complicated lower intervals, we consider a lower interval generated by any pairing with a cactus graph representation that has no internal vertices has a nice description in terms of the cactus graph. For the proof below given a pairing τ define $\mathcal{E}(\tau)$ to be the set of noncrossing partitions that correspond to some atom a of UC_n with $a \leq \tau$.

Proposition 2.4.5. *Let τ be a pairing with an associated cactus graph G which has no internal vertices. The lower interval $[\widehat{0}, \tau]$ is isomorphic to the poset of simple minors of the cactus graph G.*

Proof. First we claim that every pairing $\sigma < \tau$ has a single associated cactus graph which is simple and a minor of τ. By Lemma 2.4.2 there is at most one critical cactus graph with no internal vertices associated to σ. We show that any simple minor of G is critical. Suppose that H is a minor of G and for some edge e of H that the associated pairings $\tau(H)$ and either $\tau(H/e)$ or $\tau(H \setminus e)$ are the same. The vertices of e must be on the boundary so if they are distinct the boundary partition of H/e is not the same as the boundary partition of H. Since the boundary partition of an associated cactus graph is the minimal partition of the set $\mathcal{E}(\sigma)$, if $\tau(H/e) = \tau(H)$ then we must have that e is a loop. The elements of $\mathcal{E}(\sigma)$ are the partitions of the boundary vertices obtained by contracting some subset of edges in H. Thus if $\tau(H) = \tau(H \setminus e)$ then there is a second path between the two vertices of e which only contains these two boundary vertices. Since H has no internal vertices this second path must consist of a single edge, hence H has an edge parallel to e. Therefore, if H is simple it must be critical.

The map ϕ_G is a surjection so every pairing $\sigma < \tau$ has an associated cactus graph which is a minor of G, and one may take this minor to be simple. Since this simple minor is critical, every pairing $\sigma < \tau$ has exactly one associated simple cactus graph which is a minor of G.

Let H_1 and H_2 be simple minors of G. On the one hand if H_1 is a minor of H_2 then $\tau(H_1) \leq \tau(H_2)$. On the other hand suppose that $\tau(H_1) \leq \tau(H_2)$. By Lemma 2.4.1 there exists a minor H of H_2 such that $\tau(H) = \tau(H_1)$. Since H_2 has no internal vertices neither does H, and we may assume that H is simple since simplifications do not change the associated pairing. It has been established that there is precisely one simple cactus graph with no internal vertices associated to $\tau(H_1)$ so it must be that $H = H_1$. Therefore H_1 is a minor of H_2 when $\tau(H_1) \leq \tau(H_2)$. □

Figure 2.8 depicts the poset of simple minors of a 3-cycle graph, which is isomorphic to the uncrossing poset UC_3.

We describe in detail one class of lower intervals beyond those isomorphic to the face lattice of a cube. This being the next simplest lower intervals, namely, those generated by a pairing which has an associated cactus graph which is a cycle with no internal vertices.

Recall that a zipping operation identifies three elements x, y and z in a poset P such that the following three conditions hold.

(i) The element z only covers the elements x and y.

(ii) $\{p \in P : p < x\} = \{p \in P : p < y\}$.

(iii) The element z is the join of x and y.

If P is a thin poset, that is, all length 2 intervals form a diamond, condition (ii) above follows from thinness and condition (i). Furthermore if P is graded and thin then so is the zipped poset.

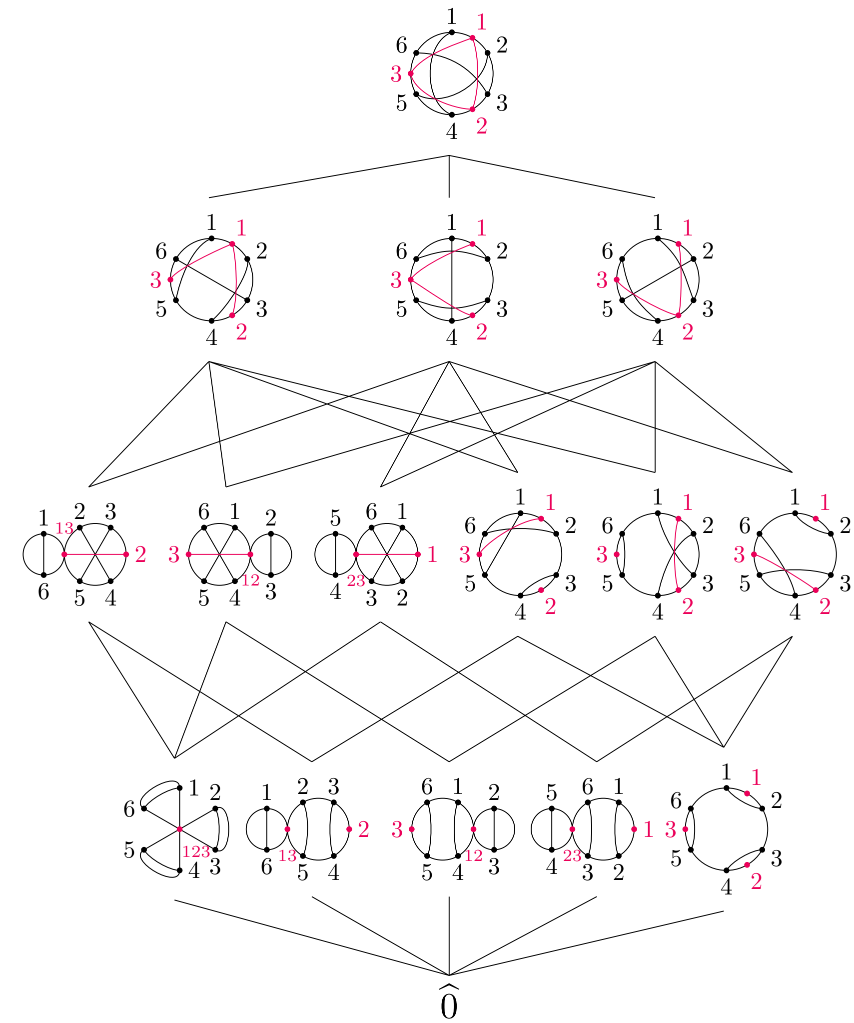


Figure 2.8: The poset of simple minors of a 3-cycle cactus graph.

Proposition 2.4.6. *Let τ be a pairing with an associated cactus graph G which is a cycle with no internal vertices. The lower interval $[\widehat{0}, \tau]$ in UC_n is isomorphic to the face poset of the regular CW complex obtained from the cube of dimension* $\mathrm{cross}(\tau)$ *by first zipping all edges incident to a chosen vertex then zipping any 2-gons formed.*

Proof. We first establish that the only vertices identified by the map ϕ_G are those incident to the vertex $(1, \ldots, 1)$ which corresponds to contracting all edges of G. Contracting all edges but one in the cycle G and deleting the remaining edge results in the same graph as contracting all edges so indeed ϕ_G does identify all vertices incident to the vertex $(1, \ldots, 1)$.

Consider the images under ϕ_G of vertices v_1 and v_2 of the cube that each have at least two 0 entries. The vertices of G are ordered $1, \ldots, n$. We order the edges as $12, 23, \ldots, n1$. Let e_{11} and e_{12} be the first two edges of G corresponding to 0 components of v_1. Similarly define e_{21} and e_{22}. The edges e_{11} and e_{12} divide G into two components, one of which consists of edges contracted to form $\phi_G(v_1)$, and the other may have edges not contracted but certainly is not identified with the first component. If the edge pairs $\{e_{11}, e_{12}\}$ and $\{e_{21}, e_{22}\}$ are not the same then the images $\phi_G(v_1)$ and $\phi_G(v_2)$ cannot be the same since the aforementioned components of G for v_1 and v_2 are not the same. Now suppose that these two edge pairs coincide. Consider the graph $H = G \setminus \{e_{11}, e_{12}\}$ and the map ϕ_H. If v_1' is the vertex obtained from v_1 by removing the components corresponding to e_{11} and e_{12} then the images $\phi_G(v_1)$ and $\phi_H(v_1')$ agree. If we similarly define v_2' then $\phi_H(v_2') = \phi_G(v_2)$ as well. The graph H is acyclic and has no internal vertices so ϕ_H is injective by Theorem 2.4.3. Thus if $\phi_G(v_1) = \phi_G(v_2)$ then $v_1' = v_2'$ which implies that $v_1 = v_2$. Therefore ϕ_G only identifies the vertices incident to the vertex $(1, \ldots, 1)$.

It remains to be seen that the identifications can be done via the prescribed sequence of zipping operations. Let $v = (1, \ldots, 1)$ and let $v_1, \ldots, v_d$ be the vertices on the d-dimensional cube incident to v, where $d = \text{cross}(\tau)$. We first must show that each edge from v to v_i is a zipper in the poset P_{i-1} obtained from the face lattice Q_d of the d-dimensional cube by identifying v with $v_1, \ldots, v_{i-1}$. The edge from v to v_i only covers v and v_i in P_{i-1}. Furthermore this edge is the join $v \vee v_i$ in Q_d. Observe the set of elements above v_i in P_{i-1} is the same as in Q_d hence the edge between v and v_i is still the join $v \vee v_i$ in P_{i-1}. Since Q_d is graded and thin so is P_{i-1} and this establishes that this edge is a zipper.

Let R_0 be the poset obtained from Q_d by zipping all edges incident to v. To construct $[\widehat{0}, \tau]$ we must now identify all cells whose vertex sets are the same. Observe each of the $\binom{d}{2}$ faces of dimension 2 which contain v are 2-gons in R_0. Hence such faces must be zipped, identifying them along with the two edges making up the cell's boundary. Any 2-dimensional face of the cube is the join of two of its edges. Since the zipping operations performed to construct R_0 have not identified any edge not incident to v to any other face, the faces which contain said edges are the same as in Q_d. Therefore the 2-dimensional faces in R_0 which contain the vertex v are each the join of their two edges.

Order the 2-dimensional cells $C_1, \ldots, C_{\binom{d}{2}}$ of R_0 which contain the vertex v via lexicographic order on the pair of vertices v_i, v_j incident to v that the 2-cell contains. We must show that C_i corresponds to a zipper in the poset R_{i-1}. Let R_i be the poset obtained from R_0 by zipping the first i of the 2-cells which contain v. Consider a 2-dimensional face of the cube which contains the vertex v. Such faces are formed by choosing two vertices v_i and v_j incident to v and the fourth vertex is the sum $v + v_i + v_j$ when considering the vectors as elements of $\mathbb{F}_2^{E(G)}$. No two of these 2-dimensional faces share the fourth vertex of the form $v_i + v_j + v$, and thus do not share the two edges incident to this fourth vertex. Let the vertices of C_i adjacent to v be u and w, thus the four vertices of C_i in Q_d are $v, u, w, v + u + w$. We have established that the zipping operations to construct R_{i-1} from Q_d do not change the upper interval generated by either of the two edges e_1 with vertices u and $v + u + w$ and e_2 with vertices w and $v + u + w$. Thus C_i is still the join of e_1 and e_2 in the poset R_{i-1}. We have already established that C_i has only these two edges in R_{i-1} hence C_i indeed corresponds to a zipper in R_{i-1}.

What remains to be seen is that the cells $C_1, \ldots, C_{\binom{d}{2}}$ are the only cells of dimension greater than one which are identified with another to construct $[\widehat{0}, \tau]$ from Q_d. Consider a 2-dimensional face $F \in Q_d$ which does not contain the vertex v. No two vertices v_i and v_j adjacent to v in Q_d are connected by an edge so F contains at most one v_i. Thus F has three vertices which are in trivial fibers of ϕ_G. No two distinct 2-dimensional faces of Q_d share three vertices so such a face F is not identified with any other face of dimension at most two by the map ϕ_G. Any face F of dimension $k \geq 3$ in Q_d contains at most $k+1$ of the vertices $v, v_1, \ldots, v_d$. Hence F has $2^k - (k+1)$ vertices which are in a trivial fiber of ϕ_G. Let V_F be the subset of the vertices of F in a trivial fiber of ϕ_G. Since no four of the vertices $v, v_1, \ldots, v_d$ lie on a 2-dimensional face of Q_d, for $k = 3$ the set V_F cannot form a face either. Thus, no two 3-dimensional faces F_1, F_2 agree on the vertex subsets V_{F_1}, V_{F_2} and must have distinct images under ϕ_G. For $k \geq 4$ since $|V_F| \geq 2^k - (k+1) > 2^{k-1}$ the vertices in V_F do not form a face, and we reach the same conclusion. This establishes that the described zipping operations indeed identify all faces in the same fiber of ϕ_G. □

Recall the **cd**-index, defined in Section 1.5, is a polynomial in the noncommutative variables **c** and **d** which enumerates the chains of a poset in a very compact manner. From this zipping construction and Theorem 1.5.3 we derive a formula for the **cd**-index of pairings represented by a cactus graph which is a cycle with no internal vertices in terms of the **cd**-index of the cube and of Boolean algebras.

Proposition 2.4.7. *Let τ be a pairing represented by a length k cycle cactus graph with no internal vertices. The* **cd***-index of the interval $[\widehat{0}, \tau]$ of the uncrossing poset is given by*

$$\Psi([\widehat{0}, \tau]) = \Psi(Q_k) - k\mathbf{d} \cdot \Psi(B_{k-1}) - \binom{k}{2}\mathbf{cd} \cdot \Psi(B_{k-2}).$$

Proof. First observe that in the construction of Proposition 2.4.6 the first k zipping operations identify two vertices and an edge. The lower intervals generated by the two vertices are each length 1 chains which have **cd**-index 1. Since these identifications all are done at the same rank and do not identify any two edges, the upper intervals generated by edges remain the same throughout. The upper interval generated by any edge in the k-dimensional cube is the Boolean algebra B_{k-1}. Thus by Theorem 1.5.3 these first k zipping operations on the level of **cd**-indices amount to subtracting $k\mathbf{d} \cdot \Psi(B_{k-1})$ in total.

Now consider the last $\binom{k}{2}$ zipping operations, each of which identifies a 2-dimensional cell with its two edges. The lower interval generated by an edge is isomorphic to the Boolean algebra B_2 which has **cd**-index **c**. The upper interval generated by any of the $\binom{k}{2}$ 2-cells that are a part of one of the zipping operations is the same as the corresponding upper interval in the k-dimensional cube. This is because none of these 2-cells are identified by the zipping operations and none of the zipping operations identify any cell of dimension greater than 2 with any other cell. Therefore, at the time of zipping any of these 2-cells together with their edges the upper interval generated by the 2-cell is isomorphic to the Boolean algebra B_{k-1}. By Theorem 1.5.3 in total these zipping operations amount to subtracting off the term $\binom{k}{2}\mathbf{cd} \cdot \Psi(B_{k-2})$ from the **cd**-index. □

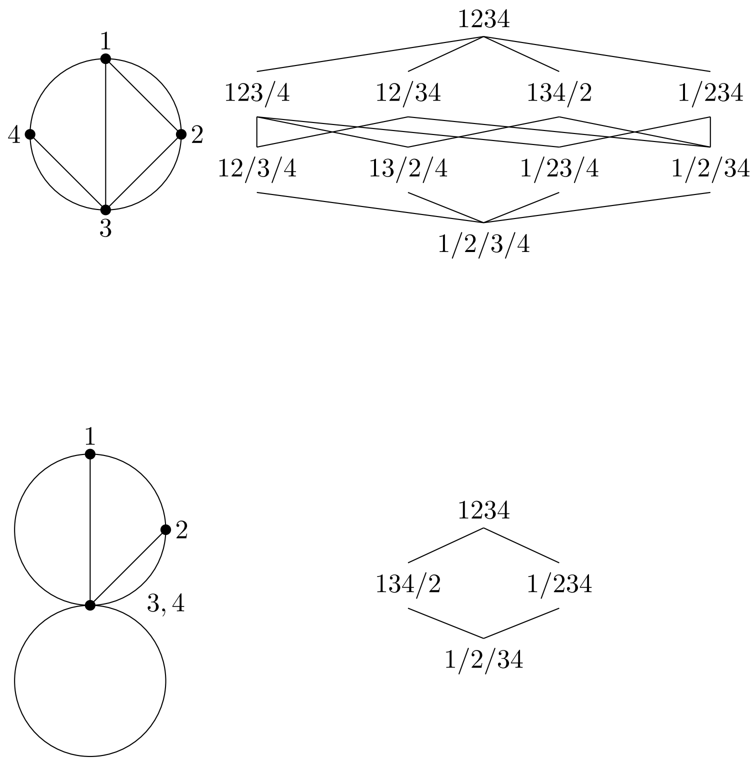


Figure 2.9: A cactus graph and a minor together with their lattice of flats.

One might hope to be able to extend the above construction beyond cactus graphs which are a cycle by somehow carefully building up the complexity of the cactus graph. Unforunately the complexity of the construction grows quite rapidly. When moving from the simplest possible case, acylic graphs with no internal vertices, to the next simplest possible case of a single cycle with no internal vertices the construction already starts to require some delicacy. The case of a cactus graph consisting of two cycles attached an edge with no internal vertices becomes quite difficult to work with from the present perspective. The issue is that the smallest complexities we can add to a graph, such as identifying two vertices to make a cycle, are in a sense too large. What is needed is a way to refine this construction, a domain of objects containing graphs through which we can pass forming intermediate steps between, say, an acyclic graph and a cycle.

Recall that the flats of a graph G may be viewed as partitions of the vertices which can be realized as the connected components of a subgraph of G, and that these partitions form a lattice under reverse refinement order. When G is a cactus graph with no internal vertices, these partitions are precisely the elements of the atom set $\mathcal{E}(\tau(G))$. We have already seen that for pairings $\sigma < \tau(G)$ the set $\mathcal{E}(\sigma)$ has some structure to it as a subset of the lattice of flats of G. Namely that $\mathcal{E}(\sigma)$ has a minimal element under reverse refinement and that the elements which cover this minimal element determine the entire set $\mathcal{E}(\sigma)$.

Small examples indicate that these sets are closed under joins and that $\mathcal{E}(\sigma)$ is the lattice of flats of the simple minor H of G with $\tau(H) = \sigma$. See Figure 2.9. These examples also indicate that deletion and contraction operations on the cactus graph G, with the results considered under electrical equivalence, correspond to removing and joining by atoms in the lattice of flats. Thus, a natural larger domain of objects within we may refine our previous construction is lattices.

For reasons that become apparent only with proper hindsight it is easier and in some ways more natural to consider lattices enriched with the structure of a generating set. Chapters 3 to 5 form a long digression in a theory of generator-enriched lattices and deletion and contraction operations as an effort to generalize the construction of Proposition 2.4.6 and yield insights for all lower intervals $[\widehat{0}, \tau]$ of UC_n where τ has an associated cactus graph with no internal vertices. Along the way we find a theory quite interesting in its own right.

Chapter 3 Generator-enriched lattices, polymatroids and minors

3.1 Introduction

In this chapter we introduce a class of objects which provide a natural setting to extend the construction of Proposition 2.4.6. Namely, the class of lattices enriched with a generating set. The motivation to study lattices originates from the observation that the minors of a cactus graph can be computed directly in terms of the lattice of flats of the graph. It becomes clear that one needs to add the structure of a generating set only after finding that the minor poset studied in Chapter 4 need not be Eulerian when there is no structure of a generating set considered.

Generator-enriched lattices are closely connected with polymatroids. In this chapter we discuss this connection. The main result is that generator-enriched lattices correspond to closure operators of polymatroids in analogy to the correspondence between geometric lattices and matroids. The precise statement is given in Theorem 3.3.7. We describe how minors of polymatroid closure operators can be described in terms of an associated generator-enriched lattice, and in Section 3.4.2 define minors of generator-enriched lattices. In Theorem 3.4.9 we show that minors of the lattice of flats of a graph are in bijection with the simple vertex labeled minors of the graph. This result connects the theory of generator-enriched lattice minors to the lower intervals of the uncrossing poset studied in Section 2.4. In Theorem 3.4.11 we generalize this result to all polymatroids.

3.2 Background

Definition 3.2.1. *A* polymatroid *on the ground set E is a function $r : B_E \to \mathbb{R}_{\geq 0}$ satisfying the following conditions for all $X, Y \subseteq E$.*

$$r(\emptyset) = 0, \tag{3.1}$$

$$\text{If } X \subseteq Y \text{ then } r(X) \leq r(Y), \tag{3.2}$$

$$r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y). \tag{3.3}$$

Condition 3.3 is referred to as *submodularity* of the function r.

Given a polymatroid r on E we say that an element $e \in E$ is a *loop* with respect to r when $r(\{e\}) = 0$. Two elements $e, f \in E$ are said to be *parallel* with respect to r when $r(\{e, f\}) = r(\{e\}) = r(\{f\})$. The *parallel class of* e with respect to r is the collection of elements in E which are parallel to e. A polymatroid is *simple* if it has no loops and all parallel classes are trivial.

Our main interest in the present work is the closure operator of polymatroids. The *closure operator* of a polymatroid r on E is the map $\overline{\cdot} : B_E \to B_E$ defined for $X \subseteq E$ by

$$\overline{X} = \{e \in E : r(X) = r(X \cup \{e\})\}.$$

The submodularity of r implies that $r(\overline{X}) = r(X)$ for any $X \subseteq E$. Sets of the form $\overline{X}$ are referred to as *r-closed sets* or *flats* of r. Edmonds showed that the set of flats of a polymatroid is closed under intersection [21, Theorem 25]. Since the set of flats of a polymatroid is finite and has a maximal element, namely E, this implies that the set of flats ordered under inclusion forms a lattice. The meet in this lattice is intersection and the join is given by $X \vee Y = \overline{X \cup Y}$.

Given a matroid r the closure operator uniquely determines r. For polymatroids this is not the case as the closure operator has no information about how much the rank of sets may differ. For example, define a polymatroid on the ground set $E = \{1, 2\}$ by assigning rank value 1 to $\{1\}$ and to $\{2\}$ and assigning any rank value in the interval $(1, 2]$ to the set $\{1, 2\}$. The resulting closure operator is the identity map on B_E regardless of the choice of the rank of $\{1, 2\}$.

3.3 Polymatroids and generator-enriched lattices

We now introduce an object which will be seen to correspond to polymatroid closure operators in the same way geometric lattices correspond to matroids.

Definition 3.3.1. *A generator-enriched lattice is a pair (L, G) in which L is a finite lattice and $G \subseteq L \setminus \{\widehat{0}\}$ generates the lattice L via the join operation.*

Note that if (L, G) is a generator-enriched lattice, the set G necessarily contains the set of join irreducibles of L which we will denote as $\mathrm{irr}(L)$. A generator-enriched lattice of the form $(L, \mathrm{irr}(L))$ will be said to be *minimally generated*.

A lattice is typically depicted via its Hasse diagram. The Hasse diagram is not enough information to specify a generator-enriched lattice since it does not describe the generating set. Instead, a generator-enriched lattice may be depicted via a diagram analogous to Cayley graphs for groups with a generating set. Given a generator-enriched lattice (L, G) the associated diagram has vertex set L, and directed edges $(\ell, \ell \vee g)$ for $\ell \in L$ and $g \in G$ such that $\ell \neq \ell \vee g$. Just as with Hasse diagrams all diagrams of generator-enriched lattices will be depicted so that the edges are directed upwards. The diagram of a generator-enriched lattice determines the underlying lattice: the order relation $\ell_1 \leq \ell_2$ holds when there is a directed path from ℓ_1 to ℓ_2 in the diagram. The minimal element $\widehat{0}$ is the unique source vertex. The generating set consists of the elements adjacent to $\widehat{0}$. See Figure 3.1 for examples of diagrams of generator-enriched lattices. Additionally, Figure A15 in Section 6.3 depicts the 10 generator-enriched lattices with 3 generators.

For every polymatroid we have an associated generator-enriched lattice.

Definition 3.3.2. *Given a polymatroid $r : E \to \mathbb{R}_{\geq 0}$ the* generator-enriched lattice of flats *is the generator-enriched lattice (L, G) where*

$$L = \{\overline{X} : X \subseteq E\},$$
$$G = \{\overline{\{e\}} : e \in E,\ r(\{e\}) \neq 0\}.$$

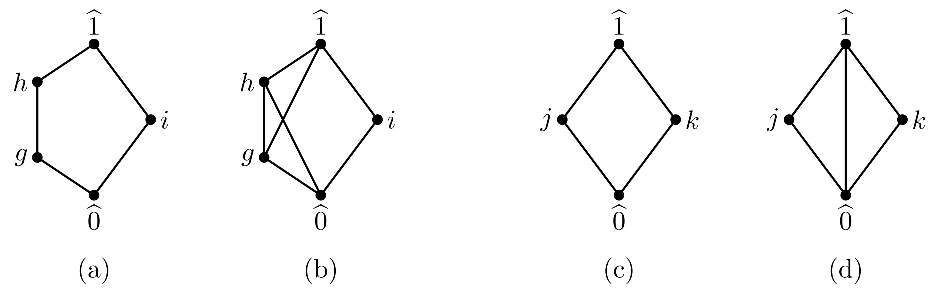


Figure 3.1: In (a) is the Hasse diagram of a lattice L with $\mathrm{irr}(L) = \{g, h, i\}$, and in (b) is the diagram of the associated minimally generated lattice $(L, \mathrm{irr}(L))$. In (c) is the Hasse diagram of the Boolean algebra B_2, which is also the diagram of the minimally generated lattice $(B_2, \{j, k\})$, and in (d) is the diagram of the generator-enriched lattice $(B_2, \{j, k, \widehat{1}\})$.

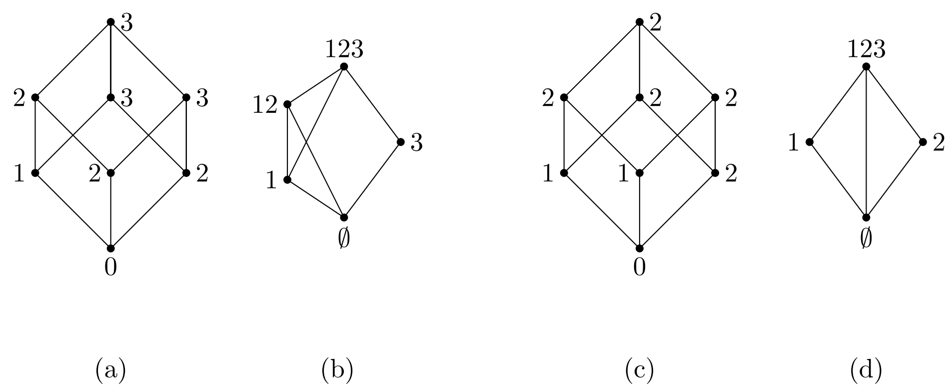


Figure 3.2: In (a) and (c) are polymatroids, and in (b) and (d) respectively are the diagrams of the generator-enriched lattice of flats.

See Figure 3.2 for examples of polymatroids and the associated generator-enriched lattice of flats.

Let r and s be two simple matroids with lattice of flats L and K respectively. A *strong map* between r and s is a map $f : L \to K$ which is join-preserving and satisfies $f(\mathrm{irr}(L)) \subseteq \mathrm{irr}(K) \cup \{\widehat{0}_K\}$. Strong maps between simple matroids were introduced by Higgs in [29] and by Crapo independently in [16]. The notion of structure preserving maps between generator-enriched lattices defined below generalizes strong maps between simple matroids.

Definition 3.3.3. *Let (L, G) and (K, H) be generator-enriched lattices. A* strong map *from (L, G) to (K, H) is a map $f : L \to K$ which is join-preserving and satisfies $f(G) \subseteq H \cup \{\widehat{0}_K\}$. This will be abbreviated by saying that $f : (L, G) \to (K, H)$ is a strong map.*

A strong map $f : (L, G) \to (K, H)$ is said to be *injective* when it is injective as a map on the underlying lattices, and *surjective* when $f(G \cup \{\widehat{0}_L\}) = H \cup \{\widehat{0}_K\}$. Two generator-enriched lattices are said to be isomorphic when there is a strong bijection between them.

Strong maps between matroids may be equivalently defined in several ways, for instance as join-preserving maps which also preserve the relation "covers or equals"; see [16, Proposition 2]. This definition does not extend to the setting of generator-enriched lattices, for example mapping atoms of a Boolean algebra to any elements of a chain will induce a strong map which need not preserve covers.

Let (L, G) be a generator-enriched lattice and let E be a ground set. Let $\mathscr{B}_E$ denote the generator-enriched lattice $(B_E, \mathrm{irr}(B_E))$. Given any map $f : E \to G \cup \{\widehat{0}_L\}$ we have an associated strong map $F : \mathscr{B}_E \to (L, G)$ defined by

$$F(X) = \bigvee_{x \in X} f(x),$$

for $X \subseteq E$. We refer to the map F as the *strong map induced by* f.

A certain nonstandard definition of matroids is useful for our lattice theoretic view of polymatroids. A matroid on a ground set E may be defined as a strong surjection f from the Boolean algebra $\mathscr{B}_E$ onto a generator-enriched lattice of the form $(L, \mathrm{irr}(L))$ for some geometric lattice L. In fact if one requires the map f to be strong in the sense of Crapo [16], the image is necessarily geometric; see [17, Proposition 9.12]. This view of matroids is briefly mentioned in [17, pp. 9.8-9.9]. Accordingly, we now turn our focus to strong surjections from Boolean algebras onto generator-enriched lattices, and showing such maps are in bijection with polymatroid closure operators (when the codomain generator-enriched lattice is considered up to isomorphism).

The following construction associates a closure operator to any strong surjection from $\mathscr{B}_E$ onto a generator-enriched lattice (L, G). This construction is standard in the theory of Galois connections. Let $\theta : \mathscr{B}_E \to (L, G)$ be a strong surjection. Define a right-sided inverse ϕ to θ by

$$\phi(\ell) = \bigcup_{X \in \theta^{-1}(\ell)} X.$$

The fact that $\theta \circ \phi$ is the identity follows directly from the fact that θ is join-preserving. We define the closure operator associated to θ to be the map $\mathrm{cl}_\theta = \phi \circ \theta : B_E \to B_E$. One may associate a generator-enriched lattice (K, H) to such a closure operator by setting

$$K = \{\mathrm{cl}_\theta(X) : X \subseteq E\},$$

and

$$H = \{\mathrm{cl}_\theta(\{e\}) : e \in E\} \setminus \{\mathrm{cl}_\theta(\emptyset)\}.$$

This generator-enriched lattice (K, H) is isomorphic to (L, G) via the isomorphism $\phi : (L, G) \to (K, H)$, which has inverse $\phi^{-1} = \theta|_K$.

The following result says that a polymatroid can be equivalently defined as a strong surjection from a Boolean algebra together with a strictly order-preserving and submodular function with nonnegative real values.

Proposition 3.3.4. *Given a generator-enriched lattice (L, G) let $\theta : \mathscr{B}_E \to (L, G)$ be a strong surjection. For any strictly order-preserving and submodular function $r : L \to \mathbb{R}_{\geq 0}$ which maps $\widehat{0}_L$ to 0, the composition $r \circ \theta : B_E \to \mathbb{R}_{\geq 0}$ is a polymatroid whose generator-enriched lattice of flats is isomorphic to (L, G). Furthermore the polymatroid is simple if and only if $\theta|_{\mathrm{irr}(B_E)\cup\{\emptyset\}}$ is injective.*

Conversely, given a polymatroid $s : B_E \to \mathbb{R}_{\geq 0}$ with generator-enriched lattice of flats (L, G), let $\theta : \mathscr{B}_E \to (L, G)$ be the strong map induced by the map $e \mapsto \overline{\{e\}}$. There is a strictly order-preserving and submodular function $r : L \to \mathbb{R}_{\geq 0}$ such that $s = r \circ \theta$.

Proof. Let s be the composition $r \circ \theta$. By assumption $s(\emptyset) = r(\widehat{0}) = 0$. The maps θ and r are order-preserving, hence s must be as well. To show that s is submodular, let X and Y be subsets of E. Since θ is join-preserving we have $s(X \cup Y) = r(\theta(X) \vee \theta(Y))$. On the other hand since θ is order-preserving, the image $\theta(X \cap Y)$ is a lower bound for both $\theta(X)$ and $\theta(Y)$, hence $\theta(X \cap Y) \leq \theta(X) \wedge \theta(Y)$. Thus $s(X \cap Y) \leq r(\theta(X) \wedge \theta(Y))$. Summing these two values results in the inequality

$$s(X \cap Y) + s(X \cup Y) \leq r(\theta(X) \wedge \theta(Y)) + r(\theta(X) \vee \theta(Y)).$$

Applying the submodularity of the function r leads to the inequality

$$s(X \cap Y) + s(X \cup Y) \leq r(\theta(X)) + r(\theta(Y)) = s(X) + s(Y).$$

Therefore the function s is a polymatroid.

To show that the generator-enriched lattice of flats of s is isomorphic to (L, G), it will suffice to show that the closure operator cl_θ is the closure operator of s. The closure of two sets X and Y with respect to s is the same if and only if $s(X) = s(X \cup Y) = s(Y)$. Since r is strictly order-preserving, this holds if and only if $\theta(X) = \theta(Y)$, which holds if and only if $\mathrm{cl}_\theta(X) = \mathrm{cl}_\theta(Y)$. By the same argument we see that s has a loop or a nontrivial parallel class precisely when $\theta|_{\mathrm{irr}(B_E)\cup\{\emptyset\}}$ is not injective.

To prove the converse, consider a polymatroid $s : B_E \to \mathbb{R}_{\geq 0}$ with generator-enriched lattice of flats (L, G). Let $\theta : \mathscr{B}_E \to (L, G)$ be the strong map induced by the map $e \mapsto \overline{\{e\}}$, and let $r = s|_L$. If $A \subsetneq B \in L$ are flats then $s(A) < s(B)$ so r is strictly order-preserving on L. Since $A \vee B = \overline{A \cup B}$ we have $s(A \vee B) = s(A \cup B)$. Therefore we have that $r(A \wedge B) + r(A \vee B) = s(A \cap B) + s(A \cup B)$, which by submodularity of s is less than or equal to $s(A) + s(B)$. This of course equals $r(A) + r(B)$ so the function r is submodular. □

Lemma 3.3.5. *For any lattice L there exists a strictly order-preserving submodular function $r : L \to \mathbb{Z}_{\geq 0}$ with $r(\widehat{0}) = 0$.*

Proof. It will suffice to construct such a function with values in $\mathbb{Q}_{\geq 0}$. Afterwards one can scale by a sufficiently large positive integer to clear denominators. Define a function $r : L \to \mathbb{Q}_{\geq 0}$ by, for $\ell \in L$ such that the largest chain in L from $\widehat{0}$ to ℓ is length k, setting $r(\ell) = 1 - 2^{-k}$. The map r is strictly order-preserving and maps $\widehat{0}$ to 0. To show r satisfies the submodularity condition, let $x, y \in L$. It may be assumed that $x \wedge y$ is neither x nor y, otherwise the submodularity inequality holds trivially for x and y. Let $r(x) = 1 - 2^{-n}$ and $r(y) = 1 - 2^{-m}$. It may also be assumed that $n \leq m$. Observe that $r(x \wedge y) \leq 1 - 2^{-n+1}$ and $r(x \vee y) \leq 1$. Adding these terms gives,

$$r(x \wedge y) + r(x \vee y) \leq 2 - 2^{-n+1} \leq 2 - 2^{-n} - 2^{-m} = r(x) + r(y).$$

Thus r is submodular and can be used to construct the desired function. $\square$

It is known that every lattice is isomorphic to the lattice of flats of some polymatroid that is integer-valued. This result is attributed to Dilworth in [34, pp. 26] and follows from Dilworth's embedding theorem [18, Theorem 14.1], which states that any finite lattice can be embedded into a geometric lattice. Below is a somewhat stronger result.

Proposition 3.3.6. *Every generator-enriched lattice is isomorphic to the generator-enriched lattice of flats of some polymatroid, which may be chosen to have integer values.*

Proof. Let (L, G) be a generator-enriched lattice. By Lemma 3.3.5 there is an integer-valued strictly order-preserving submodular function r on L. Let $\theta : \mathscr{B}_G \to (L, G)$ be the strong surjection induced by the identity map on G. By Proposition 3.3.4 the map $r \circ \theta$ is a polymatroid whose lattice of flats is isomorphic to (L, G). $\square$

Theorem 3.3.7. *Let E be a set. A function from B_E to B_E is the closure operator of a polymatroid if and only if it is the closure operator of a strong surjection $\theta : \mathscr{B}_E \to (L, G)$ onto some generator-enriched lattice (L, G).*

Proof. Let $r : B_E \to \mathbb{R}_{\geq 0}$ be a polymatroid, and let (L, G) be the generator-enriched lattice of flats of r. Let $\theta : \mathscr{B}_E \to (L, G)$ be the strong map induced by the map $e \mapsto \overline{\{e\}}$ from E to L. The image $\theta(X)$ is by definition

$$\theta(X) = \bigvee_{x \in X} \overline{\{x\}},$$

in other words, the smallest flat including $\overline{\{x\}}$ for all $x \in X$. If Y is a flat including $\overline{\{x\}}$ for all $x \in X$, then $Y \supseteq X$. Taking the closure we have $Y \supseteq \overline{X}$. Thus, the image $\theta(X)$ equals the closure $\overline{X}$. Since $L \subseteq B_E$ the closure operator cl_θ takes the same values as θ so we have shown that $\overline{\cdot} = \mathrm{cl}_\theta$.

Conversely consider a generator-enriched lattice (L, G) and a strong surjection $\theta : \mathscr{B}_E \to (L, G)$. We wish to construct a polymatroid whose closure operator coincides with the closure operator cl_θ of θ. By Lemma 3.3.5 there is a strictly order-preserving submodular function $r : L \to \mathbb{R}_{\geq 0}$. By Proposition 3.3.4 the function $s = r \circ \theta : B_E \to \mathbb{R}_{\geq 0}$ is a polymatroid on E. Furthermore the generator-enriched lattice of flats of s is isomorphic to (L, G) via the isomorphism $\overline{X} \mapsto \theta(X)$. From this it is evident that $\overline{\cdot} = \mathrm{cl}_\theta$, hence the closure operator cl_θ of θ is the closure operator of the polymatroid s. $\square$

3.4 Minors

In this section we discuss minors of polymatroids with respect to the associated closure operators and generator-enriched lattices. The underlying generator-enriched lattice of a minor does not fully depend on the original polymatroid if it is simple, only the underlying generator-enriched lattice.

In Section 3.4.2 we discuss minors of generator-enriched lattices themselves (with no structure of a strong surjection). In Theorem 3.4.9 we show that for a graphic matroid the minors of the generator-enriched lattice of flats are in bijection with the minors of the graph when the vertices are labeled and the edges are unlabeled. In Theorem 3.4.11 we prove a generalization of this result to polymatroids.

Let r be a polymatroid with ground set E. The *deletion by* $X \subseteq E$ is the polymatroid $r \setminus X : B_{E \setminus X} \to \mathbb{R}_{\geq 0}$ defined as the usual function restriction $r \setminus X = r|_{B_{E\setminus X}}$. The *contraction by* X is the polymatroid $r/X : B_{E\setminus X} \to \mathbb{R}_{\geq 0}$ defined for $Y \subseteq E \setminus X$ by setting $(r/X)(Y) = r(Y \cup X) - r(X)$. These operations correspond to restricting to a lower and upper interval of the Boolean algebra B_E respectively. Any polymatroid obtained from r via deletion and contraction operations is said to be a *minor* of r.

3.4.1 Minors of strong surjections

We begin by observing that minors of a polymatroid closure operator are well defined as the closure operator of the corresponding minor of any associated polymatroid.

Lemma 3.4.1. *Let (r, E) and (s, E) be two polymatroids with the same closure operator. For any two disjoint sets $X, Y \subseteq E$ the closure operators of the minors $(r/X)\setminus Y$ and $(s/X) \setminus Y$ are the same.*

Proof. Let $r' = (r/X) \setminus Y$ and $s' = (s/X) \setminus Y$. Let $Z \subseteq E$ and $e \in E \setminus Z$. By assumption $r(Z) = r(Z \cup \{e\})$ if and only if $s(Z) = s(Z \cup \{e\})$. The minor r' is the function defined on $E \setminus (X \cup Y)$ by $r'(Z) = r(Z) - r(X)$, and similarly for s and s'. Thus we have that $r'(Z) = r'(Z \cup \{e\})$ if and only if $s'(Z) = s'(Z \cup \{e\})$ which shows that the closure operators of r' and s' are the same. □

We now turn to defining deletion and contraction operations on strong surjections from a lattice theoretic viewpoint. In Proposition 3.4.2 we prove that these operations agree with the same operations on polymatroids. First we set up some notation.

Given a lattice L, let $H \subseteq L$ and let $z \in L$ be an element such that $z < h$ for all $h \in H$. Define the *generator-enriched lattice with generating set H and minimal element z* to be

$$\begin{aligned}\langle H|z\rangle &= \left(\{z \vee \bigvee_{x\in X} x : X \subseteq H\}, H\right)\\ &= \left(\{z\} \cup \{\bigvee_{x\in X} x : \emptyset \neq X \subseteq H\}, H\right).\end{aligned}$$

Usually when listing H explicitly the set brackets will be repressed.

Let E be a ground set, let (L, G) be a generator-enriched lattice and consider a strong surjection $\theta : \mathscr{B}_E \to (L, G)$. For $X \subseteq E$ define the *deletion of* θ *by* X to be the strong surjection

$$\theta \setminus X : \mathscr{B}_{E \setminus X} \to \langle \{\theta(\{e\}) : e \in E \setminus X\} \setminus \{\widehat{0}_L\} | \widehat{0}_L \rangle,$$

defined for $Z \subseteq E \setminus X$ by $(\theta \setminus X)(Z) = \theta(Z)$. Define the *contraction of* θ *by* X to be the strong surjection

$$\theta/X : \mathscr{B}_{E \setminus X} \to \langle \{\theta(X \cup \{e\}) : e \in E \setminus X\} \setminus \{\theta(X)\} | \theta(X) \rangle,$$

defined for $Z \subseteq E \setminus X$ by $(\theta/X)(Z) = \theta(X \cup Z)$.

Conceptually the deletion by X of a strong surjection θ is obtained by restricting θ to the lower interval $[\emptyset, E \setminus X] \subseteq B_E$, and then restricting the codomain to ensure the resulting function is a surjection. Similarly contracting by X corresponds to restricting to the upper interval $[X, E] \subseteq B_E$.

Proposition 3.4.2. *Let* $r : E \to \mathbb{R}_{\geq 0}$ *be a polymatroid with generator-enriched lattice of flats* (L, G)*, and let* $X, Y \subseteq E$ *be disjoint sets. If* $\theta : \mathscr{B}_E \to (L, G)$ *is the strong surjection associated to* r *then the closure operator of the polymatroid* $(r/X) \setminus Y$ *is equal to* $\mathrm{cl}_{(\theta/X)\setminus Y}$.

Proof. Set $r' = (r/X) \setminus Y$ and $\theta' = (\theta/X) \setminus Y$. Let $\overline{\cdot}$ denote the closure operator of r'. By definition $\overline{Z_1} = \overline{Z_2}$ if and only if $r'(Z_1) = r'(Z_1 \cup Z_2) = r'(Z_2)$ which occurs if and only if $r(Z_1 \cup X) = r(Z_1 \cup Z_2 \cup X) = r(Z_2 \cup X)$. Since cl_θ is the closure operator of r this occurs if and only if $\theta(Z_1 \cup X) = \theta(Z_1 \cup Z_2 \cup X) = \theta(Z_2 \cup X)$. This is in turn equivalent to the condition $\theta'(Z_1) = \theta'(Z_1 \cup Z_2) = \theta'(Z_2)$. Therefore, $\overline{Z_1} = \overline{Z_2}$ if and only if $\theta'(Z_1) = \theta'(Z_2)$, and thus $\mathrm{cl}_{\theta'}$ is the closure operator of r'. □

3.4.2 Minors of generator-enriched lattices

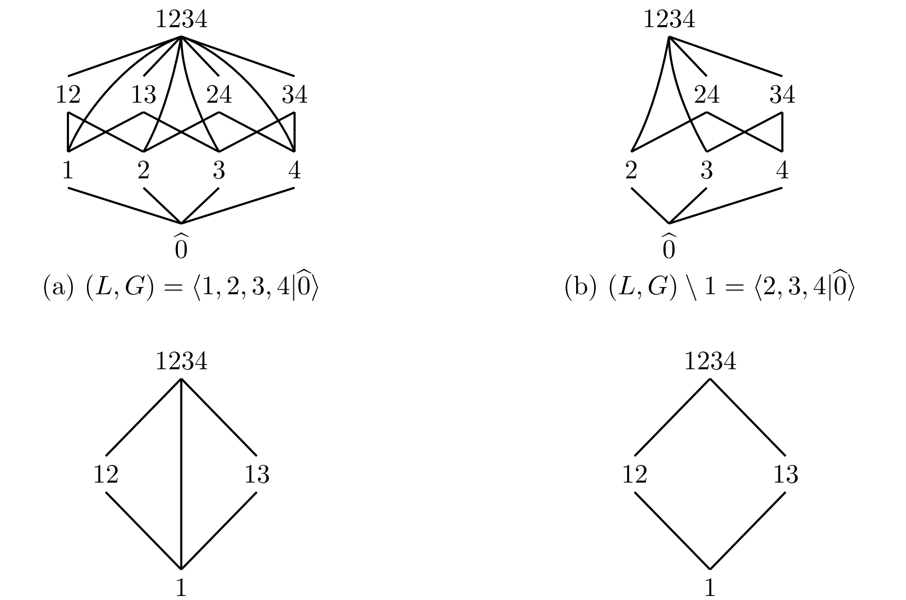
Let (L, G) be a generator-enriched lattice and $\theta : \mathscr{B}_E \to (L, G)$ be a strong surjection. When θ is simple, that is, when $\theta|_{\mathrm{irr}(B_E)\cup\{\emptyset\}}$ is injective, the codomain of the deletion $\theta \setminus X$ depends only on the set $\{\theta(\{x\}) : x \in X\}$. Similarly the codomain of the contraction θ/X depends only on the image $\theta(X)$. Thus viewing generator-enriched lattices as encoding closure operators of simple polymatroids we have a notion of deletion and contraction operations, the result of which is another generator-enriched lattice.

Let (L, G) be a generator-enriched lattice and let $I \subseteq G$. The *deletion of* (L, G) *by* I is the generator-enriched lattice

$$(L, G) \setminus I = \langle G \setminus I | \widehat{0}_L \rangle.$$

Let $i_0 = \bigvee_{i \in I} i$ and set $J = \{g \vee i_0 : g \in G\} \setminus \{i_0\}$. The *contraction of* (L, G) *by* I is the generator-enriched lattice

$$(L, G)/I = \langle J | i_0 \rangle.$$



(a) $(L, G) = \langle 1, 2, 3, 4 | \widehat{0} \rangle$ (b) $(L, G) \setminus 1 = \langle 2, 3, 4 | \widehat{0} \rangle$

(c) $(L, G)/1 = \langle 12, 13, 1234 | 1 \rangle$ (d) $((L, G)/1) \setminus (1234) = \langle 12, 13 | 1 \rangle$

Figure 3.3: The face lattice of the square and several minors.

For convenience we also define the *restriction of* (L, G) *to* I as

$$(L, G)|_I = (L, G) \setminus (G \setminus I).$$

The operations of deletion and contraction on generator-enriched lattices correspond to first choosing a simple strong surjection, performing the operations as previously defined for strong surjections, and taking a simplification of the result.

It will be convenient at times to index deletions and contractions by subsets of some ground set E, or by elements of L instead. To define the former choose a labeling of G by E so that $G = \{g_e : e \in E\}$. Given $X \subseteq E$ the deletion and contraction by X are defined as

$$(L, G) \setminus X = (L, G) \setminus \{g_x : x \in X\},$$
$$(L, G)/X = (L, G)/\{g_x : x \in X\}.$$

Given $\ell \in L$ the deletion and contraction by ℓ are defined as

$$(L, G) \setminus \ell = (L, G) \setminus \{g \in G : g \leq \ell\},$$
$$(L, G)/\ell = (L, G)/\{g \in G : g \leq \ell\}.$$

The result of any sequence of deletions and contractions applied to (L, G) is called a *minor* of (L, G). See Figure 3.3 for examples.

A few basic remarks for minors of a generator-enriched lattice (L, G) are in order.

Remark 3.4.3. *By definition the underlying lattice of a minor of* (L, G) *is a join subsemilattice of* L*. In general the underlying lattice of a minor of* (L, G) *may not*

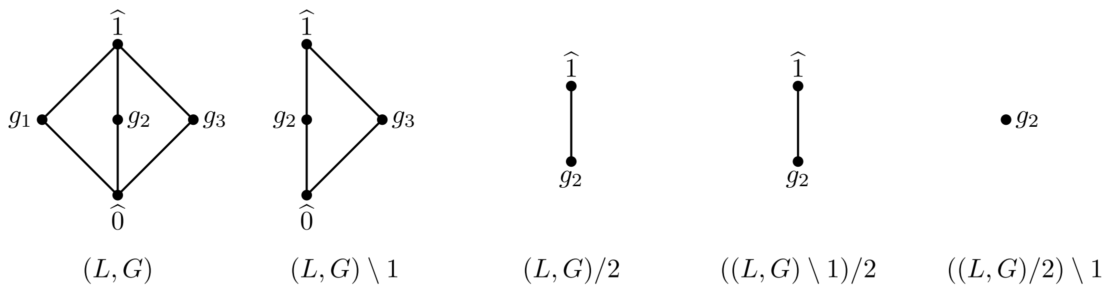


Figure 3.4: A generator-enriched lattice (L, G), where $G = \{g_1, g_2, g_3\}$ for which deletions and contractions do not commute, along with the relevant minors.

be a sublattice of L. For example, consider the partition lattice Π_4 with minimal generating set

$$\mathrm{irr}(\Pi_4) = \{12/3/4,\ 13/2/4,\ 14/2/3,\ 1/23/4,\ 1/24/3,\ 1/2/34\}.$$

Deleting the atom $13/2/4$ results in a minor which is not a sublattice of Π_4; in said minor the meet of $123/4$ and $134/2$ is the minimal partition $1/2/3/4$ as opposed to $13/2/4$ when computed in L.

Remark 3.4.4. *Any interval of L is the underlying lattice of a minor of (L, G). If $a \le b$ in L then the minor $((L, G)/a)|_b$ has underlying lattice $K = [a, b]$ of L. The example given in Remark 3.4.3 shows the converse is false, that in general not all minors of (L, G) have as underlying lattice an interval of L.*

Remark 3.4.5. *The deletion and contraction operations of generator-enriched lattices do not in general commute. Figure 3.4 depicts an example. In this example since $\widehat{1} = g_1 \vee g_2$ the element $\widehat{1}$ in the contraction $(L, G)/2$ is a generator indexed by 1. Thus deleting 1 from $(L, G)/2$ removes $\widehat{1}$, that is, $((L, G)/2) \setminus 1 = (g_2, \emptyset)$. On the other hand, since $\widehat{1} = g_3 \vee g_2$ when the deletion $(L, G) \setminus 1$ is contracted by 2 the resulting generator-enriched lattice $((L, G) \setminus 1)/2$ has $\widehat{1}$ as a generator.*

The following observation will be useful.

Lemma 3.4.6. *Any minor of a generator-enriched lattice (L, G) may be expressed as the result of a contraction followed by a deletion. Namely, a minor (K, H) of (L, G) may be expressed as $(K, H) = ((L, G)/\widehat{0}_K)|_H$.*

Proof. Let (K, H) be a minor of (L, G). By definition (K, H) may be expressed as the result of a sequence of contractions and deletions. That is, for some possibly empty sets of generators $I_1, J_1, \ldots, I_r, J_r$, that

$$(K, H) = ((\cdots(((L, G)/I_1) \setminus J_1) \cdots /I_r) \setminus J_r.$$

For $1 \le j \le r$ let i_j be the join of all elements in I_j. Set $i_0 = i_1 \vee \cdots \vee i_r$. By definition of deletion and contraction, the minimal element $\widehat{0}_K$ of K is i_0. Furthermore, the generators of (K, H) can each be expressed as $g \vee i_1 \vee \cdots \vee i_r = g \vee i_0$ for some $g \in G$. Thus each generator of (K, H) is a generator of $(L, G)/i_0$, hence $(K, H) = ((L, G)/i_0)|_H$. □

The lemma below gives an explicit description of the generating sets of minors.

Lemma 3.4.7. *For any generator-enriched lattice (L, G) the minors are precisely generator-enriched lattices of the form $\langle \ell \vee g_1, \ldots, \ell \vee g_k | \ell \rangle$ for $\ell \in L$ and $\{g_1, \ldots, g_k\} \subseteq G$ such that $g_j \not\leq \ell$ for $1 \leq j \leq k$.*

Proof. Consider a minor $(K, H) = ((L, G)/I)|_J$ of (L, G), where I and J are sets of generators. Let ℓ be the join of all elements of I and let $J = \{j_1, \ldots, j_k\}$. By definition

$$(K, H) = \langle \ell \vee j_1, \ldots, \ell \vee j_k | \ell \rangle.$$

Conversely, consider a generator-enriched lattice $(K, H) = \langle \ell \vee g_1, \ldots, \ell \vee g_k | \ell \rangle$ for some $\ell \in L$ and $g_j \in G$ with $g_j \not\leq \ell$ for $1 \leq j \leq k$. The generators of the contraction $(L, G)/\ell$ are all elements $\ell \vee g$ for $g \in G$ with $g \not\leq \ell$. Thus $\ell \vee g_1, \ldots, \ell \vee g_k$ are generators of $(L, G)/\ell$, so setting $I = \{\ell \vee g_1, \ldots, \ell \vee g_k\}$ we have that $(K, H) = ((L, G)/\ell)|_I$. □

Lemma 3.4.8. *If L is a geometric lattice then the minors of $(L, \mathrm{irr}(L))$ are the generator-enriched lattices of the form $\langle \ell_1, \ldots, \ell_k | \ell \rangle$ such that $\ell_i \succ \ell \in L$ for $1 \leq i \leq k$. In particular, every minor of $(L, \mathrm{irr}(L))$ is minimally generated and geometric.*

Proof. Since L is geometric, for any $x, y \in L$ we have $x \prec y$ if and only if $y = x \vee i$ for some $i \in \mathrm{irr}(L)$. Thus Lemma 3.4.7 specializes to the claimed form of the generating sets of minors of $(L, \mathrm{irr}(L))$. In particular, for any minor (K, H) the generating set H is the set of atoms of K.

Let $(K, \mathrm{irr}(K)) = \langle \ell_1, \ldots, \ell_k | \ell \rangle$ be a minor of $(L, \mathrm{irr}(L))$. In order to show that K is semimodular we claim that if $x \prec y$ in K then $x \prec y$ in L as well. Since $x \prec y$ there exists i such that $y = x \vee \ell_i$. We have that $\ell_i = \ell \vee a$ for some atom a of L, hence $y = x \vee a$. Since L is geometric this implies that $x \prec y$ in L.

Now let $x, y \in K$ such that $x \wedge_K y \prec x$ and $x \wedge_K y \prec y$ in K. Since K is a subposet of L we have that $x \wedge_K y \leq x \wedge_L y$. On the other hand, since $x \wedge_K y$ is covered by x and y in K, hence in L, we must have $x \wedge_K y = x \wedge_L y$. Thus $x \wedge_L y$ is covered by x and y in L, and since L is semimodular $x \vee_L y$ covers x and y in L. Then since $x \vee_L y = x \vee_K y$ this implies that $x \vee_K y$ covers x and y in K, and therefore K is geometric. □

Given a graph G, the lattice of flats L may be viewed as a lattice of partitions of the vertices of G. Each flat is associated to the partition whose blocks consist of the connected components of said flat considered as a subgraph of G. Let $L(G)$ denote the generator-enriched lattice of flats of G labeled as partitions. See Figure 3.5.

The minors of the graph G inherit a vertex labeling by blocks of a partition of the vertices of G. When an edge is contracted, the label of the new vertex is obtained by merging the two blocks labeling the vertices of the contracted edge. In this way

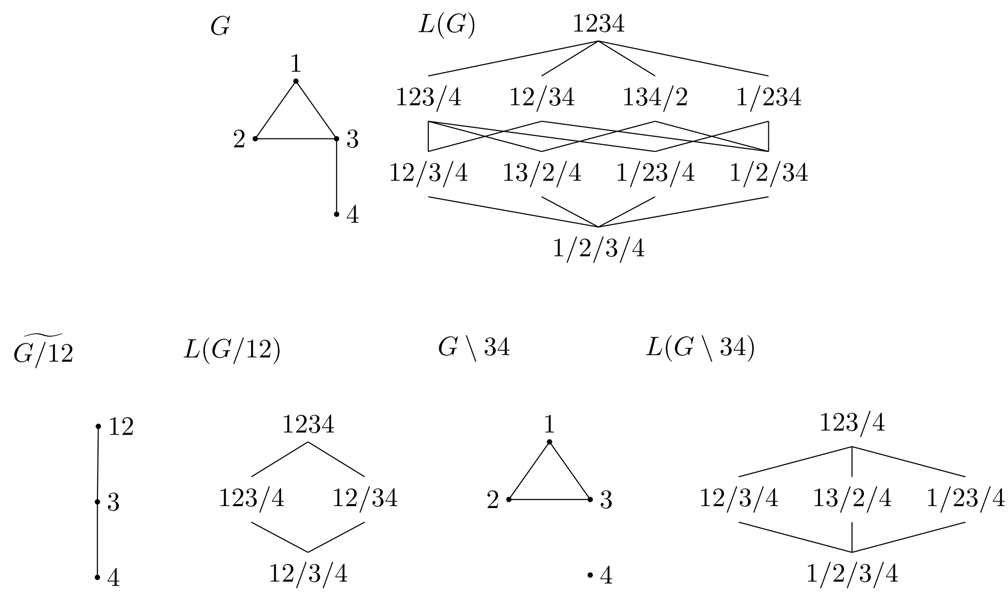


Figure 3.5: At (a) a graph G and at (b) the lattice of flats $L(G)$. At (c) the simplification of the contraction $G/\{1,2\}$ and at (d) the contraction $L(G)/(12/3/4)$.

the minors of a vertex labeled graph are considered to be themselves vertex labeled graphs.

Theorem 3.4.9. *Let G be a vertex labeled graph with unlabeled edges. The vertex labeled minors of G that are simple are in bijection with the minors of the minimally generated lattice of flats $L(G)$ via the map $H \mapsto L(H)$.*

Proof. Let L be the lattice of flats of the graph G. It may be assumed that the graph G is simple, that is, that G has no loops or multiple edges. This only changes the labeling of elements in the lattice of flats and does not change the collection of simple minors of G. The inverse of the map $H \mapsto L(H)$ will be constructed. Let $(K, \mathrm{irr}(K)) = \langle \ell_1, \ldots, \ell_k | \ell \rangle$ be a minor of $(L, \mathrm{irr}(L))$. Construct a graph H as follows. Each atom of L corresponds to an edge of G. The element ℓ corresponds to a set of edges of G; namely, those edges corresponding to an atom which is less than or equal to ℓ. Let H' be the minor of G obtained by contracting this set of edges corresponding to ℓ. The vertices of H' are labeled by the blocks of the partition ℓ. Each atom ℓ_i in K is obtained from the partition ℓ by merging two blocks, and corresponds to an edge in H'. Let H'' be the graph obtained from H' by restricting to these edges which correspond to an atom of K. The graph H is defined to be the simplification of H''.

It remains to show that the map $K \mapsto H$ constructed above and the map $H \mapsto L(H)$ are inverses. A vertex labeled graph H is determined by the labeling of its vertices and its edges. The associated lattice minor $(K, \mathrm{irr}(K))$ of $(L, \mathrm{irr}(L))$ records this same information as the minimal partition and the atoms, which in turn determines K. $\square$

The above result generalizes to polymatroids with the appropriate notion replacing vertex labeled minors.

Definition 3.4.10. *Let r be a polymatroid with ground set E. A* parallel closed pair *is a pair (F, s) such that $F \subseteq E$ is a flat of r and s is a polymatroid that may be obtained as a deletion of the polymatroid r/F satisfying the following condition:*

> *If $e \in E \setminus F$ is parallel with respect to r/F to an element f of the ground set of s then e is an element of the ground set of s as well. In other words, the ground set of s must be a union of parallel classes with respect to r/F.*

For a graphic matroid the parallel closed pairs are in bijection with the vertex labeled minors of the graph obtained by first contracting, and then deleting entire parallel classes of edges. The vertex labeling naturally encodes the flat in the parallel closed pair. Such graphs are in bijection with the simple minors when the edges are unlabeled. Without an edge labeling each such graph has one simplification, obtained by identifying parallel edges, and no two such graphs have the same simplification. Thus, the following theorem is an analogue of Theorem 3.4.9.

Theorem 3.4.11. *Let r be a polymatroid and let (L, G) be the associated generator-enriched lattice of flats. The minors of (L, G) are in bijection with the parallel closed pairs of r.*

Proof. Let E be the ground set of r. Let $\overline{\cdot} : B_E \to B_E$ be the closure operator of r. Let $\theta : \mathscr{B}_E \to (L, G)$ be the strong surjection induced by the ground set map $e \mapsto \overline{\{e\}}$.

Let (F, s) be a parallel closed pair of r and let $Y \subseteq E$ be the ground set of s. Define a map f from the set of parallel closed pairs of the polymatroid r to the set of minors of the generator-enriched lattice (L, G) by

$$f(F, s) = ((L, G)|_{Y \cup F})/F.$$

To show f is a bijection construct the inverse map g. Let (K, H) be a minor of (L, G) and let Y be the set

$$Y = \{y \in E : \overline{\{y\} \cup \widehat{0}_K} \in H\}.$$

Let $g(K, H)$ be the pair $(\widehat{0}_K, (r/\widehat{0}_K)|_Y)$. Observe that $g(K, H)$ is a parallel closed pair of r. Furthermore g is the inverse of f so the map f is a bijection. □

3.4.3 Minors of distributive lattices

In the remainder of this section we examine minors of minimally generated distributive lattices. Recall that the fundamental theorem of finite distributive lattices states that every finite distributive lattice L is isomorphic to the lattice of lower order ideals of the subposet $\mathrm{irr}(L)$ of L. The minors of a minimally generated distributive lattice have an alternative description in terms of certain pairs of subsets of the poset of irreducibles.

Definition 3.4.12. *Let P be a poset. An* order minor *of P is a pair (I, J) of disjoint subsets of P such that J is a lower order ideal of P.*

The poset P itself corresponds to the order minor $(P, \emptyset)$. The set of order minors of P is shown below to be in bijection with the minors of the minimally generated lattice of lower order ideals of P. To prove this bijection, the following lemma is needed.

Lemma 3.4.13. *If L is a distributive lattice, for any $\ell \in L$ and any distinct join irreducibles i and j such that $i \vee \ell \neq \ell$ and $j \vee \ell \neq \ell$ the elements $i \vee \ell$ and $j \vee \ell$ are distinct.*

Proof. Let L be the lattice of lower order ideals of a poset P, necessarily isomorphic to $\operatorname{irr}(L)$. It may be assumed without loss of generality that $i \not\leq j$ in L. Let $p \in P$ be the element such that the principal lower order ideal of P generated by p is the join irreducible i of L. The fact that $i \vee \ell \neq \ell$ implies that p is not contained in the ideal ℓ. Since $i \not\leq j$ the element p is not contained in the ideal j. As a consequence p is not contained in the ideal $j \vee \ell$ since the join in L corresponds to the union of lower order ideals. This establishes that $i \vee \ell \neq j \vee \ell$. □

When $(L, \operatorname{irr}(L))$ is the minimally generated lattice of lower order ideals of a poset P there is an implicit bijection between P and the generating set $\operatorname{irr}(L)$. Through this bijection deletions and contractions of $(L, \operatorname{irr}(L))$ may be indexed by subsets of P.

Proposition 3.4.14. *Let L be the lattice of lower order ideals of a poset P. The order minors of P and the minors of $(L, \operatorname{irr}(L))$ are in bijection via the map*

$$(I, J) \mapsto ((L, \operatorname{irr}(L))|_{I \cup J})/J.$$

Proof. Define a map from minors of $(L, \operatorname{irr}(L))$ to order minors of P as follows. Given a minor (K, H) of $(L, \operatorname{irr}(L))$ define J to be the subset of P corresponding to the join irreducibles in L which are less than or equal to $\widehat{0}_K$. Define I to be the subset of P consisting of all elements whose corresponding join irreducible i of L satisfies $i \vee \widehat{0}_K \in H$. Lemma 3.4.13 implies that I is the unique set satisfying $(K, H) = ((L, \operatorname{irr}(L))|_{I \cup J})/J$. The inverse of this map is given by $(I, J) \mapsto ((L, \operatorname{irr}(L)|_{I \cup J})/J$ so we have a bijection. □

Not only do the order minors of a poset index the minors of the minimally generated lattice of lower order ideals, the order minors also describe the isomorphism types of the lattice minors.

Proposition 3.4.15. *Let P be a poset and L the lattice of lower order ideals of P, and let (I, J) be an order minor of P. The minor $((L, \operatorname{irr}(L))|_{I \cup J})/J$ of $(L, \operatorname{irr}(L))$ is isomorphic to the minimally generated lattice of lower order ideals of I.*

Proof. Let (I, J) be an order minor of P and let $(K, H) = ((L, \mathrm{irr}(L))|_{I \cup J})/J$. It is claimed that K consists of the lower order ideals of P whose maximal elements are all contained in $I \cup J$ and which include J. Observe that $(L, \mathrm{irr}(L))|_{I \cup J}$ is generated by the principal lower order ideals which are themselves generated by an element of $I \cup J$. Hence this lattice consists of all lower order ideals of P whose maximal elements are contained in $I \cup J$. The generators of $((L, \mathrm{irr}(L))|_{I \cup J})/J$ are thus each the union of the lower order ideal J of P with a principal lower order ideal of P which is generated by an element of I. Such lower order ideals as a join subsemilattice of L generate the set of lower order ideals of P which include J and whose maximal elements are contained in $I \cup J$.

Let $(M, \mathrm{irr}(M))$ be the minimally generated lattice of lower order ideals of the subposet I of P. Define a map $f : (K, H) \to (M, \mathrm{irr}(M))$ by $f(\Lambda) = \Lambda \cap I$. Define a map $g : (M, \mathrm{irr}(M)) \to (K, H)$ by letting $g(\Lambda)$ be the lower order ideal of P generated by $\Lambda \cup J$. Observe this is the inverse of f since every ideal in L includes J and has maximal elements which are contained in $I \cup J$. Therefore K is isomorphic to M as claimed. □

Chapter 4 The minor poset

4.1 Introduction

Recall in Proposition 2.4.6 it was shown that if τ is a pairing that can be depicted by a medial graph with one internal region then the lower interval $[\widehat{0},\tau]$ in the uncrossing poset UC_n is isomorphic to the poset of simple minors of the associated cactus graph G. Minors of a cactus graph have labeled vertices and unlabeled edges so Theorem 3.4.9 implies this interval $[\widehat{0},\tau]$ can be computed in terms of the lattice of flats of the graph G. Motivated by this observation, in this chapter we introduce minor posets of generator-enriched lattices. This broader class of objects provides a natural setting to extend the zipping construction of Proposition 2.4.6.

In Section 4.2 the minor poset is introduced. The class of generator-enriched lattices for which the minor poset is itself a lattice is characterized in terms of forbidden minors in Proposition 4.2.18 and Theorem 4.2.24. A decomposition of the minor poset into Boolean algebras is given in Theorem 4.2.28. This decomposition is used to give expressions for the rank generating function of minor posets associated to geometric lattices with generating set consisting of the join irreducibles and to lattices with the no parallels property (defined in Definition 4.2.17). This class includes distributive lattices with generating set consisting of the join irreducibles. See Theorems 4.2.30 and 4.2.32.

In Section 4.3 the minor poset of any generator-enriched lattice is shown to be isomorphic to the face poset of a regular CW sphere; see Corollary 4.3.9. This is done via a construction using the zipping operation of Reading [41] in Theorem 4.3.7. Using this construction in Corollaries 4.3.10 and 4.3.11, we derive inequalities between the coefficients of **cd**-indices of minor posets when there is a structure preserving surjection between the associated generator-enriched lattices. In particular, we show that the **cd**-index of the n-dimensional cube is the coefficientwise maximum of **cd**-indices of minor posets of rank $n+1$.

4.2 The minor poset

In this section a partial order structure on the set of minors of a given generator-enriched lattice called the minor poset is studied. We begin with some basic results for minor posets. In Section 4.2.2 we discuss a few operations on generator-enriched lattices and their effects on the minor poset. In Section 4.2.3 a characterization is given of the generator-enriched lattices for which the minor poset is itself a lattice. In Section 4.2.4 a decomposition theorem is presented which is used to derive expressions for the rank generating function of minor posets in special cases.

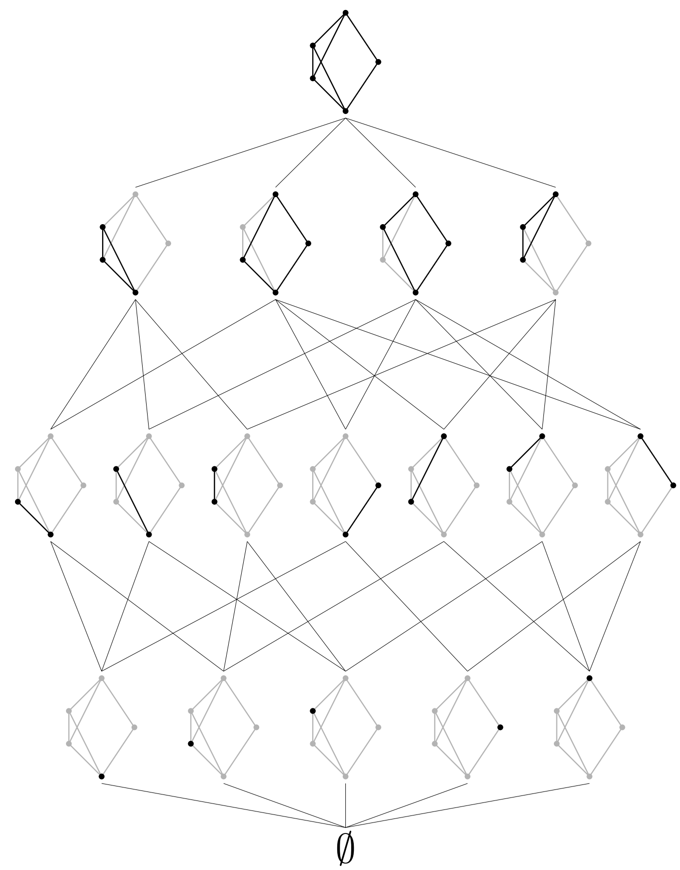


Figure 4.1: The minor poset of a generator-enriched lattice. The elements are depicted via their diagrams.

4.2.1 Basic results

Definition 4.2.1. *Let (L, G) be a generator-enriched lattice. The minor poset, denoted* $\mathrm{M}(L, G)$*, is the poset consisting of a unique minimal element* $\emptyset$ *and the minors of* (L, G) *and with order relation defined by* $(K_1, H_1) \le (K_2, H_2)$ *when* (K_1, H_1) *is a minor of* (K_2, H_2).

As an immediate observation note that the lower interval $[\emptyset, (K, H)]$ in the minor poset $\mathrm{M}(L, G)$ is the minor poset $\mathrm{M}(K, H)$. See Figure 4.1 for an example of a minor poset. The Hasse diagrams of minor posets of generator-enriched lattices with 3 generators are depicted in Figures A16-A25 of Appendix A.

Recall a ranked poset is said to be *thin* if all length 2 intervals are isomorphic to the diamond poset, that is, the Boolean algebra B_2. The following lemma is used to show that minor posets are thin and graded.

Lemma 4.2.2. *Let (L, G) be a generator-enriched lattice and let (K_1, H_1) and (K_2, H_2) be minors of (L, G) such that $(K_1, H_1) < (K_2, H_2)$ in the minor poset* $\mathrm{M}(L, G)$*. If $|H_2| - |H_1| = 2$ then the interval $[(K_1, H_1), (K_2, H_2)]$ of* $\mathrm{M}(L, G)$ *forms a diamond.*

Proof. The minor (K_1, H_1) may be presented as $((K_2, H_2)/I) \setminus J$ for some sets of generators I and J. We proceed by considering the different possibilities for I and J. When I is empty the set J must contain two elements, say j_1 and j_2. Then $(K_1, H_2) = (K_2, H_2) \setminus \{j_1, j_2\}$ and the open interval $((K_1, H_1), (K_2, H_2))$ consists of the two minors $(K_2, H_2) \setminus \{j_1\}$ and $(K_2, H_2) \setminus \{j_2\}$.

Now consider the case where J is empty. In this case I may consist of either one element or two elements. First suppose $I = \{i_1, i_2\}$. As a further subcase assume that $i_1 \not< i_2$ and $i_1 \not> i_2$. Since by assumption $|H_2| - |H_1| = 2$, every generator j of (K_1, H_1) corresponds to a unique generator i of (K_2, H_2) such that $i \vee i_1 \vee i_2 = j$. Due to this uniqueness no deletion of (K_2, H_2) has (K_1, H_1) as a minor. If $(K, H) = (K_2, H_2) \setminus \{j'\}$ then $j' \vee i_1 \vee i_2$ is not an element of $(K, H)/\{i_1, i_2\}$ but is an element of K_1. By similar reasoning no contraction of (K_2, H_2) other than the single contractions $(K_2, H_2)/i_1$ and $(K_2, H_2)/i_2$ as well as (K_1, H_1) itself has (K_1, H_1) as a minor. This establishes that the open interval $((K_1, H_1), (K_2, H_2))$ consists solely of the minors $(K_2, H_2)/i_1$ and $(K_2, H_2)/i_2$.

Now return to the case $I = \{i_1, i_2\}$ and now suppose that $i_1 < i_2$. In this case $i_1 \vee i_2 = i_2$ so $(K_2, H_2)/\{i_1, i_2\} = (K_2, H_2)/\{i_2\}$. By a similar argument as used in the previous subcase each generator j of (K_1, H_1) corresponds to a unique generator j' of (K_2, H_2) such that $j' \vee i_2 = j$. Due to this uniqueness no deletion of (K_2, H_2) other than $(K_2, H_2) \setminus i_1$ contains (K_1, H_1) as a minor. Similarly, no contraction of (K_2, H_2) with the exceptions of $(K_2, H_2)/i_1$ and (K_1, H_1) contain (K_1, H_1) as a minor. Thus, the open interval $((K_1, H_1), (K_2, H_2))$ consists solely of the minors $(K_2, H_2)/i_1$ and $(K_2, H_2) \setminus i_1$.

Suppose $I = \{i\}$ and $J = \emptyset$. If there exists $i' \in H_2$ such that $i' < i$ then we may take $I = \{i, i'\}$ which falls under a previous case. Otherwise, there is one generator $j \in H_1$ that corresponds to two generators $j_1, j_2 \in H_2$. All other generators in H_1 correspond to a unique generator in H_2. Thus in this case

$$((K_1, H_1), (K_2, H_2)) = \{(K_2, H_2) \setminus \{j_1\}, (K_2, H_2) \setminus \{j_2\}\}.$$

What remains is the case when both I and J are singletons. Suppose $I = \{i\}$ and $J = \{i \vee j\}$ for some generator j of (K_2, H_2). Once again since $|H_2| - |H_1| = 2$, every generator of (K_1, H_1) corresponds to a unique generator of (K_2, H_2), hence for all generators j_1, j_2 of (K_2, H_2) the joins $i \vee j_1$ and $i \vee j_2$ are distinct. Hence, the open interval $((K_1, H_1), (K_2, H_2))$ consists of the minors $(K_2, H_2) \setminus j$ and $(K_2, H_2)/i$. □

Lemma 4.2.3. *For any generator-enriched lattice (L, G) the minor poset $\mathrm{M}(L, G)$ is graded by $\mathrm{rk}(K, H) = |H| + 1$ and is thin.*

Proof. First we observe that the atoms of the minor poset $\mathrm{M}(L, G)$ are the minors of (L, G) that contain one element. A generator-enriched lattice with a single element has no generators, and thus has no minors. Thus, such elements cover the minimal element $\emptyset$ of $\mathrm{M}(L, G)$. On the other hand if a minor of (L, G) covers the minimal element $\emptyset$ of $\mathrm{M}(L, G)$, then by definition it must have no minors. A generator-enriched lattice with no minors must have no generators and thus consists of a single element.

By Lemma 4.2.2 whenever $(K_1, H_1) \prec (K_2, H_2)$ the difference $|H_2| - |H_1|$ must be less than 2. Clearly this difference is positive, so it must be equal to 1. Since every atom has zero generators every saturated chain from the minimal element $\emptyset$ to a minor (K, H) must have the same length, namely $|H| + 1$.

Having proven the minor poset $\mathrm{M}(L, G)$ is graded by $\mathrm{rk}(K, H) = |H|+1$, it follows from Lemma 4.2.2 that the poset $\mathrm{M}(L, G)$ is thin. □

The order relation of the minor poset of a generator-enriched lattice that satisfies the join irreducible lift property is somewhat simpler.

Definition 4.2.4. *A generator-enriched lattice (L, G) is said to* lift join irreducibles *if for all $\ell \in L$ and $i \in G$ the element $i \vee \ell$ is join irreducible in $[\ell, \widehat{1}]$.*

Taking $\ell = \widehat{0}$ in the above definition it is seen that a generator-enriched lattice that lifts join irreducibles must be minimally generated. This property may be equivalently stated as every minor of (L, G) is minimally generated. In particular a minimally generated distributive lattice has the join irreducible lift property by Proposition 3.4.15, as do minimally generated geometric lattices by Lemma 3.4.8. Figure 3.3 shows that the face lattice of the square is an example of a minimally generated lattice which does not lift join irreducibles.

Proposition 4.2.5. *If (L, G) is a generator-enriched lattice with the join irreducible lift property, and (K_1, H_1) and (K_2, H_2) are minors of (L, G) then we have $(K_1, H_1) \leq (K_2, H_2)$ if and only if $K_1 \subseteq K_2$.*

Proof. If $(K_1, H_1) \leq (K_2, H_2)$, that is, (K_1, H_1) is a minor of (K_2, H_2), then clearly $K_1 \subseteq K_2$. Now assume conversely that $K_1 \subseteq K_2$. Set $k_1 = \widehat{0}_{K_1}$ and $k_2 = \widehat{0}_{K_2}$. Let $M_1 = [k_1, \widehat{1}]$ and $M_2 = [k_2, \widehat{1}]$. Since (L, G) has the join irreducible lift property, $H_1 = \mathrm{irr}(K_1) \subseteq \mathrm{irr}(M_1)$. Furthermore

$$\mathrm{irr}(M_1) = \{i \vee k_1 : i \in \mathrm{irr}(L)\} = \{i \vee k_1 : i \in \mathrm{irr}(M_2)\}.$$

By assumption $K_1 \subseteq K_2$. Thus $\mathrm{irr}(K_1) \subseteq \mathrm{irr}(M_1) \cap K_2$. Irreducibility in M_1 implies irreducibility in $M_1 \cap K_2$. Thus $\mathrm{irr}(K_1) \subseteq \mathrm{irr}(M_1 \cap K_2)$, hence (K_1, H_1) is a deletion of

$$(M_1 \cap K_2, \mathrm{irr}(M_1 \cap K_2)) = (K_2, H_2)/k_1.$$

Therefore $(K_1, H_1) \leq (K_2, H_2)$ in $\mathrm{M}(L, G)$. □

At this point the isomorphism type of the minor posets of Boolean algebras and chains are readily determined.

Proposition 4.2.6. *The minor poset $\mathrm{M}(B_n, \mathrm{irr}(B_n))$ of the Boolean algebra B_n with minimal generating set is isomorphic to the face lattice of the n-dimensional cube.*

Proof. Recall that the face lattice of the n-dimensional cube is isomorphic to the poset of intervals of B_n with the empty interval as the unique minimal element [46, Chapter 3, Exercise 177]. It is easy to see from Lemma 3.4.8 that the minors of $(B_n, \mathrm{irr}(B_n))$ are exactly the intervals of B_n. Proposition 4.2.5 implies the order relations are the same. □

Proposition 4.2.7. *The minor poset of the length n chain is isomorphic to the rank $n+1$ Boolean algebra.*

Proof. In a chain every element except the minimal element is join irreducible and hence must be a generator. As a result every subset of a chain is a minor. Proposition 4.2.5 implies the minors are ordered by inclusion. □

Minor posets of minimally generated distributive lattices can be described in terms of order minors.

Corollary 4.2.8. *Let $(L, \mathrm{irr}(L))$ be the minimally generated lattice of lower order ideals of a poset P. The minor poset $\mathrm{M}(L, \mathrm{irr}(L))$ is isomorphic to the poset of order ideals of P ordered via $(I_1, J_1) \leq (I_2, J_2)$ when $I_1 \subseteq I_2$ and the maximal elements of J_1 are contained in $I_2 \cup J_2$.*

Proof. Recall from Proposition 3.4.14 that an order minor (I, J) corresponds to the lattice minor $((L, \mathrm{irr}(L))|_{I \cup J})/J$. Let

$$(K_1, \mathrm{irr}(K_1)) = ((L, \mathrm{irr}(L))|_{I_1 \cup J_1})/J_1,$$

and

$$(K_2, \mathrm{irr}(K_2)) = ((L, \mathrm{irr}(L))|_{I_2 \cup J_2})/J_2.$$

We have $(K_1, \mathrm{irr}(K_1)) \leq (K_2, \mathrm{irr}(K_2))$ if and only if $\widehat{0}_{K_1} \in K_2$ and

$$\mathrm{irr}(K_1) \subseteq \{i \vee \widehat{0}_{K_1} : i \in \mathrm{irr}(K_2)\}.$$

The former occurs precisely when the maximal elements of J_1 are contained in $I_2 \cup J_2$. The latter condition holds if and only if $I_1 \subseteq I_2$. □

See Figure 4.2 for an example.

The lowest three ranks of minor posets are easily described. In particular from Lemma 3.4.7 the rank 1 elements of $\mathrm{M}(L, G)$ are in bijection with the elements of L and the rank 2 elements of $\mathrm{M}(L, G)$ are in bijection with the edges of the diagram of (L, G). Somewhat more can be said for the rank 2 and 3 minors when (L, G) is minimally generated and geometric. The following lemma will be used for this purpose.

Lemma 4.2.9. *A generator-enriched lattice is minimally generated and geometric if and only if it has no minors isomorphic to the length 2 chain.*

Proof. The class of minimally generated geometric lattices is closed under taking minors and the length 2 chain is not geometric. Conversely, consider a generator-enriched lattice (L, G) which is not a minimally generated geometric lattice. First consider the case in which (L, G) has some generator g that is not an atom. There must be some atom a with $a < g$. Observe that the minor $L|_{\{a,g\}}$ is isomorphic to the length 2 chain. Now suppose that L is not upper semimodular. Let $x, y \in L$ such that $x \wedge y \prec x$ and $x \wedge y \prec y$ but $x \not\prec x \vee y$. Since y covers $x \wedge y$ it is a generator in the contraction $(L, G)/(x \wedge y)$. This implies that $x \vee y$ is a generator

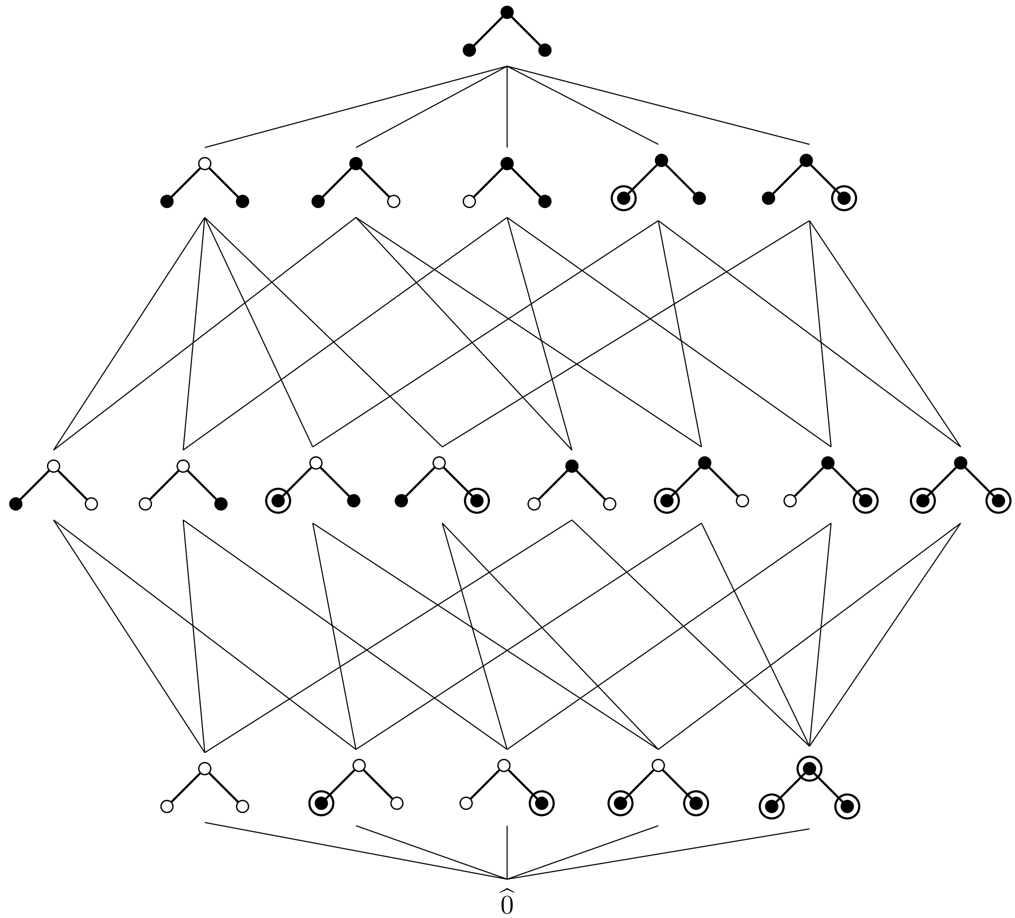


Figure 4.2: The poset of order minors of a 3 element poset P. An order minor (I, J) is depicted by coloring the elements of I black, circling the elements of J and coloring the elements of $P \setminus (I \cup J)$ white.

of the minor $(L, G)/x = ((L, G)/(x \wedge y))/x$. Since $x \not\prec x \vee y$ the minor $(L, G)/x$ has a generator which is not an atom. By the preceding case $(L, G)/x$ has a minor isomorphic to the length 2 chain which is a minor of (L, G) as well. □

Proposition 4.2.10. *The diagram of a generator-enriched lattice (L, G) is equal to the Hasse diagram of L if and only if (L, G) is minimally generated and geometric.*

Proof. First assume (L, G) is minimally generated and geometric. A minor (L, G) with one generator is of the form $\langle b|a\rangle$ for $a, b \in L$ with $a \prec b$. Thus all edges of the diagram are edges of the Hasse diagram of L. Since the converse always holds the two graphs are equal.

Now let (L, G) be a generator-enriched lattice that either is not minimally generated or is not geometric. By Lemma 4.2.9 the generator-enriched lattice (L, G) must have a minor isomorphic to a length 2 chain. The diagram of a minor of (L, G) is a subgraph of the diagram of (L, G), so there must be an edge in the diagram of (L, G) that is not an edge of the Hasse diagram of L. □

Proposition 4.2.11. *For any generator-enriched lattice* (L, G) *the lower intervals* $[\widehat{0}, (K, H)]$ *where* $\mathrm{rk}(K, H) = 3$ *of the minor poset* $\mathrm{M}(L, G)$ *are each isomorphic to the Boolean algebra* B_3 *or the face lattice* Q_2 *of a square. Moreover, all intervals* $[\widehat{0}, (K, H)]$ *with* $\mathrm{rk}(K, H) = 3$ *are isomorphic to* Q_2 *if and only if* (L, G) *is minimally generated and geometric.*

Proof. The only possible isomorphism types of a generator-enriched lattice with 2 generators are the minimally generated Boolean algebra $(B_2, \mathrm{irr}(B_2))$ and the length 2 chain. The minor poset of the Boolean algebra B_2 is isomorphic to the face lattice of a square while the minor poset of the length 2 chain is isomorphic to the Boolean algebra B_3. The second statement follows from Lemma 4.2.9. □

4.2.2 Operations on generator-enriched lattices

We next discuss several operations on generator-enriched lattices and how these affect the associated minor posets. Recall that for two posets P and Q the *diamond product* $P \diamond Q$ is defined to be the poset $((P \setminus \{\widehat{0}_P\}) \times (Q \setminus \{\widehat{0}_Q\})) \cup \{\widehat{0}\}$. We define the diamond product on generator-enriched lattices as

$$(L, G) \diamond (K, H) = (L \diamond K, G \times H).$$

The pyramid and prism operations are defined on a poset P as $\mathrm{Pyr}(P) = P \times B_1$ and $\mathrm{Prism}(P) = P \diamond B_2$. We define Pyr on generator-enriched lattices in the same manner as $\mathrm{Pyr}(L, G) = (L, G) \times (B_1, \mathrm{irr}(B_1))$. For a discussion of the diamond product, see [25, end of Section 2].

Given two generator-enriched lattices (L, G) and (K, H) define the Cartesian product $(L, G) \times (K, H)$ to be the generator-enriched lattice

$$(L, G) \times (K, H) = (L \times K, (G \times \{\widehat{0}_K\}) \cup (\{\widehat{0}_L\} \times K)).$$

This operation of Cartesian product behaves nicely on the minor posets, it corresponds to the diamond product.

Proposition 4.2.12. *For any generator-enriched lattices* (L, G) *and* (K, H) *we have that*

$$\mathrm{M}((L, G) \times (K, H)) \cong \mathrm{M}(L, G) \diamond \mathrm{M}(K, H).$$

In particular $\mathrm{M}(\mathrm{Pyr}(L, G)) \cong \mathrm{Prism}(\mathrm{M}(L, G))$ *holds.*

Proof. Let $\pi_1 : (L, G) \diamond (K, H) \to (L, G)$ and $\pi_2 : (L, G) \diamond (K, H) \to (L, G)$ be the projection maps. These induce order-preserving maps $\overline{\pi}_1$ and $\overline{\pi}_2$ between the minor posets. Neither of these maps send a minor to the minimal element $\emptyset$, so we have an order-preserving map $\phi : \mathrm{M}((L, G) \times (K, H)) \to \mathrm{M}(L, G) \diamond \mathrm{M}(K, H)$ defined on minors by $\phi(M, I) = (\overline{\pi}_1(M, I), \overline{\pi}_2(M, I))$.

The inverse of ϕ is the map ψ defined for pairs $((L', G'), (K', H'))$ by

$$\psi((L', G'), (K', H')) = \langle (G' \times \{\widehat{0}_{K'}\}) \cup (\{\widehat{0}_{L'}\} \times H') | (\widehat{0}_{L'}, \widehat{0}_{K'}) \rangle.$$

To see that the image under ψ is indeed a minor, let $G'' \subseteq G$ such that

$$G' = \{g \vee \widehat{0}_{L'} : g \in G''\}.$$

Similarly define H''. The generating set of the image under ψ can be described as

$$\{(g, \widehat{0}_K) \vee (\widehat{0}_{L'}, \widehat{0}_{K'}) : g \in G''\} \cup \{(\widehat{0}_L, h) \vee (\widehat{0}_{L'}, \widehat{0}_{K'}) : h \in H''\}.$$

Thus the image is indeed a minor of the Cartesian product.

Observe that for $I \subseteq G'$ we have

$$\psi((L', G') \setminus I, (K', H')) = \psi((L', G'), (K', H')) \setminus (I \times \{\widehat{0}'_K\}),$$

and

$$\psi((L', G')/I, (K', H')) = \psi((L', G'), (K', H'))/(I \times \{\widehat{0}'_K\}.$$

A similar statement holds for deletions and contractions of (K', H'). Thus the map ψ is order-preserving. □

We now consider the minor poset of the result of adjoining a new maximal element to a generator-enriched lattice. Given a generator-enriched lattice (L, G) let $(\widehat{L}, \widehat{G})$ denote the generator-enriched lattice obtained by adjoining a new maximal element which, necessarily, is an element of $\widehat{G}$.

Proposition 4.2.13. *Let (L, G) be a generator-enriched lattice, then we have the isomorphism*

$$\mathrm{M}(\widehat{L}, \widehat{G}) \cong \mathrm{Pyr}(\mathrm{M}(L, G)).$$

Proof. Let m be the maximal element of $\widehat{L}$. Observe since m is join irreducible and maximal in $\widehat{L}$, for any minor $(K, H) \neq (m, \emptyset)$ of $(\widehat{L}, \widehat{G})$, both $(K \cup \{m\}, H \cup \{m\})$ and $(K \setminus \{m\}, H \setminus \{m\})$ are minors of (L, G). Define a map

$$f : \mathrm{M}(\widehat{L}, \widehat{G}) \to \mathrm{Pyr}(\mathrm{M}(L, G))$$

by setting $f(\emptyset) = (\emptyset, \widehat{0})$, and for minors (K, H) of (L, G) setting

$$f(K, H) = \begin{cases} (K, H, \widehat{0}) & \text{if } m \notin K, \\ (K \setminus \{m\}, H \setminus \{m\}, \widehat{1}) & \text{if } m \in H, \\ (\emptyset, \widehat{1}) & \text{if } K = \{m\},\ H = \emptyset. \end{cases}$$

The map f is order-preserving. Define $g : \mathrm{Pyr}(\mathrm{M}(L, G)) \to \mathrm{M}(\widehat{L}, \widehat{G})$ by setting $g(\emptyset, \widehat{0}) = \emptyset$ and $g(\emptyset, \widehat{1}) = (m, \emptyset)$ and setting

$$g(K, H, \varepsilon) = \begin{cases} (K, H) & \text{if } \varepsilon = \widehat{0}, \\ (K \cup \{m\}, H \cup \{m\}) & \text{if } \varepsilon = \widehat{1}. \end{cases}$$

Clearly the map g is order-preserving and is the inverse of f. □

It would be interesting if minor posets of more generator-enriched lattices of the form $(L, G) \circ (K, H)$ could be described. Of course when K is a chain the minor poset $\mathrm{M}((L, G) \circ (K, H))$ is an iterated pyramid of $\mathrm{M}(L, G)$. In general this operation appears to be more nuanced. For example, considering the generator-enriched lattice $(C_1, \mathrm{irr}(C_1)) \circ (B_2, \mathrm{irr}(B_2))$ the minor poset is not simply a pyramid over $\mathrm{M}(B_2, \mathrm{irr}(B_2))$ but is instead the result of merging two triangular facets of this pyramid.

Lastly we discuss an operation we call a *mapping pyramid*, which is defined for two generator-enriched lattices with a strong surjection between them.

Definition 4.2.14. *Let (L, G) and (K, H) be two disjoint generator-enriched lattices and let $f : (L, G) \to (K, H)$ be a strong surjection. Extend the join operations of L and K to a join operation on $L \cup K$; for $\ell \in L$ and $k \in K$ let $\ell \vee k = f(\ell) \vee k$. The* mapping pyramid *of (L, G) with respect to f is the generator-enriched lattice $\mathrm{Pyr}_f(L, G) = (L \cup K, G \cup \{\widehat{0}_K\})$.*

As an example the generator-enriched lattice $\langle a, b, c | \widehat{0} \rangle$ whose minor poset is depicted in Figure 4.1 is isomorphic to the mapping pyramid $\mathrm{Pyr}_f(\langle a, b | \widehat{0} \rangle)$ in which f is the map $\ell \mapsto \ell \vee c$.

In order to describe the minor posets of mapping pyramids we need the following definition of a mapping prism of posets.

Definition 4.2.15. *Let P and Q be disjoint posets, each with a unique minimal element, and let $f : P \to Q$ be an order-preserving map. The* mapping prism $\mathrm{Prism}_f(P)$ *has underlying set*

$$\mathrm{Prism}_f(P) = (((P \setminus \{\widehat{0}\}) \times B_1) \cup (Q \setminus \{\widehat{0}\})) \cup \{\widehat{0}\}.$$

The order relations of $\mathrm{Prism}_f(P)$ are those induced by the subposets $(P \setminus \{\widehat{0}\}) \times B_1$ and $Q \setminus \{\widehat{0}\}$, along with the relations $q \le (p, \widehat{1})$ if and only if $q \le f(p)$.

Proposition 4.2.16. *Let (L, G) and (K, H) be two generator-enriched lattices and let $f : (L, G) \to (K, H)$ be a strong surjection. Let $F : \mathrm{M}(L, G) \to \mathrm{M}(K, H)$ be the order-preserving surjection induced by f. We have that*

$$\mathrm{M}(\mathrm{Pyr}_f(L, G)) \cong \mathrm{Prism}_F(\mathrm{M}(L, G)).$$

Proof. Consider applying a single deletion or contraction to $\mathrm{Pyr}_f(L, G)$. If $\widehat{0}_K$ is deleted the result is (L, G), while if $\widehat{0}_K$ is contracted the result is (K, H). If some generator $g \in G$ of $\mathrm{Pyr}_f(L, G)$ is contracted or deleted, the result is $\mathrm{Pyr}_f((L, G)/g)$ or $\mathrm{Pyr}_f((L, G) \setminus \{g\})$ respectively. Thus, we have three different types of minors of $\mathrm{Pyr}_f(L, G)$ which we may distinguish by whether the minor of $\mathrm{Pyr}_f(L, G)$ is a subset of L, of K or of neither. Define a map $g : \mathrm{M}(\mathrm{Pyr}_f(L, G)) \to \mathrm{Prism}_F(\mathrm{M}(L, G))$ for $(M, I) \in \mathrm{M}(\mathrm{Pyr}_f(L, G)) \setminus \{\emptyset\}$ by setting

$$g(M, I) = \begin{cases} (M, I) & \text{if } M \subseteq K, \\ (M, I, \widehat{0}) & \text{if } M \subseteq L, \\ (M \setminus K, I \setminus \{\widehat{0}_K\}, \widehat{1}) & \text{otherwise.} \end{cases}$$

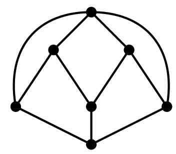


Figure 4.3: The diagram of a generator-enriched lattice with no parallels which is not distributive.

Additionally define $g(\emptyset) = \widehat{0}$. Define a map $h : \mathrm{Prism}_f(\mathrm{M}(L, G)) \to \mathrm{M}(\mathrm{Pyr}_f(L, G))$ by for $(M, I) \in \mathrm{M}(L, G) \setminus \{\emptyset\}$ setting

$$h(M, I, \widehat{0}) = (M, I),$$
$$h(M, I, \widehat{1}) = \mathrm{Pyr}_f(M, I),$$

and for $(M, I) \in \mathrm{M}(K, H) \setminus \{\emptyset\}$ setting

$$h(M, I) = (M, I).$$

Additionally, set $h(\widehat{0}) = \emptyset$. Observe the maps g and h are inverses and are each order-preserving hence g is the desired isomorphism. □

4.2.3 The lattice property for the poset of minors

In this subsection we characterize the generator-enriched lattices where the associated poset of minors is itself a lattice. This characterization (Theorem 4.2.24 and Proposition 4.2.18) is in terms of five forbidden minors.

To prove the characterization the following concept of a lattice with no parallels is required. This property is weaker than the property that the minor poset is a lattice, and is crucial to the proof of Theorem 4.2.24.

Definition 4.2.17. *A generator-enriched lattice (L, G) has no parallels if for any element $\ell \in L$ and generators $g, h \in G$ whenever $g \vee \ell \neq \ell$ then $g \vee \ell \neq h \vee \ell$. If $g \vee \ell = h \vee \ell \neq \ell$ holds the generator-enriched lattice (L, G) is said to have a parallel.*

A parallel in a minimally generated geometric lattice corresponds to a nontrivial parallel class in a contraction of the associated simple matroid. Lemma 3.4.13 shows that any minimally generated distributive lattice has no parallels. Figure 4.3 gives an example of a generator-enriched lattice which is not distributive and has no parallels. The same example also shows a lattice with no parallels need not be semimodular, but it will be seen lattices with no parallels are dual semimodular. As it turns out the generator-enriched lattice depicted in Figure 4.3 occurs as a minor of every generator-enriched lattice with no parallels that is not distributive, this is Proposition 4.2.22 below. On the other hand the generator-enriched lattice shown in Figure 4.4 (a) is an example of a minimally generated modular lattice which has a parallel.

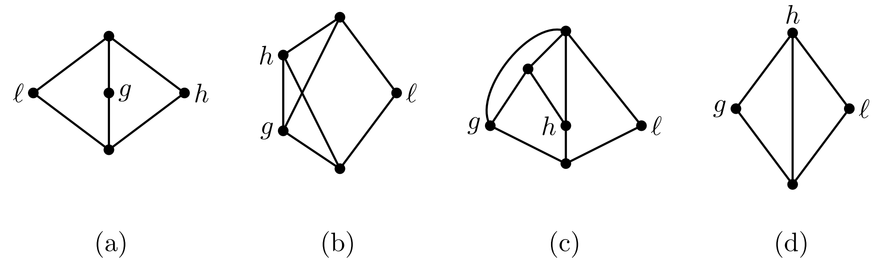


Figure 4.4: The forbidden minors for the no parallels property.

The definition above of the no parallels property is a mild restatement of the anti-exchange property for closure operators introduced by Edelman in [20]. Anti-exchange closure operators are also referred to as convex geometries and are dual to antimatroids. An antimatroid consists of a collection of feasible sets which are the complements of the closed sets of an anti-exchange closure operator. Edelman showed that a closure operator has the anti-exchange property if and only if the lattice of closed sets is meet distributive. A lattice is *meet distributive* if every interval of the form $[\bigwedge_{y \prec x} y, x]$ is Boolean. Given a generator-enriched lattice (L, G) we must have $G = \mathrm{irr}(L)$, this is seen by taking $\ell = \widehat{0}$ in the definition, hence a generator-enriched lattice has no parallels if and only if L is meet distributive and $G = \mathrm{irr}(L)$.

Proposition 4.2.18. *A generator-enriched lattice has no parallels if and only if it has no minor isomorphic to one of the four generator-enriched lattices whose diagrams are depicted in Figure 4.4.*

Proof. It is clear from the definition that the class of generator-enriched lattices with no parallels is closed under taking minors. Observe that each of the four depicted generator-enriched lattices has a parallel formed by the elements labeled g, h and ℓ. Thus any generator-enriched lattice with a minor isomorphic to one of the four generator-enriched lattices depicted in Figure 4.4 has a parallel.

Let (L, G) be a generator-enriched lattice with a parallel. By definition there exists an element $\ell \in L$ and generators $g_1, g_2 \in G$ such that $g_1 \vee \ell = g_2 \vee \ell \neq \ell$. Choose ℓ' to be minimal among such ℓ, that is, $g_1 \vee \ell' = g_2 \vee \ell' \neq \ell'$ and for all $\ell < \ell'$ the elements $g_1 \vee \ell$ and $g_2 \vee \ell$ are distinct. Choose an element ℓ_0 of L such that there is a generator h of (L, G) with $h \vee \ell_0 = \ell'$. Now consider the contraction $(L, G)/\ell_0$. The elements $g_1 \vee \ell_0$, $g_2 \vee \ell_0$ and ℓ' are distinct and each is a generator of the contraction $(L, G)/\ell_0$. Let $(K, H) = ((L, G)/\ell_0)|_{\{g_1 \vee \ell_0, g_2 \vee \ell_0, \ell'\}}$. The generator-enriched lattice (K, H) has a parallel and three generators. It is readily checked that the only possibilities for (K, H) are the four generator-enriched lattices depicted in Figure 4.4. □

Recall a generator-enriched lattice (L, G) is said to lift join irreducibles if for any generator g and element $\ell \in L$ the element $g \vee \ell$ is join irreducible in the interval $[\ell, \widehat{1}]$. Equivalently a generator-enriched lattice lifts join irreducibles whenever every minor

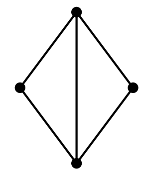


Figure 4.5: The forbidden minor for the join irreducible lift property.

is minimally generated. The following forbidden minor characterization shows this property is weaker than the property of no parallels.

Lemma 4.2.19. *A generator-enriched lattice lifts join irreducibles if and only if it has no minors isomorphic to the generator-enriched lattice whose diagram is depicted in Figure 4.5.*

Proof. The generator-enriched lattice depicted in Figure 4.5 has underlying lattice isomorphic to B_2 and has three generators. Since this generator-enriched lattice is not minimally generated, it does not lift join irreducibles. Since the join irreducible lift property is closed under taking minors, no generator-enriched lattice with a minor isomorphic to B_2 with three generators lifts join irreducibles.

Let (L, G) be a generator-enriched lattice that does not lift join irreducibles. There is some minor (K, H) of (L, G) that is not minimally generated. Let h be a generator of (K, H) that is not join irreducible. The element h may be expressed as the join of join irreducibles of K, say

$$h = i_1 \vee \cdots \vee i_r.$$

Assume this set $S = \{i_1, \ldots, i_r\}$ is minimal in the sense that for any proper subset of S the join is not equal to h. Consider the contraction $(K', H') = (K, H)/(i_1 \vee \cdots \vee i_{r-2})$. Note that the element h is equal to $h \vee i_1 \vee \cdots \vee i_{r-2}$ hence h is a generator of (K', H'). By the minimality of S the generators $a = i_1 \vee \cdots \vee i_{r-1}$ and $b = i_1 \vee \cdots \vee i_{r-2} \vee i_r$ of (K', H') are not equal to h. Furthermore, a and b must be incomparable since $a \vee b = h$. This shows that the minor $(K', H')|_{\{a,b,h\}}$ of (L, G), which has three generators, has underlying lattice isomorphic to B_2. □

The following lemma gives a third equivalent definition of the no parallels property. This is needed to prove Theorem 4.2.24. This lemma is essentially due to Avann [1, Theorem 5.5] who gave several equivalent characterizations of meet distributive lattices. Dilworth first examined the duals of lattices satisfying the below alternative description of the no parallels property in [19] where he characterized such lattices as semimodular lattices whose modular sublattices are all distributive.

Lemma 4.2.20. *Let (L, G) be a generator-enriched lattice with n generators and let $\theta : B_n \to L$ be the canonical strong surjection. The generator-enriched lattice (L, G) has no parallels if and only if for every $\ell \in L$ the set $\theta^{-1}(\ell)$ has a unique minimal element in the Boolean algebra.*

Proof. Label the elements of G as $g_1, \ldots, g_n$. Assume that (L, G) has a parallel, namely for some element $\ell \in L$ and $i, j \in [n]$ we have $g_i \vee \ell = g_j \vee \ell \neq \ell$. Choose some $X \subseteq [n]$ such that $\theta(X) = \ell$. Observe that $\theta(X \cup \{i\}) = \theta(X \cup \{j\}) = g_i \vee \ell$. For any sets $A \subseteq B$ that are elements of the fiber $\theta^{-1}(g_i \vee \ell)$, we have $[A, B] \subseteq \theta^{-1}(g_i \vee \ell)$. Since the preimage set contains $X \cup \{i\}$ and $X \cup \{j\}$ but not the intersection X, it does not have a unique minimal element.

To show the converse, suppose for some $\ell \in L$ that $\theta^{-1}(\ell)$ has two minimal elements X and Y. If either X or Y is of cardinality 1 then ℓ is a generator of (L, G). Furthermore the existence of a second minimal element of the preimage $\theta^{-1}(\ell)$ implies that ℓ is join reducible in L. Thus in this case (L, G) does not lift join irreducibles, hence (L, G) has a parallel. Now assume $|X| \geq 2$ and $|Y| \geq 2$. Choose some $x \in X \setminus Y$ and set $X' = X \setminus \{x\}$. Since X is minimal in $\theta^{-1}(\ell)$ the element $\theta(X')$ is not equal to ℓ. Consider the contraction $(L, G)/X'$. Since $\theta(X') \vee \theta(\{x\}) = \ell$ the element ℓ is a generator of $(L, G)/X'$. On the other hand $\ell = \theta(Y) = \theta(X') \vee \theta(Y)$ so

$$\ell = \theta(X') \vee \bigvee_{y \in Y} \theta(\{y\}).$$

Either $\theta(X') \vee \theta(\{y\}) = \ell$ for some $y \in Y$ or ℓ is not join irreducible in $(L, G)/X'$. The first case shows (L, G) has a parallel and the second shows that (L, G) does not lift join irreducibles. In either case (L, G) has a parallel. □

At this point we take a small detour to use Lemma 4.2.20 to give a forbidden minor characterization of distributive lattices.

Lemma 4.2.21. *Given a generator-enriched lattice (L, G) with no parallels the lattice L is distributive if and only if the unique minimal expression of any element $\ell \in L$ as a join of generators from Lemma 4.2.20 is the set of maximal elements of $G \cap [\widehat{0}_L, \ell]$.*

Proof. A distributive lattice satisfies the above condition since any lower order ideal is expressed minimally as the union of principal lower order ideals generated by maximal elements of the ideal. Now suppose (L, G) has no parallels and each element $\ell \in L$ has as its unique minimal expression as a join of generators the maximal elements of $G \cap [\widehat{0}_L, \ell]$. Let J be the distributive lattice consisting of all lower order ideals of the poset G. We have a join-preserving surjection from J onto L defined by mapping a lower order ideal to the join of its elements computed in L.

We claim this surjection is also injective. Consider a lower order ideal I of G and the join $\ell = \bigvee_{i \in I} i$. By assumption the set I must contain the set of maximal elements of the lower order ideal $G \cap [\widehat{0}_L, \ell]$. Since we also have $I \subseteq G \cap [\widehat{0}_L, \ell]$ this establishes equality. Thus, each element $\ell \in L$ has one lower order ideal in its preimage, so the map is injective. □

Proposition 4.2.22. *A lattice L is distributive if and only if the generator-enriched lattice $(L, \mathrm{irr}(L))$ has no parallels and no minor isomorphic to the generator-enriched lattice depicted in Figure 4.3.*

Proof. Minors of minimally generated distributive lattices were shown to be themselves minimally generated and distributive in Proposition 3.4.15. Now let (L, G) be a generator-enriched lattice with no parallels such that L is not distributive. By

Lemma 4.2.21 there exists an element $\ell \in L$ such that its unique minimal expression as a join of generators is not the set of maximal elements of $G \cap [\widehat{0}_L, \ell]$. Assume ℓ is minimal in L with this property. Let $M = \{m_1, \ldots, m_r\}$ be the set of maximal elements of $G \cap [\widehat{0}_L, \ell]$. Suppose m_1 is not part of the minimal expression for ℓ and m_2 and m_3 are both included in the minimal expression. Set

$$(K, H) = ((L, G)|_M)/\{m_4, \ldots, m_r\}.$$

We claim (K, H) is isomorphic to the generator-enriched lattice depicted in Figure 4.3. Set $m = m_4 \vee \cdots \vee m_r$ and set $i_1 = m_1 \vee m$, set $i_2 = m_2 \vee m$ and set $i_3 = m_3 \vee m$. Observe $\ell = i_1 \vee i_2 \vee i_3 = i_2 \vee i_3$. We need to show the other 5 nonempty joins of i_1,i_2 and i_3 are all distinct.

Proceeding upwards in the number of elements joined firstly, we have $i_2 \neq i_3$ since $i_2 \vee i_3 = \ell$. Each of i_1,i_2, and i_3 have been expressed as a join of generators not including the set M so $\ell \neq i_1, i_2, i_3$. Furthermore, $i_1 \not\geq m_2$ and $i_3 \not\geq m_2$ so $i_1 \vee i_3 \neq \ell$. Similarly, $i_1 \vee i_2 \neq \ell$. This also implies $i_1 \vee i_3 \neq i_1 \vee i_2$.

What remains to be seen is that $i_1 \vee i_2$ is neither i_1 nor i_2 and similarly for $i_1 \vee i_3$. Since $i_1 \vee i_3 \neq \ell$ but $i_1 \vee i_3 = \ell$ we have $i_1 \vee i_2 \neq i_1$. Similarly $i_1 \vee i_3 \neq i_1$. Recall ℓ was assumed minimal so the unique expression of $i_1 \vee i_2$ as a join of generators consists of the maximal elements of $G \cap [\widehat{0}_L, i_1 \vee i_2]$. Since m_1 was maximal in the larger set $G \cap [\widehat{0}_L, \ell]$ is also maximal in $G \cap [\widehat{0}_L, i_1 \vee i_2]$. We have expressed i_2 as a join of a set of generators not containing m_1, namely

$$i_2 = m_2 \vee (m_4 \vee \cdots \vee m_r).$$

Thus, $i_2 \not\geq i_1 \vee i_2$ so these elements are distinct. Similarly, $i_1 \vee i_3 \neq i_3$ which establishes (K, H) is isomorphic to the generator-enriched lattice depicted in Figure 4.3. □

The following lemma characterizes the join operation in the minor poset for any generator-enriched lattice.

Lemma 4.2.23. *Let (L, G) be a generator-enriched lattice, let (K_1, H_1) and (K_2, H_2) be minors of (L, G) and set $\ell_0 = \widehat{0}_{K_1} \wedge \widehat{0}_{K_2}$. The join $(K_1, H_1) \vee (K_2, H_2)$ in the minor poset $\mathrm{M}(L, G)$ exists if and only if the following three conditions hold:*

1. *Let θ be the canonical strong map onto $(L, G)/\ell_0$. The fibers $\theta^{-1}(\widehat{0}_{K_1})$ and $\theta^{-1}(\widehat{0}_{K_2})$ each have a unique minimal element, say X_1 and X_2 respectively.*

2. *For each generator h of (K_1, H_1) there is a unique generator g of $(L, G)/\ell_0$ with $g \vee \widehat{0}_{K_1} = h$, and similarly for (K_2, H_2).*

3. *Any minor (K, H) of (L, G) such that*

$$(K, H) \geq (K_1, H_1) \text{ and } (K, H) \geq (K_2, H_2)$$

contains the element ℓ_0.

Let I be the set

$$\begin{aligned} I = &\{\theta(\{x\}) : x \in X_1 \cup X_2\} \\ &\cup \{g \vee \ell_0 : g \in G \text{ and } g \vee \widehat{0}_{K_1} \in H_1\} \\ &\cup \{g \vee \ell_0 : g \in G \text{ and } g \vee \widehat{0}_{K_2} \in H_2\}. \end{aligned}$$

If the above conditions are satisfied then the join $(K_1, H_1) \vee (K_2, H_2)$ is the minor $((L, G)/\ell_0)|_I$.

Proof. Assume that the join $(K_1, H_1) \vee (K_2, H_2)$ exists in $\mathrm{M}(L, G)$. Both of the minors (K_1, H_1) and (K_2, H_2) are a minor of the contraction $(L, G)/\ell_0$, hence the join $(K_1, H_1) \vee (K_2, H_2)$ is a minor of $(L, G)/\ell_0$ as well. Let z be the minimal element of the underlying lattice of $(K_1, H_1) \vee (K_2, H_2)$. We have that $z \geq \ell_0$. On the other hand $\widehat{0}_{K_1} \geq z$ and $\widehat{0}_{K_2} \geq z$ so $z \leq \ell_0$. Therefore $z = \ell_0$. The join $(K_1, H_1) \vee (K_2, H_2)$ is thus a deletion of the minor $(L, G)/\ell_0$, say $(K_1, H_1) \vee (K_2, H_2) = ((L, G)/\ell_0)|_J$ for some set J of generators. The set J must be the unique minimal set with the properties $\widehat{0}_{K_1}$ can be expressed as a join of elements in J, the set H_1 is included in the set $\{j \vee \widehat{0}_{K_1} : j \in J\}$, and the corresponding statements hold for (K_2, H_2). For if J' is another set with these properties then $((L, G)/\ell_0)|_{J'}$ is greater than or equal to both (K_1, H_1) and (K_2, H_2); thus $((L, G)/\ell_0)|_J \leq ((L, G)/\ell_0)|_{J'}$ hence $J \subseteq J'$. The existence of the set J implies that Conditions 1 and 2 hold. Condition 3 holds since the join $(K_1, H_1) \vee (K_2, H_2)$ contains the element ℓ_0.

Now suppose Conditions 1 through 3 are satisfied. Let I be the set defined in the statement and let $(K_0, H_0) = ((L, G)/\ell_0)|_I$. Consider a minor (K, H) of (L, G) with $(K, H) \geq (K_1, H_1)$ and $(K, H) \geq (K_2, H_2)$. By Condition 3 the minor (K, H) contains ℓ_0. The contraction $(K, H)/\ell_0$ satisfies $(K, H)/\ell_0 \geq (K_1, H_1)$ and $(K, H)/\ell_0 \geq (K_2, H_2)$. Thus the set

$$\{g \vee \widehat{0}_{K_1} : g \text{ is a generator of } (K, H)/\ell_0)\}$$

includes H_1 and similarly for (K_2, H_2). Furthermore,

$$\widehat{0}_{K_1} \in (L, G)/\ell_0 \text{ and } \widehat{0}_{K_2} \in (L, G)/\ell_0$$

so by condition 1 the generating set of $(L, G)/\ell_0$ includes $\{\theta(x) : x \in X_1 \cup X_2\}$. By condition 2 the generating set of $(L, G)/\ell_0$ and H include I, hence

$$(K, H) \geq ((L, G)/\ell_0)|_I.$$

Therefore, $((L, G)/\ell_0)|_I$ is the join $(K_1, H_1) \vee (K_2, H_2)$. □

Theorem 4.2.24. *For any generator-enriched lattice (L, G) the poset of minors $\mathrm{M}(L, G)$ is a lattice if and only if (L, G) has no parallels and no minor isomorphic to the generator-enriched lattice whose diagram is depicted in Figure 4.6.*

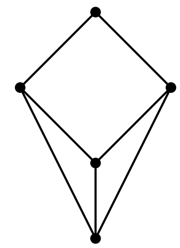


Figure 4.6: A generator-enriched lattice whose minor poset is not a lattice. The other four obstructions can be found in Figure 4.4.

Proof. Since for any minor (K, H) of (L, G) the minor poset $\mathrm{M}(K, H)$ is a lower interval in $\mathrm{M}(L, G)$, the property that the minor poset is a lattice is closed under taking minors. It is readily checked that for each of the five generator-enriched lattices depicted in Figures 4.4 and 4.6 the minor poset is not a lattice.

Let (L, G) be a generator-enriched lattice with no parallels, and such that there are two minors (K_1, H_1) and (K_2, H_2) of (L, G) for which the join $(K_1, H_1) \vee (K_2, H_2)$ in $\mathrm{M}(L, G)$ does not exist. Set $\ell_1 = \widehat{0}_{K_1}$, $\ell_2 = \widehat{0}_{K_2}$ and $\ell_0 = \ell_1 \wedge \ell_2$. It will be shown that (L, G) must have a minor isomorphic to the generator-enriched lattice depicted above. Since (L, G) has no parallels conditions 1 and 2 in Lemma 4.2.23 must be satisfied; condition 1 follows from Lemma 4.2.20 and condition 2 follows from the definition of the no parallels property. Thus condition 3 must fail, that is, there is some minor (K, H) with $(K, H) \geq (K_1, H_1)$ and $(K, H) \geq (K_2, H_2)$, but where ℓ_0 is not an element of K. In particular this implies the element ℓ_0 is neither ℓ_1 nor ℓ_2.

Let θ be the canonical map from $B_{|G|}$ onto (L, G). The fiber $\theta^{-1}(\ell_0)$ contains a unique minimal element, say X. Let I_0 be the θ images of all singletons included in X, that is,

$$I_0 = \{\theta(\{x\}) : x \in X\}.$$

Similarly define I_1, I_2 and I for ℓ_1, ℓ_2 and $\widehat{0}_K$ respectively. By choice of I_0 the join of a set of generators is greater than or equal to ℓ_0 only if said set includes I_0. A similar statement holds for I_1 and I_2. Since $\ell_0 \notin K$ the set

$$I \cup \{g \in G : g \vee \widehat{0}_K \in H\}$$

does not include I_0. On the other hand since ℓ_1 and ℓ_2 are elements of K the sets I_1 and I_2 are included in the above set. Thus, there is some $i_0 \in I_0 \setminus (I_1 \cup I_2)$. Recall ℓ_0 is neither ℓ_1 nor ℓ_2. Thus $\ell_1 \not\leq \ell_2 = \ell_2 \vee \ell_0$ so I_1 cannot be included in $I_0 \cup I_2$. Thus there is some element $i_1 \in I_1 \setminus (I_0 \cup I_2)$. Similarly, there is some element $i_2 \in I_2 \setminus (I_0 \cup I_1)$.

Let i be the join of all elements in the set

$$J = I_0 \cup I_1 \cup I_2 \setminus \{i_0, i_1, i_2\}.$$

Since i is the join of the elements of J, and this set includes none of I_0, I_1 or I_2 the element i is not greater than or equal to any of i_0, i_1 or i_2. Thus the elements $i \vee i_0$, $i \vee i_1$, and $i \vee i_2$ are each distinct from i, they are generators of the contraction $(L, G)/i$, and by the no parallels assumption must be distinct. We claim the minor $((L,G)/i)|_{\{i_0 \vee i, i_1 \vee i, i_2 \vee i\}}$ is isomorphic to the generator-enriched lattice depicted in Figure 4.6. Note that $i_j \vee i = \ell_j \vee i$ for $j = 0, 1, 2$. Since taking the join with i is an order-preserving map, we have $i_0 \vee i \leq i_1 \vee i$ and $i_0 \vee i \leq i_2 \vee i$. It remains to show that $i_1 \vee i$ and $i_2 \vee i$ are incomparable. The element $i_2 \vee i$ is the join of the elements of $I_2 \cup J$ which does not contain i_1 hence does not include I_1. Thus $i_2 \vee i \not\geq i_1$ hence $i_2 \vee i \not\geq i_1 \vee i$. Similarly $i_1 \vee i \not\geq i_2 \vee i$. □

The four generator-enriched lattices depicted in Figure 4.4 and the generator-enriched lattice depicted in Figure 4.6 together form a forbidden minor characterization of the generator-enriched lattices for which the minor poset is a lattice. Since all five of these generator-enriched lattices have three generators, the following corollary is immediate.

Corollary 4.2.25. *For any generator-enriched lattice (L,G) the minor poset $\mathrm{M}(L,G)$ is a lattice if and only if every interval $[\emptyset,(K,H)]$ such that $\mathrm{rk}(K,H) = 4$ is a lattice.*

The following restatement of Theorem 4.2.24 makes it clear generator-enriched lattices whose minor poset is a lattice are incredibly sparse.

Corollary 4.2.26. *Given a generator-enriched lattice (L,G) with no parallels the minor poset $\mathrm{M}(L,G)$ is a lattice if and only if for each $g \in G$ there is a unique element of G that covers g.*

Proof. Observe if (L,G) has no parallels for any minor (K,H) the generating set H is isomorphic to a subposet of G, this is clear from the definition of no parallels. Conversely, simply by using deletions for any subposet of G there is a minor whose generating set is isomorphic said subposet. Furthermore, any generator-enriched lattice (K,H) with no parallels such that H is isomorphic as a poset to the generating set of the generator-enriched lattice depicted in Figure 4.6 is isomorphic to this same generator-enriched lattice. Thus, the minor poset $\mathrm{M}(L,G)$ is a poset if and only if for each $x \in G$ and $y, z \geq x$ either $y \leq z$ or $z \leq y$. This statement is equivalent to the condition: for each $g \in G$ there is a unique element of G that covers g. □

Recall given a poset P the *order polytope* $\mathcal{O}(P)$ is the polytope consisting of all order-preserving functions from P to $[0,1]$. The order polytope $\mathcal{O}(P)$ is defined by the linear inequalities

$$\begin{aligned} x_p \leq x_q \qquad & \text{for } p \leq q \text{ in } P \\ 0 \leq x_p \leq 1 \ & \text{for all } p \text{ in } P. \end{aligned}$$

For details of order polytopes see [43].

Proposition 4.2.27. *If L is a distributive lattice such that the generator-enriched lattice $(L, \mathrm{irr}(L))$ has no minors isomorphic to the generator-enriched lattice depicted in Figure 4.6 then the minor poset $\mathrm{M}(L, \mathrm{irr}(L))$ is isomorphic to the face poset of the order polytope $\mathcal{O}(\mathrm{irr}(L)^*)$.*

Proof. The minors of $(L, \mathrm{irr}(L))$ can each be uniquely expressed as $((L, \mathrm{irr}(L)) \setminus I)/J$ for $I, J \subseteq \mathrm{irr}(L)$ with $I \cap J = \emptyset$ and J a lower order ideal. Conversely, every such pair of subsets of $\mathrm{irr}(L)$ gives a minor. This is just a slight restatement of Proposition 3.4.14 that is more convenient for the present purposes. Recall the order polytope $\mathcal{O}(\mathrm{irr}(L)^*)$ is defined by the inequalities

$$\begin{aligned} &0 \le x_i \le 1 \text{ for all } i \in \mathrm{irr}(L),\\ &x_i \le x_j \qquad \text{for all } i \ge j \in \mathrm{irr}(L). \end{aligned}$$

Given a minor $((L, \mathrm{irr}(L)) \setminus I)/J$ we associate the face of the order polytope defined by the equations

$$\begin{aligned} &x_i = 0 \text{ for } i \in \max(\mathrm{irr}(L)) \cap I,\\ &x_i = x_{\mathrm{cov}(i)} \text{ for } i \in I \setminus \max(\mathrm{irr}(L)),\\ &x_j = 1 \text{ for } j \in J. \end{aligned}$$

The face may satisfy more equations induced by transitivity and the second type of equation, but surely this defines a face.

Convsersely, given a face F of the order polytope we associate the minor $((L, \mathrm{irr}(L)) \setminus I)/J$ where $I, J \subseteq \mathrm{irr}(L)$ are defined by

$$\begin{aligned} I &= \{i \in \mathrm{irr}(L) : x_i = 0 \text{ or } x_i = x_{\mathrm{cov}(i)} \neq 1 \text{ for all } x \in F\},\\ J &= \{j \in \mathrm{irr}(L) : x_j = 1 \text{ for all } x \in F\}. \end{aligned}$$

Plainly $I \cap J = \emptyset$ and J is a lower order ideal of $\mathrm{irr}(L)$ since $x \in F$ are order-reversing functions on $\mathrm{irr}(L)$.

The two correspondences described are inverses, so we have a bijection between the face poset of the order polytope and the minor poset. Corollary 4.2.8, in the present setting, says we have $((L, \mathrm{irr}(L)) \setminus I_1)/J_1 \le ((L, \mathrm{irr}(L)) \setminus I_2)/J_2$ if and only if

$$\begin{aligned} &J_1 \supseteq J_2 \text{ and}\\ &I_1 \supseteq I_2. \end{aligned}$$

This is equivalent to the set of defining equations for the face associated to

$$((L, \mathrm{irr}(L)) \setminus I_2)/J_2$$

containing the defining equations for the face associated to

$$((L, \mathrm{irr}(L)) \setminus I_1)/J_1.$$

□

4.2.4 A decomposition theorem

In this subsection we describe a decomposition of minor posets into a disjoint union of Boolean algebras. This decomposition is leveraged to derive formulas for the rank generating function of minor posets of minimally generated geometric lattices, and of generator-enriched lattices with no parallels.

To state the decomposition theorem some notation is needed. Let (L, G) be a generator-enriched lattice and let $\ell \in L$. Define $\mathrm{M}(L, G, \ell)$ to be the subposet of $\mathrm{M}(L, G)$ consisting of the minors of (L, G) for which the minimal element is ℓ.

Theorem 4.2.28. *Let (L, G) be a generator-enriched lattice. The minor poset of (L, G) decomposes as a disjoint union of the minimal element $\emptyset$ and the subposets $\mathrm{M}(L, G, \ell)$, that is,*

$$\mathrm{M}(L, G) = \{\emptyset\} \cup \dot{\bigcup}_{\ell \in L} \mathrm{M}(L, G, \ell).$$

Furthermore, the subposet $\mathrm{M}(L, G, \ell)$ is the interval $[(\ell, \emptyset), (L, G)/\ell]$ of $\mathrm{M}(L, G)$, and each such interval is isomorphic to a Boolean algebra.

Proof. Clearly the union is disjoint and consists of all elements of the minor poset $\mathrm{M}(L, G)$. Fix $\ell \in L$ and $(K, H) \in \mathrm{M}(L, G, \ell)$. Deleting all generators from (K, H) results in the minor $(\widehat{0}_K, \emptyset)$ consisting of only the minimal element ℓ of K. Thus $(K, H) \geq (\ell, \emptyset)$. By Lemma 3.4.6 the minor (K, H) can be expressed as $((L, G)/\ell) \setminus I$ for some set I. Thus $(K, H) \leq (L, G)/\ell$ and $\mathrm{M}(L, G, \ell) \subseteq [(\ell, \emptyset), (L, G)/\ell]$. The minimal element of any minor of $(L, G)/\ell$ is greater than or equal to ℓ. On the other hand if (K, H) is a minor of (L, G) such that $(\ell, \emptyset)$ is a minor of (K, H), then $\widehat{0}_K \leq \ell$. Thus $[(\ell, \emptyset), (L, G)/\ell] \subseteq \mathrm{M}(L, G, \ell)$ so equality holds.

Since the contraction operation changes the minimal element of a generator-enriched lattice, all relations $(K_1, H_1) \leq (K_2, H_2)$ in the interval $\mathrm{M}(L, G, \ell)$ must be induced by deletions. Thus, $(K_1, H_1) \leq (K_2, H_2)$ in $\mathrm{M}(L, G, \ell)$ if and only if $H_1 \subseteq H_2$. Therefore $\mathrm{M}(L, G, \ell)$ is isomorphic to the Boolean algebra $B_{\mathrm{rk}((L,G)/\ell)-1}$. □

See Figure 4.7 for an example.

For a ranked poset P we denote the rank generating function by $F(P; q)$. The rank generating function of the Boolean algebra B_n is $F(B_n; q) = \sum_{X \subseteq [n]} q^{|X|} = (1 + q)^n$. Since the minimal element of each interval $\mathrm{M}(L, G, \ell)$ is rank 1 in $\mathrm{M}(L, G)$, the decomposition theorem above implies the following.

Lemma 4.2.29. *For any generator-enriched lattice (L, G) the rank generating function of the poset of minors $\mathrm{M}(L, G)$ is given by*

$$F(M(L, G); q) = 1 + q \sum_{\ell \in L} (1 + q)^{\alpha(\ell)},$$

where $\alpha(\ell) = |\{g \vee \ell : g \in G\} \setminus \{\ell\}|$.

We now proceed to derive a more compact formula for minimally generated geometric lattices, and minimally generated distributive lattices. The formula for the first of these is in terms of the incidence algebra of the lattice. See [46, Section 3.6] for a more thorough introduction to incidence algebras.

Recall the incidence algebra of a finite poset P is the set of all maps from the set of nonempty intervals of P to the complex numbers $\mathbb{C}$. The incidence algebra is equipped with a product, known as the convolution:

$$(f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y).$$

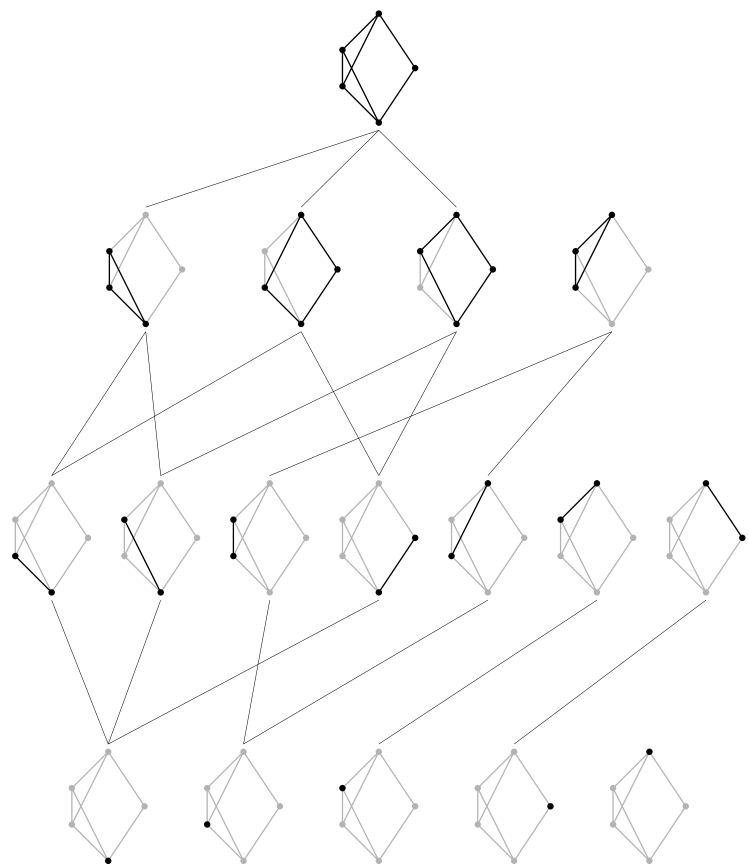


Figure 4.7: The Boolean decomposition of the minor poset depicted in Figure 4.1.

The identity is the map δ defined by $\delta(x,x) = 1$ for all $x \in P$ and $\delta(x,y) = 0$ for all $x \neq y$ in P. The zeta function $\zeta : \mathrm{Int}(P) \to \mathbb{C}$ is defined by $\zeta(x,y) = 1$. In order to state the rank generating function formula an exponentiation operation is needed. This exponentiation is the operation defined by

$$f^g(x,y) = \prod_{x \leq z \leq y} f(x,z)^{g(z,y)}.$$

We also need the map κ which encodes cover relations, defined by $\kappa(x,y) = 1$ when $x \prec y$ and $\kappa(x,y) = 0$ otherwise.

Theorem 4.2.30. *Let L be a geometric lattice. The rank generating function of the minor poset* $\mathrm{M}(L, \mathrm{irr}(L))$ *is*

$$F(\mathrm{M}(L, \mathrm{irr}(L)); q) = 1 + q(\zeta * (\zeta + q\kappa)^{\zeta})(\widehat{0}_L, \widehat{1}_L).$$

Proof. Since the lattice L is geometric, the generators of the contraction $(L,G)/\ell$ are precisely the elements that cover the minimal element ℓ. Thus

$$\sum_{\ell \in L} (1+q)^{\mathrm{rk}(L/\ell)-1} = \sum_{\ell \in L} (1+q)^{|\{\ell' \in L : \ell' \succ \ell\}|}.$$

Expanding the term inside the sum gives

$$\begin{aligned}\sum_{\ell \in L}(1+q)^{|\{\ell' \in L : \ell' \succ \ell\}|} &= \sum_{\ell \in L}\prod_{\ell' \geq \ell}\begin{cases}1+q & \text{if } \ell' \succ \ell,\\ 1 & \text{if } \ell' \not\succ \ell,\end{cases}\\ &= \sum_{\ell \in L}\prod_{\ell' \geq \ell}(\zeta + q\kappa)(\ell, \ell')\\ &= \sum_{\ell \in L}\prod_{\ell' \geq \ell}(\zeta + q\kappa)^{\zeta(\ell', \widehat{1}_L)}(\ell, \ell')\\ &= \sum_{\ell \in L}(\zeta + q\kappa)^{\zeta}(\ell, \widehat{1})\\ &= (\zeta * (\zeta + q\kappa)^{\zeta})(\widehat{0}, \widehat{1}).\end{aligned}$$ □

For generator-enriched lattices with no parallels, we give an expression for the rank generating function of $\mathrm{M}(L, G)$ in terms of the rank generating function of the dual lattice L^*. First we establish that a lattice with no parallels is indeed ranked. In fact, we give a description of the rank function which gives another characterization of lattices with no parallels. Avann also established this condition as equivalent to meet distributivity in [1, Theorem 5.5].

Proposition 4.2.31. *A generator-enriched lattice (L, G) has no parallels if and only if the lattice L is graded and the rank function is given by $\mathrm{rk}(\ell) = |\{g \in G : g \leq \ell\}|$.*

Proof. For $\ell \in L$ let $r(\ell) = |\{g \in G : g \leq \ell\}|$. First assume that (L, G) has a parallel, say for $g, h \in G$ and $\ell \in L$ we have $\ell \vee g = \ell \vee h \neq \ell$. If $x \prec \ell \vee g$ then $g, h \not\leq x$. Thus we must have $r(\ell \vee g) - r(x) \geq 2$ so r cannot be the rank function of L.

Now assume that (L, G) has no parallels. For any atom $a \in L$ clearly $r(a) = 1$, so it will suffice to show that whenever $x \prec y$ we have $r(y) - r(x) = 1$. Since $x \prec y$ for any $g \in G$ such that $g \leq y$ but $g \not\leq x$ we have $y = x \vee g$. If $y = x \vee g = x \vee h$ then since (L, G) has no parallels we have $g = h$. Thus there is exactly one such generator and $r(y) - r(x) = 1$. □

Theorem 4.2.32. *If L is a lattice with no parallels then the rank generating function of the poset of minors $\mathrm{M}(L, \mathrm{irr}(L))$ is given by*

$$F(\mathrm{M}(L, \mathrm{irr}(L)); q) = 1 + qF(L^*; 1+q).$$

Proof. Since $(L, \mathrm{irr}(L))$ has no parallels

$$\mathrm{rk}_{\mathrm{M}(L,\mathrm{irr}(L))}((L, \mathrm{irr}(L))/\ell) = |\{i \in \mathrm{irr}(L) : i \not\leq \ell\}|.$$

On the other hand, the rank of the element $\ell \in L$ is given by

$$\mathrm{rk}(\ell) = |\{i \in \mathrm{irr}(L) : i \leq \ell\}|.$$

The cardinality of the set $\{i \in \mathrm{irr}(L) : i \not\leq \ell\}$ is the corank of ℓ, that is, $\mathrm{rk}(\widehat{1}) - \mathrm{rk}(\ell)$. This difference is equal to the rank of ℓ in the dual lattice L^*. Substituting the rank of ℓ in L^* for the exponent in Lemma 4.2.29 results in the desired expression. □

4.3 The zipping construction

In this section a construction is given for strong minor posets using the zipping operation introduced by Reading in [41]. This construction implies that any minor poset is isomorphic to the face poset of a regular CW sphere, and in particular is Eulerian. The construction also yields inequalities for the **cd**-indices of minor posets.

4.3.1 Factoring strong surjections

In this subsection we provide a process to factor any strong surjection into strong surjections that only identify two elements. For the maps appearing in this factorization we give a description of the fibers of the induced map between the minor posets which is needed for the proof of Theorem 4.3.7.

Definition 4.3.1. *Let $\mathcal{E}(L,G)$ denote the poset consisting of edges of the diagram of a generator-enriched lattice (L,G), that is, pairs $(\ell,\ell\vee g)$ for $\ell\in L$ and $g\in G$ such that $\ell\vee g\neq\ell$. Partially order the elements of $\mathcal{E}(L,G)$ by $(\ell,\ell\vee g)\le(\ell\vee a,\ell\vee g\vee a)$ for $a\in L$.*

An oriented edge in the diagram of a generator-enriched lattice (L,G) is determined by its vertices. For notational convenience the elements of $\mathcal{E}(L,G)$ will typically be considered to be unordered pairs.

Definition 4.3.2. *Given an equivalence relation ϕ on a generator-enriched lattice (L,G), an* edge *of ϕ is an edge $(\ell,\ell\vee g)$ of (L,G) such that $\ell\equiv\ell\vee g(\phi)$. Let $\mathcal{E}(\phi)$ denote the set of edges of ϕ. The equivalence relation ϕ is said to be* connected *if for all $a\equiv b(\phi)$ there is a sequence $a=c_0,c_1,\dots,c_k=b$ such that for $1\le i\le k$ the pair $\{c_{i-1},c_i\}$ is an edge of ϕ.*

Note that a connected relation is determined by its set of edges.

Lemma 4.3.3. *Let ϕ be an equivalence relation on a generator-enriched lattice (L,G). The relation ϕ is join-preserving if and only if it is connected and the set of edges $\mathcal{E}(\phi)$ forms an upper order ideal in the poset $\mathcal{E}(L,G)$ of edges of (L,G).*

Proof. Let ϕ be a join-preserving equivalence relation on (L,G). To show that ϕ is connected, let $a,b\in(L,G)$ with $a\equiv b(\phi)$. Then $a\equiv a\vee b\equiv b(\phi)$. Choose some sequence $g_1,\dots,g_r$ of generators of (L,G) such that

$$a<a\vee g_1<\cdots<a\vee(g_1\vee\cdots\vee g_r)=a\vee b.$$

Each term of the sequence must be congruent to a since said terms lie in the interval $[a,a\vee b]$. Thus each pair of subsequent terms in the sequence is an edge of ϕ. There exists a similarly defined sequence from b to $a\vee b$. Concatenating these two sequences gives a sequence from a to b consisting of edges of ϕ. Therefore ϕ is connected.

In order to show that $\mathcal{E}(\phi)$ forms an upper order ideal of $\mathcal{E}(L,G)$, let $\{a,b\}$ be an edge of ϕ and let $\ell\in L$. Since $a\equiv b(\phi)$ and ϕ is join-preserving $a\vee\ell\equiv b\vee\ell(\phi)$.

Hence if $a \vee \ell \neq b \vee \ell$ then $\{a \vee \ell, b \vee \ell\}$ is an edge of ϕ. Thus $\mathcal{E}(\phi)$ is an upper order ideal of $\mathcal{E}(L, G)$.

Conversely, consider a connected equivalence relation ϕ on L such that the set of edges $\mathcal{E}(\phi)$ forms an upper order ideal in the poset $\mathcal{E}(L, G)$ of edges of (L, G). Let $a, b \in L$ such that $a \equiv b(\phi)$. Since ϕ is connected there is a sequence $a = c_0, c_1, \ldots, c_k = b$ such that for $1 \leq i \leq k$ the pair $\{c_{i-1}, c_i\}$ is an edge of ϕ. Taking the join with any $\ell \in L$ results in a sequence $a \vee \ell = c_0 \vee \ell, c_1 \vee \ell, \ldots, c_k \vee \ell = b \vee \ell$. For $1 \leq i \leq k$ either $c_{i-1} \vee \ell = c_i \vee \ell$ or $\{c_{i-1} \vee \ell, c_i \vee \ell\}$ is an edge of L. In the second case since $\mathcal{E}(\phi)$ is an upper order ideal, the edge $\{c_{i-1}, c_i\} \in \mathcal{E}(\phi)$ implies that $\{c_{i-1} \vee \ell, c_i \vee \ell\} \in \mathcal{E}(\phi)$. After removing repeated terms there is a sequence from $a \vee \ell$ to $b \vee \ell$ consisting of edges of ϕ. Thus $a \vee \ell \equiv b \vee \ell(\phi)$, hence ϕ is a join-preserving equivalence relation. □

Lemma 4.3.4. *Let (L, G) and (K, H) be generator-enriched lattices and consider a strong map $f : (L, G) \to (K, H)$. If the map f has a single nontrivial fiber $\{x, y\}$ in L, then the element x is only covered by y or vice versa.*

Proof. Since $f(x) = f(y)$ it must be that $f(x \vee y) = f(x) \vee f(y) = f(x)$. Since $f^{-1}(x) = \{x, y\}$, the element $x \vee y$ is either x or y. Without loss of generality we may assume that $x < y$. In fact $x \prec y$, since if $x \leq z \leq y$ then $f(x) \leq f(z) \leq f(y) = f(x)$ so $f(z) = f(x)$; hence z must be equal to x or to y. If $z > x$ then $f(z) = f(x \vee z) = f(y \vee z)$. The map f is invertible when restricted to $L \setminus \{x\}$ so this implies that $y \vee z = z$ hence $y \leq z$. Therefore x is only covered by y. □

Lemma 4.3.5. *Let (L, G) and (K, H) be generator-enriched lattices. Any strong surjection $f : (L, G) \to (K, H)$ can be factored as $f = f_r \circ \cdots \circ f_1$ where each map f_i is a strong surjection that identifies only two elements.*

Proof. Consider the map f as an equivalence relation on L defined by $a \equiv b(f)$ when $f(a) = f(b)$. By Lemma 4.3.3 this equivalence relation is connected and the edges form an upper order ideal in the poset $\mathcal{E}(L, G)$ of edges of (L, G). Choose a linear extension of $\mathcal{E}(L, G)$. Order the edges of f as $\{x_1, y_1\}, \ldots, \{x_r, y_r\}$. Define equivalence relations f_i for $1 \leq i \leq r$ by letting f_i be the transitive closure of the relation defined by $x_j \equiv y_j$ for $1 \leq j \leq i$. By definition f_i is connected and the edges of f_i form an upper order ideal in $\mathcal{E}(L, G)$ so f_i is join-preserving. Thus, each relation f_i defines a generator-enriched lattice, namely the quotient $(L, G)/f_i$. Furthermore for each i either $(L, G)/f_{i+1} = (L, G)/f_i$ or $(L, G)/f_{i+1}$ is obtained from $(L, G)/f_i$ by identifying two elements, namely $f_i(x_{i+1})$ and $f_i(y_{i+1})$. Thus the strong map f factors as the product $f_r \circ \cdots \circ f_1$ of the quotient maps. Removing any of the maps f_i which are the identity map gives the desired factorization. □

Lemma 4.3.6. *Let (L, G) and (K, H) be generator-enriched lattices and let $f : (L, G) \to (K, H)$ be a strong map. If the map f has a single nontrivial fiber $x \prec y$ then the induced map $F : \mathrm{M}(L, G) \to \mathrm{M}(K, H)$ has nontrivial fibers of the form*

$$\{(M,I),(M_x,I_x),(M_y,I_y)\},$$

where

$$\begin{aligned} x,y &\in I \cup \{\widehat{0}_M\}, \\ (M_x,I_x) &= (M,I)\setminus\{y\}, \\ (M_y,I_y) &= \begin{cases} (M,I)\setminus\{x\} & \text{if } x \in I, \\ (M,I)/\{y\} & \text{if } x = \widehat{0}_M. \end{cases} \end{aligned}$$

Proof. Clearly such a triple $\{(M,I),(M_x,I_x),(M_y,I_y)\}$ consists of a single fiber of F, and given any minor (M,I) with $x,y \in I \cup \{\widehat{0}_M\}$ there is such a triple. Furthermore, the fiber containing (M,I) is precisely the set $\{(M,I),(M_x,I_x),(M_y,I_y)\}$. Now suppose that (M_1,I_1) and (M_2,I_2) are minors of (L,G) in the same fiber of F such that neither $I_1 \cup \{\widehat{0}_{M_1}\}$ nor $I_2 \cup \{\widehat{0}_{M_2}\}$ contain both x and y. By assumption $f(\widehat{0}_{M_1}) = f(\widehat{0}_{M_2})$. Either $\widehat{0}_{M_1} = \widehat{0}_{M_2}$ or one is x and the other y. Consider the case $\widehat{0}_{M_1} = \widehat{0}_{M_2}$. The equality $F(M_1,I_1) = F(M_2,I_2)$ holds, thus $f(I_1) = f(I_2)$. Hence these sets differ by exchanging x and y. There is a minor (M,I) of (L,G) with $I = I_1 \cup I_2$ and $\widehat{0}_M = \widehat{0}_{M_1}$. This satisfies $F(M,I) = F(M_1,I_1) = F(M_2,I_2)$ and (M_1,I_1) and (M_2,I_2) are the minors $(M,I)\setminus\{x\}$ and $(M,I)\setminus\{y\}$.

Now consider the case where $\widehat{0}_{M_1} \neq \widehat{0}_{M_2}$. Since $f(\widehat{0}_{M_1}) = f(\widehat{0}_{M_2})$ one must equal x and the other must equal y. Say $\widehat{0}_{M_1} = x$. Since $x \prec y$ the element y is a generator of the contraction $(L,G)/x$. Consequently there is a minor (M,I) of (L,G) with $\widehat{0}_M = x$ and $I = I_1 \cup \{y\}$. Furthermore $(M_1,I_1) = (M,I)\setminus\{y\}$ and $(M_2,I_2) = (M,I)/\{y\}$. □

4.3.2 The zipping construction for minor posets and inequalities

Theorem 4.3.7. *Let (L,G) and (K,H) be generator-enriched lattices such that there is a strong surjection from (L,G) onto (K,H). The minor poset $\mathrm{M}(K,H)$ can be obtained from $\mathrm{M}(L,G)$ via a sequence of zipping operations.*

Proof. By Lemma 4.3.5 we may assume f has a single nontrivial fiber $\{x \prec y\}$. Let $F : \mathrm{M}(L,G) \to \mathrm{M}(K,H)$ be the map induced by f. By Lemma 4.3.6 the nontrivial fibers of F are of the form $\{(M,I),(M_x,I_x),(M_y,I_y)\}$ in which $(M_x,I_x) = (M,I)\setminus\{y\}$ and $(M_y,I_y) = (M,I)\setminus\{x\}$ when $x \neq \widehat{0}_M$ and $(M_y,I_y) = (M,I)/\{y\}$ when $x = \widehat{0}_M$. Let Z be the set of maximal elements of these nontrivial fibers of F. Choose some total ordering $<_{\text{zip}}$ of Z with the property that for (M_1,I_1) and (M_2,I_2) in Z with $\mathrm{rk}(M_1,I_1) < \mathrm{rk}(M_2,I_2)$ then $(M_1,I_1) <_{\text{zip}} (M_2,I_2)$. The poset $\mathrm{M}(K,H)$ will be obtained from $\mathrm{M}(L,G)$ by identifying each of the triples $\{(M,I),(M_x,I_x),(M_y,I_y)\}$ as described above. It will be shown these identifications may be made by zipping each element of Z with respect to the order $<_{\text{zip}}$.

Let $(M,I) \in Z$ and let (M_x,I_x) and (M_y,I_y) be the other elements of the same fiber as (M,I) as before. Let P be the poset obtained from $\mathrm{M}(L,G)$ by zipping elements of Z in increasing order with respect to $<_{\text{zip}}$ up to but not including the step of zipping the element (M,I). The minor poset $\mathrm{M}(L,G)$ is graded and thin by Lemma 4.2.3, so Proposition 1.4.4 implies that P is graded and thin as well. Let $\pi : \mathrm{M}(L,G) \to P$ be the projection map induced by the zipping operations. To show that the triple $\pi(M_x,I_x), \pi(M_y,I_y), \pi(M,I)$ forms a zipper in P it suffices to show that $\pi(M,I)$ only covers the elements $\pi(M_x,I_x)$ and $\pi(M_y,I_y)$, and show that the join $\pi(M_x,I_x) \vee \pi(M_y,I_y)$ in P is $\pi(M,I)$, by Remark 1.4.3.

We claim any minor (M',I') other than (M_x,I_x) or (M_y,I_y) covered by (M,I) satisfies $x, y \in I' \cup \{\widehat{0}_{M'}\}$. This claim is obvious when (M',I') is a deletion of (M,I). Since x is only covered by y, for any $z \in L$ either $z \vee x = z \vee y$ or $z \leq x$. Thus contracting (M,I) by an element $\ell \neq x$ either fixes both elements x and y or removes more than one generator. We conclude any minor $(M',I') \prec (M,I)$ satisfies $x, y \in I' \cup \{\widehat{0}_{M'}\}$. By construction such a minor (M',I') was the maximal element of a zipper in the construction of P and thus $\mathrm{rk}_P(\pi(M',I')) < \mathrm{rk}_{\mathrm{M}(L,G)}(M',I')$. Hence the element $\pi(M',I')$ is not covered by $\pi(M,I)$. Therefore $\pi(M,I)$ only covers the elements $\pi(M_x,I_x)$ and $\pi(M_y,I_y)$ in the poset P.

It remains to show $\pi(M,I) = \pi(M_x,I_x) \vee \pi(M_y,I_y)$ in P. To this end, we first observe $(M,I) = (M_x,I_x) \vee (M_y,I_y)$ in $\mathrm{M}(L,G)$. From the definitions of (M_x,I_x) and (M_y,I_y) either $\widehat{0}_{M_x} = \widehat{0}_{M_y}$ or $\widehat{0}_{M_x} = x$. In the case the minimal element $\widehat{0}_{M_x} = x$ this is only covered by $\widehat{0}_{M_y} = y$ in (M,I). With these relations in mind it is straightforward to check via Lemma 4.2.23 that $(M,I) = (M_x,I_x) \vee (M_y,I_y)$ in $\mathrm{M}(L,G)$.

Now consider an element $p \in P$ that is an upper bound for the elements $\pi(M_x,I_x)$ and $\pi(M_y,I_y)$. By construction the fibers $\pi^{-1} \circ \pi(M_x,I_x)$ and $\pi^{-1} \circ \pi(M_y,I_y)$ are trivial. The fact that $p > \pi(M_x,I_x)$ implies there is some minor (N_x,J_x) of (L,G) such that $(N_x,J_x) > (M_x,I_x)$ and $\pi(N_x,J_x) = p$. Similarly there is a minor (N_y,J_y) of (L,G) such that $(N_y,J_y) > (M_y,I_y)$ and $\pi(N_y,J_y) = p$. If $(N_y,J_y) = (N_x,J_x)$ then $(N_x,J_x) \geq (M,I)$, hence $p \geq \pi(M,I)$ as desired. It remains to show this equality must hold. Since $p > \pi(M_x,I_x)$ we have that $\mathrm{rk}_P(p) \geq \mathrm{rk}_P(\pi(M_x,I_x)) + 1 = \mathrm{rk}(M,I)$. By construction no minors of (L,G) of rank greater than $\mathrm{rk}(M,I)$ were the maximal element of a zipper in the construction of P, so $\pi^{-1}(p)$ consists of minors with rank at most $\mathrm{rk}(M,I)$. If the fiber $\pi^{-1}(p)$ was not trivial then since P was constructed via zipping operations we have $\mathrm{rk}_P(p) \leq \mathrm{rk}(M,I) - 1 = \mathrm{rk}_P(\pi(M_x,I_x))$. Since $p > \pi(M_x,I_x)$ this cannot be the case. Hence, the fiber $\pi^{-1}(p)$ is trivial and the equality $(N_x,J_x) = (N_y,J_y)$ holds, hence $p \geq \pi(M,I)$. Therefore $\pi(M,I)$ is the join of $\pi(M_x,I_x)$ and $\pi(M_y,I_y)$ in P. This establishes the triple $\pi(M_x,I_x), \pi(M_y,I_y), \pi(M,I)$ forms a zipper in P and completes the proof. □

Figure 4.8 depicts an example of the zipping construction.

Corollary 4.3.8. *For any generator-enriched lattice (L,G) the minor poset $\mathrm{M}(L,G)$ is Eulerian, and if $G \neq \emptyset$ the proper part of $\mathrm{M}(L,G)$ is a PL-sphere.*

Proof. Let $n = |G|$. Proposition 4.2.6 says the minor poset $\mathrm{M}(B_n, \mathrm{irr}(B_n))$ is isomorphic to the face lattice Q_n of the n-dimensional cube. In particular this poset is Eulerian and the proper part is a PL-sphere. There is a strong surjection from $(B_n, \mathrm{irr}(B_n))$ onto (L,G), namely the canonical strong map. Thus, Theorem 4.3.7 implies that the minor poset $\mathrm{M}(L,G)$ can be obtained from Q_n via zipping

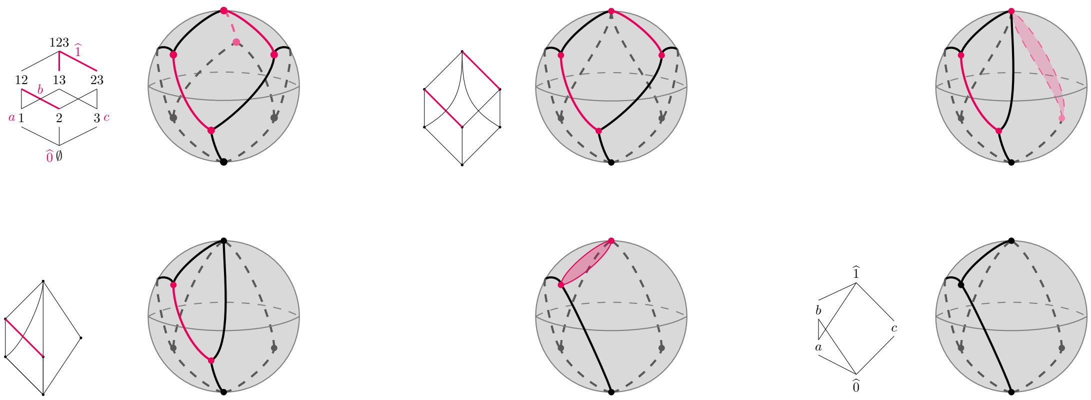


Figure 4.8: An example of the zipping construction.

operations. Theorems 1.4.6 and 1.5.3 imply that the minor poset $\mathrm{M}(L, G)$ is Eulerian and if $G \neq \emptyset$ then its proper part is a PL-sphere. □

Corollary 4.3.9. *For any generator-enriched lattice (L, G) with $G \neq \emptyset$ the minor poset $\mathrm{M}(L, G)$ is isomorphic to the face poset of a regular CW sphere.*

Proof. Since lower intervals of a minor poset are themselves minor posets, Corollary 4.3.8 along with Theorem 1.4.8 implies the result. □

Recall that the rank 1 elements of the minor poset $\mathrm{M}(L, G)$ are in bijection with the elements of (L, G), and the rank 2 elements are in bijection with the edges of the diagram of (L, G). It is interesting to note that the regular CW complex whose face poset is isomorphic to $\mathrm{M}(L, G)$ has a 1-skeleton isomorphic to the diagram of (L, G). In particular, when (L, G) is minimally generated and geometric, the 1-skeleton is isomorphic to the Hasse diagram of L. In any case, when $|G| = 3$ the associated CW complex can be constructed simply by embedding the diagram of (L, G) into the 2-sphere.

Corollary 4.3.10. *Let (L, G) be a generator-enriched lattice with n generators. The following inequalities hold coefficientwise among the* **cd***-indices:*

$$0 \leq \Psi(\mathrm{M}(L, G)) \leq \Psi(Q_n).$$

Proof. The left-hand inequality is implied by Corollary 4.3.9 and Theorem 1.5.2. Theorem 1.5.2 also applies to the face lattice of the n-dimensional cube, to the intermediate posets in the zipping construction of $\mathrm{M}(L, G)$ and to all intervals of these posets; as the proper parts of these posets are PL-spheres. Theorem 1.5.3 implies that the **cd**-index of the minor poset $\mathrm{M}(L, G)$ may be obtained from the **cd**-index of the face lattice of the n-dimensional cube by subtracting terms which all have nonnegative coefficients. □

Corollary 4.3.11. *Let (L, G) and (K, H) be generator-enriched lattices such that there is a join-preserving surjection from (L, G) onto (K, H). The following inequality* *of* **cd***-indices is satisfied coefficientwise:*

$$\Psi(\mathrm{M}(K,H)) \cdot \mathbf{c}^{|G|-|H|} \leq \Psi(\mathrm{M}(L,G)). \tag{4.1}$$

Proof. First consider the case where $|G| = |H|$. By Theorem 4.3.7 we have a sequence of zipping operations that takes the minor poset $\mathrm{M}(L,G)$ to $\mathrm{M}(K,H)$. The minor poset $\mathrm{M}(L,G)$ has proper part a PL-sphere, hence so does every intermediate poset resulting from the sequence of zipping operations. By Theorem 1.5.2 every intermediate poset, and every interval of every intermediate poset, has a **cd**-index with nonnegative coefficients. By assumption $\mathrm{rk}(\mathrm{M}(L,G)) = \mathrm{rk}(\mathrm{M}(K,H))$ so no zipping operation involves the maximal element. Thus, Theorem 1.5.3(a) implies each zipping operation corresponds, on the level of **cd**-indices, to subtracting off some **cd**-polynomial with nonnegative coefficients. Therefore, we have $\Psi(\mathrm{M}(K,H)) \leq \Psi(\mathrm{M}(L,G))$ coefficientwise when $|G| = |H|$.

Now consider the case where $|G| = |H| + 1$. Let $f : (L,G) \to (K,H)$ be a strong surjection. Factor the map f as in Lemma 4.3.5, and then group maps which do not decrease the number of generators. This results in a factorization $f = f_3 \circ f_2 \circ f_1$, for strong surjections

$$\begin{aligned} f_1 &: (L,G) \to (M,I), \\ f_2 &: (M,I) \to (N,J), \\ f_3 &: (N,J) \to (K,H), \end{aligned}$$

such that $|G| = |I| = |J|+1$. By the previous case we have $\Psi(\mathrm{M}(M,I)) \leq \Psi(\mathrm{M}(L,G))$ and $\Psi(\mathrm{M}(K,H)) \leq \Psi(\mathrm{M}(N,J))$ coefficientwise. Consider the zipping sequence induced by the map f_2. Since $|I| - |J| = 1$ we have $\mathrm{rk}(\mathrm{M}(M,I)) - \mathrm{rk}(\mathrm{M}(N,J)) = 1$, thus there is one zipping operation which involves the maximal element. By the construction given in Theorem 4.3.7 this is the final zipping operation. Let P be the result of the zipping operations applied to $\mathrm{M}(M,I)$ with the exception of the final zipping operation. The zipping operations used to construct P all correspond, on the level of **cd**-indices, to subtracting off a **cd**-polynomial with nonnegative coefficients. Thus, we have $\Psi(P) \leq \Psi(\mathrm{M}(M,I))$. The minor poset $\mathrm{M}(N,J)$ is constructed from P by zipping the maximal element $\hat{1}_P$. By Theorem 1.5.3(b) we have $\Psi(P) = \Psi(\mathrm{M}(N,J)) \cdot \mathbf{c}$. We have established the following inequalities:

$$\Psi(\mathrm{M}(K,H)) \cdot \mathbf{c} \leq \Psi(\mathrm{M}(N,J)) \cdot \mathbf{c} \leq \Psi(\mathrm{M}(M,I)) \leq \Psi(\mathrm{M}(L,G)).$$

Finally, consider the case where $|G| - |H| > 1$. Factor the map f as in Lemma 4.3.5 and group maps so that each group ends with a map decreasing the number of generators by one. This results in a factorization $f = f_r \circ \cdots \circ f_1$ consisting of strong surjections $f_i : (M_i, I_i) \to (M_{i+1}, I_{i+1})$ such that

$$\begin{aligned} |I_i| - |I_{i+1}| &= 1, \\ (M_1, I_1) &= (L,G), \\ (M_{r+1}, I_{r+1}) &= (K,H). \end{aligned}$$

The previous case shows $\Psi(\mathrm{M}(M_{i+1}, I_{i+1})) \cdot \mathbf{c} \leq \Psi(\mathrm{M}(M_i, I_i))$ for $i = 1, \ldots, r$. It follows that $\Psi(\mathrm{M}(K,H)) \cdot \mathbf{c}^r \leq \Psi(\mathrm{M}(L,G))$, here $r = |G| - |H|$. □

Remark 4.3.12. *It would be too much to expect the converse of the above corollary to hold, namely,* **cd***-index inequalities for minor posets would imply the existence of strong surjections between the associated generator-enriched lattices. Indeed, this converse is false. As a counterexample, let (K, H) be the generator-enriched lattice labeled (a) in Proposition 4.2.18 and (L, G) the generator-enriched lattice labeled (b) in Proposition 4.2.18. There is no strong surjection from (L, G) onto (K, H), but the inequality is satisfied since*

$$\Psi(\mathrm{M}(L, G)) = \mathbf{c}^3 + 2\mathbf{cd} + 3\mathbf{dc},$$
$$\Psi(\mathrm{M}(K, H)) = \mathbf{c}^3 + \mathbf{cd} + 3\mathbf{dc}.$$

To end this section we consider a special case of inequality (4.1). Namely, it is shown that any generator-enriched lattice with no parallels admits a strong surjection onto the chain with the same number of generators. Thus, the set of **cd**-indices of minor posets of generator-enriched lattices with no parallels and n generators is minimized by the **cd**-index of the Boolean algebra B_{n+1}. Restricting to those minor posets that are lattices is a special case of [24, Corollary 1.3] which says that the set of **cd**-indices of Gorenstein* lattices is minimized by the Boolean algebra. By Theorem 4.2.24 not every minor poset of a generator-enriched lattice with no parallels is itself a lattice. In contrast, every minor poset that is a lattice is the minor poset of a generator-enriched lattice with no parallels.

Lemma 4.3.13. *If (L, G) is a generator-enriched lattice with n generators and no parallels then there is a strong surjection onto the length n chain.*

Proof. Construct a linear extension $g_1, \ldots, g_n$ of the set G of generators as follows. For g_1 choose any element minimal in the subposet G of L. Consider the contraction $(L, G)/g_1$. By the no parallels assumption, and the fact that g_1 is minimal, the contraction $(L, G)/g_1$ must have $n - 1$ generators. Next choose any minimal generator g of $(L, G)/g_1$. The element g corresponds to a unique generator g_2 of (L, G). Now the process repeats considering the contraction $(L, G)/\{g_1, g_2\}$. This process ends with an ordering $g_1, \ldots, g_n$ of G. This ordering has the property that for $i < j$

$$g_1 \vee \cdots \vee g_i \not\geq g_j \vee (g_1 \vee \cdots \vee g_{i-1}).$$

In particular, the ordering is a linear extension of the poset G.

Label the elements of the length n chain C_n as $0 < 1 < \cdots < n$. Define a map $f : (L, G) \to (C_n, \mathrm{irr}(C_n))$ by

$$f(\ell) = \max(\{0\} \cup \{i : g_i \leq \ell\}).$$

To show that f is join-preserving, let $a, b \in L$ and suppose that $f(a) \vee f(b) = i$. Let $j > i$, by construction $g_j \not\leq g_1 \vee \cdots \vee g_i \geq a \vee b$. Thus $g_j \not\leq a \vee b$, and $f(a \vee b) \not> f(a) \vee f(b)$. Clearly $f(a \vee b) \geq f(a) \vee f(b)$ so $f(a \vee b) = f(a) \vee f(b)$. Since the ordering $g_1, \ldots, g_n$ is a linear extension of the subposet G of L, the image $f(g_i)$ is i, hence f is a surjection. □

Corollary 4.3.14. *The class of* **cd***-indices of minor posets of generator-enriched lattices with n generators and no parallels is coefficientwise minimized by the* **cd***-index of the rank $n+1$ Boolean algebra.*

Proof. By Lemma 4.3.13 any generator-enriched lattice (L, G) with no parallels and $|G| = n$ admits a strong surjection onto the length n chain $(C_n, \mathrm{irr}(C_n))$. Corollary 4.3.10 implies that

$$\Psi(B_{n+1}) = \Psi(\mathrm{M}(C_n, \mathrm{irr}(C_n))) \leq \Psi(\mathrm{M}(L, G)). \qquad \square$$

Chapter 5 The weak minor poset

5.1 Introduction

The deletion and contraction operations of generator-enriched lattices studied in Chapter 4 do not in general commute. Let us illustrate this with an example.

Let $L = \{\widehat{0} < g_1, g_2, g_3 < \widehat{1}\}$, that is, L is the rank 2 lattice with 3 atoms. Let $G = \{g_1, g_2, g_3\}$. The contraction $(L, G)/1$ has two elements g_1 and $\widehat{1}$. Since $\widehat{1} = g_1 \vee g_2 = g_1 \vee g_3$ it is indexed both by 2 and 3 and $((L, G)/1) \setminus 2$ has only a single element g_1. On the other hand, the minor $(L, G) \setminus 2$ has the four elements $\widehat{0}, g_1, g_3$ and $\widehat{1}$. Contracting by 1 the minor $((L, G) \setminus 2)/1$ has two elements g_1 and $\widehat{1} = g_1 \vee g_3$.

The essential reason the operations do not commute in the above example is because the deleted generator g_2 can be replaced by g_3 when g_1 is contracted since $g_1 \vee g_2 = g_1 \vee g_3$. In this chapter we introduce a form of modified contractions called weak contractions which are engineered to commute with deletions. To make weak contractions commute with deletions we define weak contractions so that this replacement phenomenon is ruled out and if a generator is deleted so is any generator it maps to under a weak contraction. Along with the weak contraction operation we study an associated poset, the weak minor poset. The weak minor poset of a generator-enriched lattice resembles the minor poset, but the relations are induced by deletions and weak contractions in place of contractions so the weak minor poset has a weaker order relation.

In Section 5.2 we introduce weak contractions and the weak minor poset. We show that the weak minor poset of any generator-enriched lattice is a lattice and describe the join and meet operations. In analogy with Theorem 4.3.7 we show strong maps between generator-enriched lattices induce meet-preserving maps between the weak minor posets.

In Section 5.3 we examine the effects on the weak minor poset of the operations of Cartesian product and adjoining a new maximum. Unlike minor posetes weak minor posets are not in general graded. In Section 5.4 we characterize graded weak minor posets as those induced by generator-enriched lattices with no parallels. In Section 5.5 we induce a dual lexicographic shelling for the weak minor poset of a generator-enriched lattice whose minors are all lexicographically shellable, along with the corresponding dual construction. In particular, this shelling shows graded weak minor posets are Cohen-Macaulay.

5.2 The weak minor poset

We begin with defining the weak contraction operation. First, we formalize the notion of the generators deleted to form a minor as the minor's deletion set defined below.

Definition 5.2.1. *Let (L, G) be a generator-enriched lattice and let (K, H) be a* *minor of* (L, G). *Define the* deletion set *of* (K, H) *to be*

$$\mathrm{Del}(K, H) = \{g \in G : g \vee \widehat{0}_K \notin H \cup \{\widehat{0}_K\}\}.$$

Note a minor of (L, G) is determined by its deletion set and its minimal element, if (K, H) is a minor then

$$\begin{aligned}(K, H) &= ((L, G)/\widehat{0}_K) \setminus \{g \vee \widehat{0}_K : g \in \mathrm{Del}(K, H)\} \\ &= ((L, G) \setminus \mathrm{Del}(K, H))/\widehat{0}_K.\end{aligned}$$

Conversely, choosing any element $\ell \in L$ and a set $D \subseteq G$ such that $D \cap [\widehat{0}_L, \ell] = \emptyset$ determines a minor with minimal element ℓ and deletion set D. Now we can give the definition of weak contractions.

Definition 5.2.2. *Let* (L, G) *be a generator-enriched lattice, let* $I \subseteq H$ *and set* $i_0 = \bigvee_{i \in I} i$. *If* $\mathrm{Del}(K, H) \cap [\widehat{0}, i_0] = \emptyset$ *then the* weak contraction of (K, H) by I with respect to (L, G) *is*

$$(K, H)/_{(L,G)} I = ((K, H)/I) \setminus \{g \vee i_0 : g \in \mathrm{Del}(K, H)\}.$$

Otherwise $(K, H)/_{(L,G)} I = \emptyset$ *holds.*

In the above definition the set $G \cap [\widehat{0}, i_0]$ is the set of generators that need to be contracted to form the minor $(K, H)/_{(L,G)} I$. The condition $\mathrm{Del}(K, H) \cap [\widehat{0}, i_0] = \emptyset$ can be interpreted as saying no generator was both deleted and contracted. If a generator was both deleted and contracted the operation does not really make sense and we declare the result to be $\emptyset$ for convenience; the element $\emptyset$ will be an artificial minimum in the weak minor poset.

Note a weak contraction applied to (L, G) itself is the same as a usual contraction. We will frequently express minors as the result of a contraction followed by a deletion.

If (K, H) can be obtained from (M, I) using deletions and weak contractions with respect to (L, G) then we say (K, H) is a *weak* (L, G)*-minor* of (M, I).

Let us return momentarily to the example from the beginning of the section. Recall $L = \{\widehat{0} < g_1, g_2, g_3 < \widehat{1}\}$ and $G = \{g_1, g_2, g_3\}$. If we consider the same operations as before except with weak contractions in place of contractions we find the operations do commute as claimed. The minor $((L, G)/_{(L,G)} 1) \setminus 2 = \langle \emptyset | g_1 \rangle$ as before. But now since $\mathrm{Del}((L, G) \setminus 2) = \{g_2\}$ when applying the weak contraction by 1 we get

$$\begin{aligned}((L, G) \setminus 2)/_{(L,G)} 1 &= ((L, G) \setminus 2)/1) \setminus \{g_2 \vee g_1 = \widehat{1}\} \\ &= \langle \emptyset | g_1 \rangle.\end{aligned}$$

We now verify that the operations of deletion and weak contraction commute in general.

Proposition 5.2.3. *Let (L,G) be a generator-enriched lattice and fix a labeling $G = \{g_1,\ldots,g_n\}$ of the generating set. Deletions and weak contractions commute, in that, for any $X, Y \subseteq [n]$ such that $g_y \not\leq \bigvee_{x\in X} g_x$ for all $y \in Y$ and any minor (K,H) of (L,G)*

$$((K,H)/_{(L,G)}X) \setminus Y = ((K,H) \setminus Y)/_{(L,G)}X.$$

As an aside, the hypothesis $g_y \not\leq \bigvee_{x\in X} g_x$ for $y \in Y$ could be removed if one defined the result of deleting the minimal element from a generator-enriched lattice to be the empty set. We opt instead to not alter the definition of deletions for simplicity.

Proof. Set $\ell = \bigvee_{x\in X} X$. By definition

$$\begin{aligned}((K,H)/_{(L,G)}X) \setminus Y = \langle\{&\ell \vee h : h \in H\} \\ &\setminus (\{\ell \vee g : g \in \mathrm{Del}(K,H)\} \cup \{\ell \vee g_y : y \in Y\} \cup \{\ell\})|\ell\rangle.\end{aligned}$$

On the other hand the minor $((K,H) \setminus Y)/_{(L,G)}X$ is equal to

$$\begin{aligned}&\langle\{\ell \vee h : h \in H \setminus \{\widehat{0}_K \vee g_y : y \in Y\}\} \setminus (\{\ell \vee g : g \in \mathrm{Del}((K,H) \setminus Y)\} \cup \{\ell\})|\ell\rangle \\ &= \langle\{\ell \vee h : h \in H\} \setminus (\{\ell \vee g : g \in \mathrm{Del}((K,H) \setminus Y)\} \cup \{\ell\})|\ell\rangle.\end{aligned}$$

The two minors are equal since $\mathrm{Del}((K,H) \setminus Y) = \mathrm{Del}(K,H) \cup \{g_y : y \in Y\}$. □

Definition 5.2.4. *Given a generator-enriched lattice (L,G) the* weak minor poset $\mathrm{WM}(L,G)$ *consists of a unique minimal element $\emptyset$ and all minors of (L,G) with the order relation $(K_1,H_1) \leq (K_2,H_2)$ if and only if (K_1,H_1) is a weak (L,G)-minor of (K_2,H_2).*

Figure 5.1 depicts an example of a weak minor poset. The weak minor posets of generator-enriched lattices with 3 generators are depicted in Appendix A. The following lemma gives a useful alternative definition of the order relation of weak minor posets.

Lemma 5.2.5. *Given a generator-enriched lattice (L,G) and minors (K,H) and (M,I) we have $(K,H) \leq (M,I)$ if and only if the following two conditions hold:*

$$\widehat{0}_M \leq \widehat{0}_K,$$
$$\mathrm{Del}(M,I) \subseteq \mathrm{Del}(K,H).$$

Proof. First suppose $(K,H) \leq (M,I)$. Then (K,H) is a weak minor of (M,I) so we can write

$$(K,H) = ((M,I)/_{(L,G)}\widehat{0}_K) \setminus \{g \vee \widehat{0}_K : g \in \mathrm{Del}(K,H)\}.$$

Clearly this implies $\widehat{0}_M \leq \widehat{0}_K$ and $\mathrm{Del}(M,I) \subseteq \mathrm{Del}(K,H)$ as deletions and weak contractions can only increase each of these attributes.

Conversely suppose $\widehat{0}_M \leq \widehat{0}_K$ and $\mathrm{Del}(M,I) \subseteq \mathrm{Del}(K,H)$. We wish to show

$$(K,H) = ((M,I)/_{(L,G)}\widehat{0}_K) \setminus \{g \vee \widehat{0}_K : g \in \mathrm{Del}(K,H)\}.$$

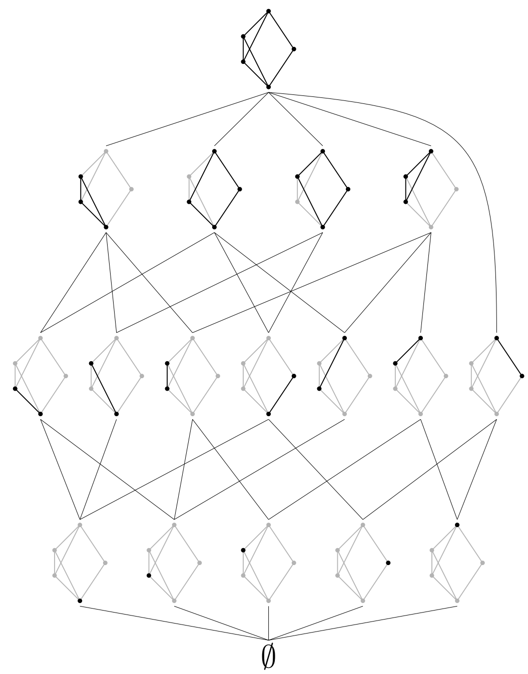


Figure 5.1: An example of a weak minor poset.

First, observe since $\mathrm{Del}(M, I) \subseteq \mathrm{Del}(K, H) \subseteq G \setminus [\widehat{0}_L, \widehat{0}_K]$ and $\widehat{0}_M \leq \widehat{0}_K$ we have $\widehat{0}_K \in M$. Thus,

$$(M, I)/_{(L,G)}\widehat{0}_K = ((L, G)/\widehat{0}_K) \setminus \{g \vee \widehat{0}_K : g \in \mathrm{Del}(M, I)\}.$$

Since $\mathrm{Del}(M, I) \subseteq \mathrm{Del}(K, H)$ the last minor above is an upper bound for (K, H). □

Now we show that weak minor posets are lattices and describe the join and meet operations.

Theorem 5.2.6. *For any generator-enriched lattice (L, G) the weak minor poset $\mathrm{WM}(L, G)$ is a lattice. Furthermore, given minors (K, H) and (M, I) of (L, G):*

1. *The meet $(K, H) \wedge (M, I)$ is the minor $(N^\wedge, J^\wedge)$ defined by*

$$\widehat{0}_{N^\wedge} = \widehat{0}_K \vee \widehat{0}_M,$$
$$\mathrm{Del}(N^\wedge, J^\wedge) = \mathrm{Del}(K, H) \cup \mathrm{Del}(M, I)$$

if $(\mathrm{Del}(K, H) \cup \mathrm{Del}(M, I)) \cap [\widehat{0}_L, \widehat{0}_K \vee \widehat{0}_M] = \emptyset$ and otherwise $(K, H) \wedge (M, I) = \emptyset$.

2. *The join $(K, H) \vee (M, I)$ is the minor $(N^\vee, J^\vee)$ defined by*

$$\widehat{0}_{N^\vee} = \widehat{0}_K \wedge \widehat{0}_M,$$
$$\mathrm{Del}(N^\vee, J^\vee) = \mathrm{Del}(K, H) \cap \mathrm{Del}(M, I).$$

Proof. We begin by proving the given formula for meets. First suppose there is some $d \in \mathrm{Del}(K,H) \cup \mathrm{Del}(M,I)$ such that $d \leq \widehat{0}_K \vee \widehat{0}_M$. Without loss of generality suppose $d \in \mathrm{Del}(K,H)$. If there were a minor below both (K,H) and (M,I) its minimal element must be an upper bound for $\widehat{0}_K \vee \widehat{0}_M$. Thus, any common lower bound of (K,H) and (M,I) is also a lower bound for $(K,H)/_{(L,G)}\widehat{0}_M$. By assumption $(K,H)/_{(L,G)}\widehat{0}_M = \emptyset$ so the meet $(K,H) \wedge (M,I)$ is $\emptyset$ as well.

Now suppose $d \not\leq \widehat{0}_K \vee \widehat{0}_M$ for $d \in \mathrm{Del}(K,H) \cup \mathrm{Del}(M,I)$. Let $(N^\wedge, J^\wedge)$ be the minor of (L,G) defined by

$$\widehat{0}_{N^\wedge} = \widehat{0}_K \vee \widehat{0}_M,$$
$$\mathrm{Del}(N^\wedge, J^\wedge) = \mathrm{Del}(K,H) \cup \mathrm{Del}(M,I).$$

Suppose (N,J) is a minor of (L,G) such that $(N,J) \leq (K,H)$ and $(N,J) \leq (M,I)$. By Lemma 5.2.5 this implies $\widehat{0}_N \geq \widehat{0}_K \vee \widehat{0}_M$ and $\mathrm{Del}(N,J) \supseteq \mathrm{Del}(K,H) \cup \mathrm{Del}(M,I)$. This implies $(N,J) \leq (N^\wedge, J^\wedge)$ so $(N^\wedge, J^\wedge) = (K,H) \wedge (M,I)$.

Now we prove the given formula for joins. Observe $(N^\vee, J^\vee)$ is a well defined minor since

$$\mathrm{Del}(N^\vee, J^\vee) \cap [\widehat{0}_L, \widehat{0}_{N^\vee}] \subseteq \mathrm{Del}(N^\vee, J^\vee) \cap [\widehat{0}_L, \widehat{0}_K] = \emptyset.$$

We clearly have $(N^\vee, J^\vee) \geq (K,H)$ and $(N^\vee, J^\vee) \geq (M,I)$ by Lemma 5.2.5. The same lemma also easily shows any upper bound for both (K,H) and (M,I) is an upper bound for $(N^\vee, J^\vee)$. □

Theorem 5.2.6 in particular, shows for any generator-enriched lattice (L,G) the lattice L is isomorphic to the sublattice of $\mathrm{WM}(L,G)$ consisting of contractions.

Proposition 5.2.7. *Let (L,G) be a generator-enriched lattice.*

1. *The meet irreducibles of the weak minor poset $\mathrm{WM}(L,G)$ are the deletions $(L,G) \setminus \{g\}$ for $g \in G$ and the contractions $(L,G)/\{i\}$ for $i \in \mathrm{irr}(L)$.*
2. *The join irreducibles of the weak minor poset $\mathrm{WM}(L,G)$ are the minors of (L,G) that contain a single element and the minors of the form $\langle y|x\rangle$ such that $x \not\lessdot y$ in L.*

In particular, $\mathrm{WM}(L,G)$ is coatomic if and only if G is the set of atoms of L and $\mathrm{WM}(L,G)$ is atomic if and only if L is geometric and $G = \mathrm{irr}(L)$.

Proof. It is clear from the description of meets in Theorem 5.2.6 that the meet irreducibles consist of the minors (K,H) such that either $\widehat{0}_K$ is join irreducible and $\mathrm{Del}(K,H) = \emptyset$ or $\widehat{0}_K = \widehat{0}_L$ and $\mathrm{Del}(K,H)$ is a singleton. These are precisely the minors claimed to be the meet irreducibles of $\mathrm{WM}(L,G)$.

An element (K,H) is join irreducible in $\mathrm{WM}(L,G)$ if and only if (K,H) covers one element. This is of course the case if (K,H) is an atom, suppose (K,H) is not an atom. If $|H| \geq 2$ then (K,H) covers at least two elements, since for $h \in H$ we have $(K,H) \succ (K,H) \setminus \{h\}$. Suppose $H = \{h\}$. In this case (K,H) is join irreducible if and only if $(K,H)/\{h\} = \emptyset$. We have $(K,H)/\{h\} = \emptyset$ if and only if $\mathrm{Del}(K,H) \cap \{g \in G : g \leq h\} \neq \emptyset$. Since $H = \{h\}$ we have

$$\mathrm{Del}(K,H) = \{g \in G : g \not\leq \widehat{0}_K \text{ and } g \vee \widehat{0}_K \neq h\}.$$

Thus, (K,H) is join irreducible if and only if for all $g \in G$ the conditions $g \not\leq \widehat{0}_K$ and $g \leq h$ imply $g \vee \widehat{0}_K = h$. This is equivalent to the condition $\widehat{0}_K \prec h$.

It is clear $\mathrm{WM}(L,G)$ is coatomic if and only if G is the set of atoms of L since each deletion $(L,G) \setminus \{g\}$ is a coatom and the sublattice of contractions is anti-isomorphic to L. The fact that $\mathrm{WM}(L,G)$ is atomic if and only if L is geometric and $G = \mathrm{irr}(L)$ follows from Proposition 4.2.10. □

In Theorem 4.3.7 it was shown that strong surjections between generator-enriched lattices induce certain order-preserving maps called zipping operations between the minor posets. We prove an analagous result for weak minor posets, that strong maps between generator-enriched lattices induce meet-preserving maps between the weak minor posets.

A strong map $f : (L,G) \to (K,H)$ induces a map $F : \mathrm{WM}(L,G) \to \mathrm{WM}(K,H)$ between the weak minor posets defined by

$$F(((L,G)/\ell) \setminus I) = ((K,H)/f(\ell)) \setminus f(I)$$

if $f(i) \not\leq f(\ell)$ for all $i \in I$ and otherwise is the minimal element $\emptyset$ of $\mathrm{WM}(K,H)$. Equivalently, the image $F(M,I)$ of a minor (M,I) of (L,G) is defined by

$$\widehat{0}_{F(M,I)} = f(\widehat{0}_M),$$
$$\mathrm{Del}(F(M,I)) = f(\mathrm{Del}(M,I))$$

if $f(\mathrm{Del}(M,I)) \cap [\widehat{0}_K, f(\widehat{0}_M)] = \emptyset$ and otherwise $F(M,I) = \emptyset$.

Theorem 5.2.8. *Given generator-enriched lattices (L,G) and (K,H) and a strong map $f : (L,G) \to (K,H)$ the induced map $F : \mathrm{WM}(L,G) \to \mathrm{WM}(K,H)$ is meet-preserving. Furthermore, if f is injective (surjective) then F is injective (surjective).*

Proof. Let (M,I) and (N,J) be minors of (L,G) and assume $(M,I) \wedge (N,J) \neq \emptyset$. Recall, $F(M,I) \wedge F(N,J) = \emptyset$ if and only if

$$(f(\mathrm{Del}(M,I)) \cup f(\mathrm{Del}(N,J))) \cap [\widehat{0}_K, f(\widehat{0}_M) \vee f(\widehat{0}_N)] \neq \emptyset$$

holds.

If this condition holds then by the definition of F and applying Theorem 5.2.6 we also have $F((M,I) \wedge (N,J)) = \emptyset$. On the other hand, if $F(M,I) \wedge F(N,J) \neq \emptyset$ then

$$(f(\mathrm{Del}(M,I)) \cup f(\mathrm{Del}(N,J))) \cap [\widehat{0}_K, f(\widehat{0}_M) \vee f(\widehat{0}_N)] = \emptyset.$$

This implies $f(\mathrm{Del}((M,I) \wedge (N,J))) \cap [\widehat{0}_K, \widehat{0}_{(M,I)\wedge(N,J)}] \neq \emptyset$. Thus, $F((M,I) \wedge (N,J))$ has deletion set $f(\mathrm{Del}(M,I)) \cup f(\mathrm{Del}(N,J))$ and minimal element $f(\widehat{0}_M) \vee f(\widehat{0}_N)$ and is equal to $F(M,I) \wedge F(N,J)$.

Now assume the strong map f is injective. We wish to show the induced map F is injective as well. Since the map f is join-preserving the inverse map $f^{-1} : f(L) \to L$ is join-preserving as well. Now let (M, I) be a minor of (L, G). We have $f(\mathrm{Del}(M, I)) \cap [\widehat{0}_K, \widehat{0}_M] = \emptyset$ if and only if $f^{-1}f(\mathrm{Del}(M, I)) \cap f^{-1}([\widehat{0}_K, \widehat{0}_M]) = \emptyset$ which is equivalent to the condition $\mathrm{Del}(M, I) \cap [\widehat{0}_L, \widehat{0}_M] = \emptyset$ which always holds. Thus, only $\emptyset$ is mapped to $\emptyset$ by F. Now, given a minor (N, J) of (L, G) such that $F(M, I) = F(N, J)$ we have

$$f(\mathrm{Del}(M, I)) = f(\mathrm{Del}(N, J))$$

and $f(\widehat{0}_M) = f(\widehat{0}_N)$. Since f is injective this implies $(M, I) = (N, J)$ so F is injective as well.

Now suppose f is surjective. To show F is surjective let (M, I) be a minor of (K, H). Choose $\ell \in L$ such that $f(\ell) = \widehat{0}_M$ and choose a set $D \subseteq G$ such that $f(D) = \mathrm{Del}(M, I)$. For all $h \in \mathrm{Del}(M, I)$ we have $h \not\leq \widehat{0}_M$. Since f is order-preserving this implies $g \not\leq \ell$ for any $g \in G$ such that $f(g) = h$. Thus, there is a minor of (L, G) with minimal element ℓ and deletion set D and its image is (M, I). □

Recall a lattice L is said to be *complemented* if for all $\ell \in L$ there exists an element $k \in L$ such that $\ell \vee k = \widehat{1}$ and $\ell \wedge k = \widehat{0}$; the element k is said to be a *complement* of ℓ. As it turns out weak minor posets are complemented lattices. See [10, Chapter 1, Section 9] for details regarding complements in lattices.

Proposition 5.2.9. *Given a generator-enriched lattice (L, G) the weak minor poset* $\mathrm{WM}(L, G)$ *is a complemented lattice. Given a minor $(K, H) \neq (L, G)$ a minor (M, I) is a complement of (K, H) if and only if the following three conditions hold:*

$$(\mathrm{Del}(K, H) \cup \mathrm{Del}(M, I)) \cap ([\widehat{0}_L, \widehat{0}_K] \cup [\widehat{0}_L, \widehat{0}_M]) \neq \emptyset, \tag{5.1}$$

$$\mathrm{Del}(K, H) \cap \mathrm{Del}(M, I) \subseteq [\widehat{0}_L, \widehat{0}_K] \cup [\widehat{0}_L, \widehat{0}_M], \tag{5.2}$$

$$\widehat{0}_K \wedge \widehat{0}_M = \widehat{0}_L. \tag{5.3}$$

Proof. First assume (K, H) and (M, I) are complements. Since

$$(K, H) \vee (M, I) = (L, G),$$

by Theorem 5.2.6 we must have $\widehat{0}_K \wedge \widehat{0}_M = \widehat{0}_L$ and

$$\mathrm{Del}(K, H) \cap \mathrm{Del}(M, I) \subseteq [\widehat{0}_L, \widehat{0}_K] \cup [\widehat{0}_L, \widehat{0}_M].$$

Since $(K, H) \wedge (M, I) = \emptyset$ by Theorem 5.2.6 either

$$\mathrm{Del}(K, H) \cap [\widehat{0}_L, \widehat{0}_M] \neq \emptyset$$

or

$$\mathrm{Del}(M, I) \cap [\widehat{0}_L, \widehat{0}_K] \neq \emptyset,$$

hence $(\mathrm{Del}(K, H) \cup \mathrm{Del}(M, I)) \cap ([\widehat{0}_L, \widehat{0}_M] \cup [\widehat{0}_L, \widehat{0}_K]) \neq \emptyset$.

Conversely, assume the following Conditions 5.1 through 5.3 hold. Then

$$(\mathrm{Del}(K,H) \cap \mathrm{Del}(M,I)) \setminus ([\widehat{0}_L, \widehat{0}_K] \cup [\widehat{0}_L, \widehat{0}_M]) = \emptyset$$

and since $\widehat{0}_K \wedge \widehat{0}_M = \widehat{0}_L$ Theorem 5.2.6 implies $(K,H) \vee (M,I) = (L,G)$. Since

$$\mathrm{Del}(K,H) \cap [\widehat{0}_L, \widehat{0}_K] = \emptyset$$

and

$$\mathrm{Del}(M,I) \cap [\widehat{0}_L, \widehat{0}_M] = \emptyset$$

our assumptions imply either

$$\mathrm{Del}(K,H) \cap [\widehat{0}_L, \widehat{0}_M] \neq \emptyset$$

or

$$\mathrm{Del}(M,I) \cap [\widehat{0}_L, \widehat{0}_K] \neq \emptyset.$$

In either case $(K,H) \wedge (M,I) = \emptyset$ holds. □

5.3 Operations

In this section we show the weak minor poset of a Cartesian product of generator-enriched lattices is the diamond product of the weak minor posets; and that the weak minor poset of a generator-enriched lattice with a new maximum attached is a certain weak subposet of the pyramid over the weak minor poset. This first result is the same as for minor posets (Proposition 4.2.12).

Definition 5.3.1. *Given two generator-enriched lattices* (L,G) *and* (K,H) *the* Cartesian product *is the generator-enriched lattice*

$$(L,G) \times (K,H) = (L \times K, (G \times \{\widehat{0}_K\}) \cup (\{\widehat{0}_L\} \times K)).$$

Recall given two posets P and Q each with a unique minimal element the *diamond product* is defined as

$$P \diamond Q = ((P \setminus \{\widehat{0}_P\}) \times (Q \setminus \{\widehat{0}_Q\})) \cup \{\widehat{0}\}.$$

The diamond product corresponds to direct products of polytopes and of regular CW complexes.

Given a poset P the *pyramid over* P is the poset $\mathrm{Pyr}(P) = P \times B_1$ and the *prism over* P is the poset $\mathrm{Prism}(P) = P \diamond B_2$. We define the pyramid operator on generator-enriched lattices in the same way $\mathrm{Pyr}(L,G) = (L,G) \times (B_1, \mathrm{irr}(B_1))$.

Proposition 5.3.2. *For any two generator-enriched lattices* (L,G) *and* (K,H)

$$\mathrm{WM}((L,G) \times (K,H)) \cong \mathrm{WM}(L,G) \diamond \mathrm{WM}(K,H).$$

In particular, the isomorphism $\mathrm{WM}(\mathrm{Pyr}(L,G)) \cong \mathrm{Prism}(\mathrm{WM}(L,G))$ *holds.*

Proof. Let $\pi_L : L \times K \to L$ and $\pi_K : L \times K \to K$ be the projection maps. Define a map $\phi : \mathrm{WM}((L, G) \times (K, H)) \to \mathrm{WM}(L, G) \diamond \mathrm{WM}(K, H)$ on minors (M, I) by

$$\phi(M, I) = (\pi_L(M), \pi_L(I), \pi_K(M), \pi_K(I)).$$

We also set $\phi(\emptyset) = \emptyset$. The map ϕ has inverse given by

$$\phi^{-1}(L', G', K', H') = \langle (G' \times \{\widehat{0}_K\}) \cup (\{\widehat{0}_L\} \times H') | (\widehat{0}_{L'}, \widehat{0}_{K'}) \rangle.$$

Clearly the map ϕ is join-preserving from the formula in Theorem 5.2.6. □

The second operation we examine is that of adjoining a new maximum to a generator-enriched lattice. Given a generator-enriched lattice (L, G) let $\widehat{L} = L \cup \{\widehat{1}_{\widehat{L}}\}$ and $\widehat{G} = G \cup \{\widehat{1}_{\widehat{L}}\}$ for some new element $\widehat{1}_{\widehat{L}}$ which is greater than all elements of L. Note $\widehat{1}_{\widehat{L}}$ is join irreducible in $\widehat{L}$ so it must be an element of $\widehat{G}$ for $(\widehat{L}, \widehat{G})$ to be a generator-enriched lattice.

Proposition 5.3.3. *Let (L, G) be a generator-enriched lattice. The weak minor poset $\mathrm{WM}(\widehat{L}, \widehat{G})$ is isomorphic to the weak subposet of $\mathrm{Pyr}(\mathrm{WM}(L, G))$ defined by $(\mathcal{A}, \epsilon_1) \le (\mathcal{B}, \epsilon_2)$ if and only if the same relation holds in $\mathrm{Pyr}(\mathrm{WM}(L, G))$ and at least one of the following two conditions holds:*

1. $(\mathcal{A}, \epsilon_1) \ne (\emptyset, \widehat{1})$,
2. $\mathrm{Del}(\mathcal{B}) = \emptyset$.

Proof. Define a map $\phi : \mathrm{WM}(\widehat{L}, \widehat{G}) \to \mathrm{Pyr}(\mathrm{WM}(L, G))$ defined for minors (K, H) of $\mathrm{WM}(\widehat{L}, \widehat{G})$ by

$$\phi(K, H) = \begin{cases} (K, H, \widehat{0}) & \text{if } \widehat{1}_{\widehat{L}} \notin K, \\ (K \setminus \{\widehat{1}_{\widehat{L}}\}, H \setminus \{\widehat{1}_{\widehat{L}}\}, \widehat{1}) & \text{if } \widehat{1}_{\widehat{L}} \in H, \\ (\emptyset, \widehat{1}) & \text{if } \widehat{1}_{\widehat{L}} = \widehat{0}_K \end{cases}$$

and by $\phi(\emptyset) = (\emptyset, \widehat{0})$. The inverse map is described by

$$\phi^{-1}(K, H, \epsilon) = \begin{cases} (K, H) & \text{if } \epsilon = \widehat{0} \\ (K \cup \{\widehat{1}_{\widehat{L}}\}, H \cup \{\widehat{1}_{\widehat{L}}\})) & \text{if } \epsilon = \widehat{1} \end{cases}$$

by $\phi^{-1}(\emptyset, \widehat{1}) = \langle \emptyset | \widehat{1}_{\widehat{L}} \rangle$ and $\phi^{-1}(\emptyset, \widehat{0}) = \emptyset$.

Observe, if $\phi(K, H) = (K', H', \epsilon_1)$ then $\mathrm{Del}(K', H') = \mathrm{Del}(K, H) \setminus \{\widehat{1}_{\widehat{L}}\}$ whenever $\widehat{1}_{\widehat{L}} \ne \widehat{0}_K$. Setting $\phi(M, I) = (M', I', \epsilon_2)$ a similar statement holds. Thus, applying Lemma 5.2.5 we see if $\widehat{1}_{\widehat{L}} \ne \widehat{0}_K$ and $\widehat{1}_{\widehat{L}} \ne \widehat{0}_M$ then $(K, H) \le (M, I)$ holds if and only if $\phi(K, H) \le \phi(M, I)$.

Now consider the case $(K, H) = \langle \emptyset | \widehat{1}_{\widehat{L}} \rangle$, that is, the case $\phi(K, H) = (\emptyset, \widehat{1})$. Since $\mathrm{Del}(K, H) = \emptyset$ we have $(K, H) \le (M, I)$ if and only if $\mathrm{Del}(M, I) = \emptyset$; which implies the condition condition $\mathrm{Del}(M', I') = \emptyset$. On the other hand if $\mathrm{Del}(M', I') = \emptyset$ and $\phi(M, I) \ge (\emptyset, \widehat{1})$ we must have $\phi(M, I) \in \mathrm{WM}(L, G) \times \{\widehat{1}\}$. Thus,

$$\mathrm{Del}(M, I) = \mathrm{Del}(M', I') = \emptyset$$

from which we conclude $(M, I) \ge (K, H)$. □

5.4 Graded weak minor posets

In this section we examine the special case of graded weak minor posets. We begin with a characterization.

Proposition 5.4.1. *The weak minor poset* $\mathrm{WM}(L,G)$ *is graded if and only if the generator-enriched lattice* (L,G) *has no parallels. If* (L,G) *has no parallels then the rank function of* $\mathrm{WM}(L,G)$ *is given by* $\mathrm{rk}(K,H) = |H|+1$.

Proof. Let (K,H) be a minor of (L,G). In any case there is a chain of length $|H|+1$ from $\emptyset$ to (K,H), namely any chain of the form

$$\emptyset \prec \langle \emptyset | \widehat{0}_K \rangle \prec \langle h_1 | \widehat{0}_K \rangle \prec \cdots \prec \langle h_1, \dots, h_{|H|} | \widehat{0}_K \rangle = (K,H)$$

where $H = \{h_1, \dots, h_{|H|}\}$. It will suffice to show that the generator-enriched lattice (L,G) has no parallels if and only if in every cover relation $(K,H) \prec (M,I)$ in $\mathrm{WM}(L,G)$ we have $|I| - |H| = 1$.

First suppose $(K,H) \prec (M,I)$ and $|I| - |H| \geq 2$. We claim $(K,H) = (M,I)/\widehat{0}_K$. Since the two minors form a cover relation clearly (K,H) is not a deletion of (M,I) so we conclude $(K,H) = (M,I)/_{(L,G)}\widehat{0}_K$. Since

$$\begin{aligned}(K,H) &= (M,I)/_{(L,G)}\widehat{0}_K \\ &= ((M,I)/\widehat{0}_K) \setminus \{g \vee \widehat{0}_K : g \in \mathrm{Del}(K,H) \setminus \mathrm{Del}(M,I)\} \\ &= ((M,I) \setminus \{g \vee \widehat{0}_M : g \in \mathrm{Del}(K,H) \setminus \mathrm{Del}(M,I)\})/\widehat{0}_K\end{aligned}$$

we see

$$(K,H) < (M,I) \setminus \{g \vee \widehat{0}_M : g \in \mathrm{Del}(K,H) \setminus \mathrm{Del}(M,I)\}.$$

We conclude $\mathrm{Del}(K,H) = \mathrm{Del}(M,I)$ which implies $(K,H) = (M,I)/\widehat{0}_K$. We also conclude $\widehat{0}_K$ is an atom in M as otherwise the contraction by a generator $i < \widehat{0}_K$ would lie strictly in between (K,H) and (M,I). These two conclusions imply there must be two generators $i_1, i_2 \in I$ such that $i_1 \vee \widehat{0}_K = i_2 \vee \widehat{0}_K$. Thus, (M,I) has a parallel which implies (L,G) has a parallel as well.

Now assume (L,G) has a parallel, say $\ell \in L$ and $g, h \in G$ satisfy $\ell \vee g = \ell \vee h$. Furthermore, assume ℓ is minimal with respect to this property, in that, for any $\ell' < \ell$ the elements $\ell' \vee g$ and $\ell' \vee h$ are distinct. Let $\ell' \prec \ell$ and consider the contractions $(L,G)/\ell$ and $(L,G)/\ell'$. On one hand $(L,G)/\ell \prec (L,G)/\ell'$ and on the other hand $(L,G)/\ell$ has at least 2 less generators than $(L,G)/\ell'$. □

Weak minor posets admit a decomposition into Boolean algebras and this decomposition can be used to express the rank generating function of the weak minor poset of a generator-enriched lattice with no parallels in terms of the rank generating function of the original lattice. This is in fact the same decomposition for the minor poset in Theorem 4.2.28.

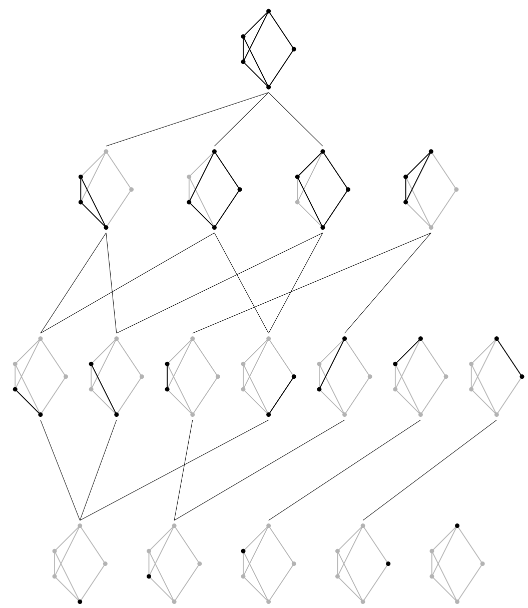


Figure 5.2: An example of the Boolean decomposition of a weak minor poset.

Proposition 5.4.2. *Given a generator-enriched lattice* (L, G) *let* $\mathrm{WM}_\ell(L, G)$ *denote the subposet of* $\mathrm{WM}(L, G)$ *consisting of minors with minimal element* ℓ*. We have the decomposition*

$$\mathrm{WM}(L, G) \setminus \{\emptyset\} = \dot{\bigcup}_{\ell \in L} \mathrm{WM}_\ell(L, G).$$

Furthermore $\mathrm{WM}_\ell(L, G) = [\langle \emptyset | \ell \rangle, (L, G)/\ell]$ *and is Boolean of rank*

$$|\{g \vee \ell : g \in G\} \setminus \{\ell\}|.$$

Proof. It is clear that the union of all posets $\mathrm{WM}_\ell(L, G)$ is disjoint and contains all minors of (L, G). Let $(K, H) \in \mathrm{WM}_\ell(L, G)$. We can express (K, H) as

$$((L, G)/\ell) \setminus \{g \vee \ell : g \in \mathrm{Del}(K, H)\}$$

so $(K, H) \leq (L, G)/\ell$. On the other hand, we have $(K, H) \setminus H = \langle \emptyset | \ell \rangle$. Thus, we conclude

$$\mathrm{WM}_\ell(L, G) \subseteq [\langle \emptyset | \ell \rangle, (L, G)/\ell].$$

Since taking the minimal element is an order-reversing map from $\mathrm{WM}(L, G)$ to L the other inclusion $[\langle \emptyset | \ell \rangle, (L, G)/\ell] \subseteq \mathrm{WM}_\ell(L, G)$ is immediate. Finally, the interval $\mathrm{WM}_\ell(L, G)$ is Boolean since all relations must come from deletions so mapping each minor to its generating set is an isomorphism between $\mathrm{WM}_\ell(L, G)$ and the Boolean algebra of subsets of $\{g \vee \ell : g \in G\} \setminus \{\ell\}$. □

We use this decomposition below to derive an expression for the rank generating function of graded weak minor posets. Given a graded poset P we let $F(P; q)$ denote the rank generating function $\sum_{p \in P} q^{\mathrm{rk}(p)}$.

Proposition 5.4.3. *For any generator-enriched lattice* (L, G) *with no parallels*

$$F(\mathrm{WM}(L,G);q) = 1 + q\sum_{\ell\in L} F(L^*; 1+q).$$

Proof. Recall the rank generating function of the Boolean algebra is

$$F(B_n;q) = (1+q)^n.$$

Thus, the Boolean decomposition of $\mathrm{WM}(L,G)$ yields

$$F(\mathrm{WM}(L,G);q) = 1 + q\sum_{\ell\in L}(1+q)^{\mathrm{rk}((L,G)/\ell)}.$$

Recall $\mathrm{rk}((L,G)/\ell) = |\{g\vee\ell : g\in G\}\setminus\{\ell\}|$. Since (L,G) has no parallels this is equal to $|\{g\in G : g\not\leq\ell\}|$. By Proposition 4.2.31 this is equal to

$$\mathrm{rk}_{L^*}(\ell) = \mathrm{rk}_L(\widehat{1}) - \mathrm{rk}_L(\ell) \qquad \square.$$

5.5 Shelling weak minor posets

In this section we describe a process to induce a dual EL-labeling on the weak minor poset of a generator-enriched lattice given EL-labelings on all the minors. Dually, we induce an EL-labeling of the weak minor poset given dual EL-labelings on all the minors. Generator-enriched lattices whose minors are all EL-labelable include distributive lattices and geometric lattices. Generator-enriched lattices whose minors are all dual EL-labelable includes generator-enriched lattices with no parallels as such lattices are lower semimodular.

Theorem 5.5.1. *Given a generator-enriched lattice* (L,G) *all of whose minors admit an EL-labeling the weak minor poset* $\mathrm{WM}(L,G)$ *is dual EL-labelable. Given a generator-enriched lattice* (L,G) *all of whose minors admit a dual EL-labeling the weak minor poset* $\mathrm{WM}(L,G)$ *is EL-labelable.*

Proof. First consider the case where all minors of (L,G) admit an EL-labeling. For $I\subseteq G$ let λ_I be the EL-labeling on $(L,G)\setminus I$ and let Λ_I be the domain of λ_I. Let $\Lambda = \bigsqcup_{I\subseteq G}\Lambda_I$. We now define a labeling λ on the covers of $\mathrm{WM}(L,G)$ with domain $2^G\sqcup\Lambda\sqcup\{0\}$. The labels are ordered via lexicographic ordering on 2^G with respect to some fixed ordering of G, via the obvious way on Λ and via the rule

$$X < 0 < \ell$$

for all $\ell\in\Lambda$ and $X\subseteq G$.

Given a cover $(K,H)\setminus\{h\}\prec(K,H)$ the image under λ is defined to be

$$\{g\in G : g\vee\widehat{0}_K = h\}.$$

The covers $\emptyset\prec\langle\emptyset|\ell\rangle$ all have image 0 under λ. The remaining covers are of the form $(K,H)/\{h\}\prec(K,H)$. Since this is a cover relation we must have $\widehat{0}_K\prec h$. The image of such a cover relation under λ is $\lambda_{\mathrm{Del}(K,H)}(\widehat{0}_K\prec h)$.

We now must show that each interval in the weak minor poset $\mathrm{WM}(L, G)$ has a unique maximal downward rising chain which is lexicographically least. First consider an interval of the form $[\emptyset, (K, H)]$. The last label of any maximal chain is 0. For a maximal to be rising it must thus only use labels preceding 0 before the last cover, hence, the only rising maximal chains consist solely of deletions of (K, H). The set of labels of chains consisting of deletions of (K, H) are the all the same and only one occurs in order, the order being lexicographic order on the sets

$$\{g \in G : g \vee \widehat{0}_K = h\}$$

for $h \in H$. Thus there is a unique rising downward maximal chain of $[\emptyset, (K, H)]$. Furthermore it is lexicographically least amongst all downward maximal chains since labels of deletions are less than labels of contractions and it is clearly lexicographically least of all chains consisting of deletions of (K, H).

Now consider an interval $[(M, I), (K, H)]$. Chains of this interval all contract the set

$$\{h \in H : h \leq \widehat{0}_M\}$$

and delete the set

$$\mathrm{Del}(M, I) \cap H = \{h \in H : h \vee \widehat{0}_M \notin I \cup \{\widehat{0}_M\}\}.$$

A rising chain must have all covers corresponding to deletions above those covers corresponding to contractions. Note this can be arranged although the reverse may not since relations are induced by weak contractions which may involve deletions. There is only way to perform the deletions in order so that the deletion portion of the chain is rising. After the deletion portion of the chain is a chain from $(K, H) \setminus \mathrm{Del}(M, I)$ to (M, I) which consists of contractions. The labels for this portion of the chain are labels under $\lambda_{\mathrm{Del}(M,I)}$ from the interval $[\widehat{0}_K, \widehat{0}_M]$ of $(L, G) \setminus \mathrm{Del}(M, I)$. The interval $[\widehat{0}_K, \widehat{0}_M]$ has a unique maximal upward rising chain which is lexicographically least. A rising chain in the interval $[(M, I), (K, H)]$ must use the corresponding contractions to be rising so there is only one rising downward maximal chain. Furthermore the chain is lexicographically least.

Now consider the case where all minors of (L, G) are dual EL-labelable. Let λ^I be the dual EL-labeling for $(L, G) \setminus I$ and let Λ^I be the domain of λ^I. As before set $\Lambda' = \bigsqcup_{I \subseteq G} \Lambda^I$. The label set is similarly $\Lambda' \sqcup 2^G \sqcup \{0\}$ with 2^G ordered lexicographically but we now have the relations

$$\ell < 0 < X$$

for $\ell \in \Lambda'$ and $X \subseteq G$. Since we now consider upward chains we have a unique rising chain for intervals $[\emptyset, (K, H)]$ which has covers consisting of deletions in lexicographic order from the bottom. Intervals $[(M, I), (K, H)]$ have a unique rising chain which, when viewed downward, begin with deletions and end with contractions with the deletions in lexicographic order from the bottom of the chain and the contractions in order of the dual EL-labeling of the interval $[\widehat{0}_K, \widehat{0}_M]$. Similar observations as above show that the rising chain is lexicographically least as well. □

We use the following result due to Björner and Wachs to relate the homology of (dual) EL-labelable weak minor posets to the homology of the associated generator-enriched lattice.

Theorem 5.5.2 (Björner–Wachs [13, Theorem 5.9])**.** *Given a poset P with an EL-labelling the ith homology group $H_i(P \setminus \{\widehat{0}, \widehat{1}\}, \mathbb{Z})$ is free of rank equal to the number of falling maximal chains in P of length $i+2$.*

Corollary 5.5.3. *Given a generator-enriched lattice (L, G) all of whose minors admit an EL-labeling or all of whose minors admit a dual EL-labelling*

$$H_{i+1}(\mathrm{WM}(L, G) \setminus \{\emptyset, (L, G)\}, \mathbb{Z}) \cong H_i(L \setminus \{\widehat{0}, \widehat{1}\}, \mathbb{Z})$$

for $i \geq 0$.

Proof. First, suppose the minors of (L, G) have EL-labellings. A downward falling maximal chain in $\mathrm{WM}(L, G)$ must consist entirely of contractions since the last label is 0. The labels of such a chain are labels from the EL-labeling on L. Thus, every falling maximal chain in $\mathrm{WM}(L, G)$ corresponds to a falling maximal chain in L. The converse holds as well, given a maximal chain in L the corresponding sequence of contractions forms a maximal chain. The falling maximal chains of L and $\mathrm{WM}(L, G)$ are thus in bijection and the falling maximal chains in $\mathrm{WM}(L, G)$ all have one additional element from the minimum $\emptyset$.

The case where the minors of (L, G) have dual EL-labelings is similar. An upward falling chain must consist of contractions since the first label is 0. We have a bijection between the falling chains in the exact same way. □

Chapter 6 Future research

In this chapter we discuss some open questions and future directions for research.

6.1 The uncrossing poset and generalized minor posets

In the minor poset of a generator-enriched lattice (L, G) the contractions are controlled by the lattice L, in that the set of contractions considered as maps with the composition operation is isomorphic to the lattice L. On the other hand deletions are always controlled by the Boolean algebra.

The uncrossing poset appears to be a poset of minors of some algebraic object generalizing generator-enriched lattices in which deletions may be controlled by any lattice. The lattice controlling contractions is the lattice of flats of any fixed cactus graph associated to the top element. The lattice controlling deletions is the lattice of flats of a dual cactus graph, this dual cactus graph is constructed from the medial graph in the same manner but the vertices are placed in regions of the opposite color. Figure 6.1 depicts an example. Figure 6.2 shows the results of contracting an edge and of deleting an edge in the cactus graph. Observe deletions in the cactus graph correspond to contractions in the dual cactus graph and vice versa. The deletion depicted in Figure 6.2 does not correspond to a cover relation, this is due to the parallel edges created in the dual cactus graph. Since the deletions are controlled by the lattice of flats of the dual cactus graph this deletion is the same as when it is followed by deleting in the dual cactus graph edges that have another edge parallel to them. These deletions correspond to resolving the double crossings of arcs in the medial diagram. The contraction depicted in Figure 6.2 also does not correspond to a cover. This is due to a more nuanced behavior.

Certain transformations called $Y - \Delta$ moves do not change the electrical equivalence class, and hence the associated pairing, of a cactus graph. This transformation replaces a 3-cycle in a cactus graph with an internal vertex at the center of the cycle that is adjacent to the vertices of the cycle. Performing this transformation on the graph depicted in Figure 6.2 (b) with the only 3-cycle results in a graph with two internal vertices of degree 2. These two vertices correspond to parallel edges in the dual cactus graph and to double crossings in the medial graph. These reductions explain why the contraction does not correspond to a cover.

This idea of an algebraic structure wherein deletions are controlled by a specified lattice should generalize minor posets of generator-enriched lattices, the special case of when deletions are as free as possible being controlled by a Boolean algebra. Perhaps a zipping construction analogous to Theorem 4.3.7 can be performed. Thus, we make the following conjecture. The conjecture holds for any pairing with no internal regions of a medial diagram representation that border each other along an arc segment.

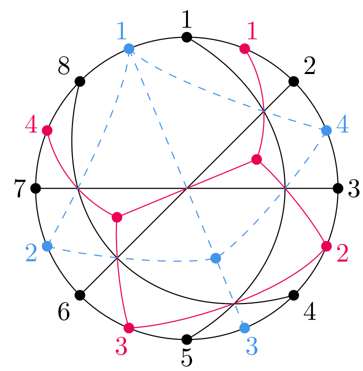


Figure 6.1: A cactus graph, depicted in red, along with the dual cactus graph, depicted in blue with dashed lines.

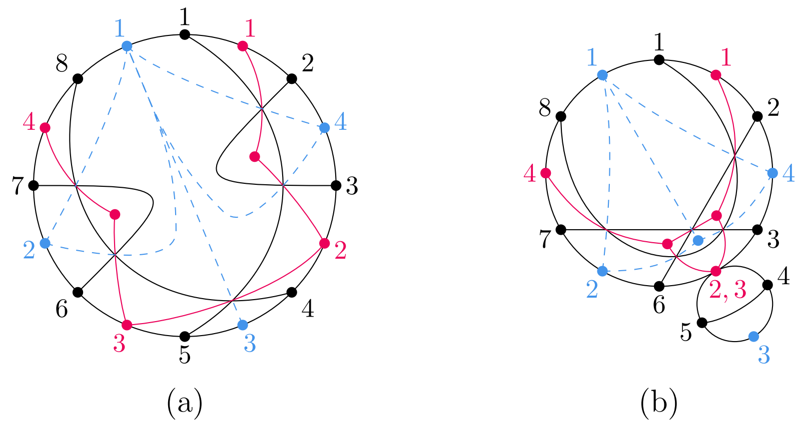


Figure 6.2: In (a) is the result of applying a deletion to the cactus graph in Figure 6.1 and in (b) is the result of applying a contraction.

Conjecture 6.1.1. *For any pairing $\tau \in \mathrm{UC}_n$ there is a sequence of zipping operations that take the face lattice $Q_{\mathrm{cross}(\tau)}$ of the* $\mathrm{cross}(\tau)$*-dimensional cube to the interval $[\widehat{0}, \tau]$ of the uncrossing poset UC_n.*

Assuming this conjecture the same proof as Corollary 4.3.8 would establish PL-sphericity of the proper part of the uncrossing poset. We explicitly pose this weaker conjecture.

Conjecture 6.1.2. *The proper part of the uncrossing poset UC_n is a PL-sphere.*

Conjecture 6.1.1 would also imply the uncrossing poset is isomorphic to the face poset of a regular CW complex and Conjecture 6.1.2 would imply the uncrossing poset is Gorenstein*. Both of these results have been established by Hersh and Kenyon in [28] by showing that the uncrossing poset is lexicographically shellable. We also conjecture **cd**-index inequalities for the uncrossing poset which would follow from Conjecture 6.1.1.

Conjecture 6.1.3. *Let $\tau \in \mathrm{UC}_n$ be a pairing and choose a cactus graph representation G for τ. Let L be the lattice of flats of G and let K be the lattice of flats of the dual cactus graph. The following* **cd***-index inequalities hold coefficientwise:*

$$\Psi([\widehat{0}, \tau]) \leq \Psi(\mathrm{M}(L, \mathrm{irr}(L)),$$
$$\Psi([\widehat{0}, \tau]) \leq \Psi(\mathrm{M}(K, \mathrm{irr}(K)).$$

In particular,

$$\Psi([\widehat{0}, \tau]) \leq \Psi(Q_{\mathrm{cross}(\tau)})$$

holds.

The inequality $\Psi([\widehat{0}, \tau]) \leq \Psi(Q_{\mathrm{cross}(\tau)})$, if true, gives a very weak bound for the **cd**-index of the interval $[\widehat{0}, \tau]$. We provide evidence for Conjecture 6.1.3 in Appendix B.

6.2 Minor posets of generator-enriched lattices

We now turn to discussing questions related to minor posets of generator-enriched lattices from Chapter 4. Corollary 4.3.8 and Corollary 4.3.9 established that any minor poset is isomorphic to the face poset of a regular CW sphere and the proper part of any minor poset is a PL-sphere. Given a generator-enriched lattice (L, G) denote this regular CW sphere by $\Gamma(L, G)$. Since the proper part of the poset $\mathrm{M}(L, G)$ is a PL-sphere the complex $\Gamma(L, G)$ can be realized with piecewise-linear cells. Is there a combinatorially meaningful way to realize the complex $\Gamma(L, G)$ with piecewise linear cells? The polytopes defined below appear to be promising candidates.

Definition 6.2.1. *Let (L, G) be a generator-enriched lattice with n generators $g_1, \ldots, g_n$, and let e_i denote the ith standard basis vector of $\mathbb{R}^n$. For $\ell \in L$ set*

$$v_\ell = \sum_{g_i \leq \ell} e_i.$$

In particular, $v_{\widehat{0}}$ is the zero vector. The flat polytope *$P(L, G)$ is the convex hull of the vectors v_ℓ for $\ell \in L$.*

The vertices of a flat polytope correspond to the flats of the closure operator associated to the canonical strong map. Figure 6.3 shows an example of a flat polytope. For any generator-enriched lattice (L, G) with 3 generators the boundary complex of the flat polytope $P(L, G)$ is a subdivision of the complex $\Gamma(L, G)$.

Conjecture 6.2.2. *For any generator-enriched lattice (L, G) the boundary complex of the closure polytope $P(L, G)$ subdivides the CW complex $\Gamma(L, G)$. That is, there is a realization of $\Gamma(L, G)$ such that the closed cells are each a union of faces of $P(L, G)$.*

When L is a distributive lattice the closure polytope $P(L, \mathrm{irr}(L))$ is the order polytope associated to the poset $\mathrm{irr}(L)^*$. Given a finite poset P the *order polytope $\mathcal{O}(P)$* consists of all order-preserving functions from P to $[0, 1] \subseteq \mathbb{R}$. The vertices of this polytope are the order-preserving functions from P to $\{0, 1\}$. The elements mapped

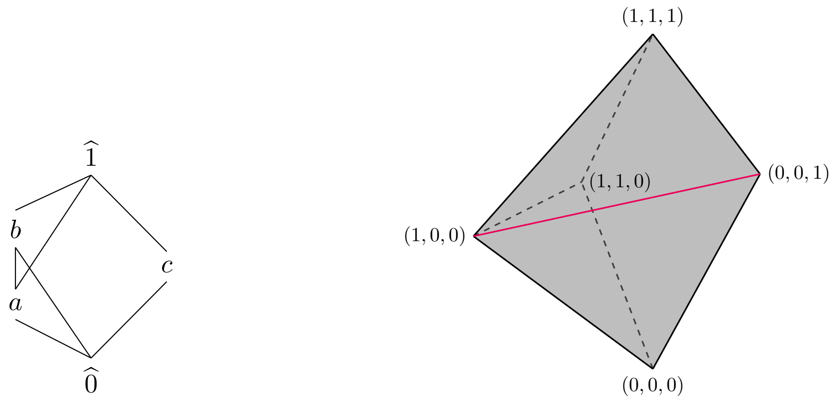


Figure 6.3: On the left a generator-enriched lattice and on the right the associated flat polytope. The black edges in the polytope correspond to edges of the diagram. Merging facets along the red edges yields the CW complex $\Gamma(L, G)$.

to 1 by a vertex form an upper order ideal of P and conversely every upper order ideal of P corresponds to a vertex of the polytope $\mathcal{O}(P)$. See [43] for a discussion of order polytopes.

When L is a geometric lattice the closure polytope $P(L, \mathrm{irr}(L))$ has vertices corresponding to the flats of the simple matroid defined by L. Several polytopes from matroids have been studied. For instance the matroid polytope associated to a matroid has vertices corresponding to the bases and the independence polytope has vertices corresponding to the independent sets. However, the polytopes $P(L, \mathrm{irr}(L))$ do not appear to have been previously studied.

Shifting focus, we pose some questions concerning the **cd**-indices of minor posets. A few bounds have already been established for minor posets. A tight upper bound of $\Psi(\mathrm{M}(L, G)) \leq \Psi(Q_n)$ holds whenever the generator-enriched lattice (L, G) has n generators, equality is achieved by the Boolean algebra $(B_n, \mathrm{irr}(B_n))$. A tight lower bound of $\Psi(B_{n+1})$ was established for all generator-enriched lattices with no parallels and n generators, the lower bound is achieved by $(C_n, \mathrm{irr}(C_n))$. In the class of all generator-enriched lattices with n generators coefficientwise minimal **cd**-indices can be achieved by many generator-enriched lattices. For example for $n = 3$ the generator-enriched lattices depicted in Figure A15 (6) and (10) have minimal **cd**-indices. Are there other nice classes of generator-enriched lattices for which tight bounds for the **cd**-indices of minor posets can be described?

One example for which it may be tractable to obtain tight bounds for these **cd**-indices is minimally generated geometric lattices with n atoms, or refining this to such lattices of rank at most r. Any geometric lattice of rank at most r with n atoms is the image under a strong map on the lattice of flats of the uniform matroid of rank r on the ground set $[n]$; this is the lattice obtained from the Boolean algebra B_n by identifying the maximal element $[n]$ with all elements of rank r or more. The minor poset of this lattice thus achieves the maximum **cd**-index among all geometric lattices of rank at most r with n atoms. On the other end, any geometric lattice

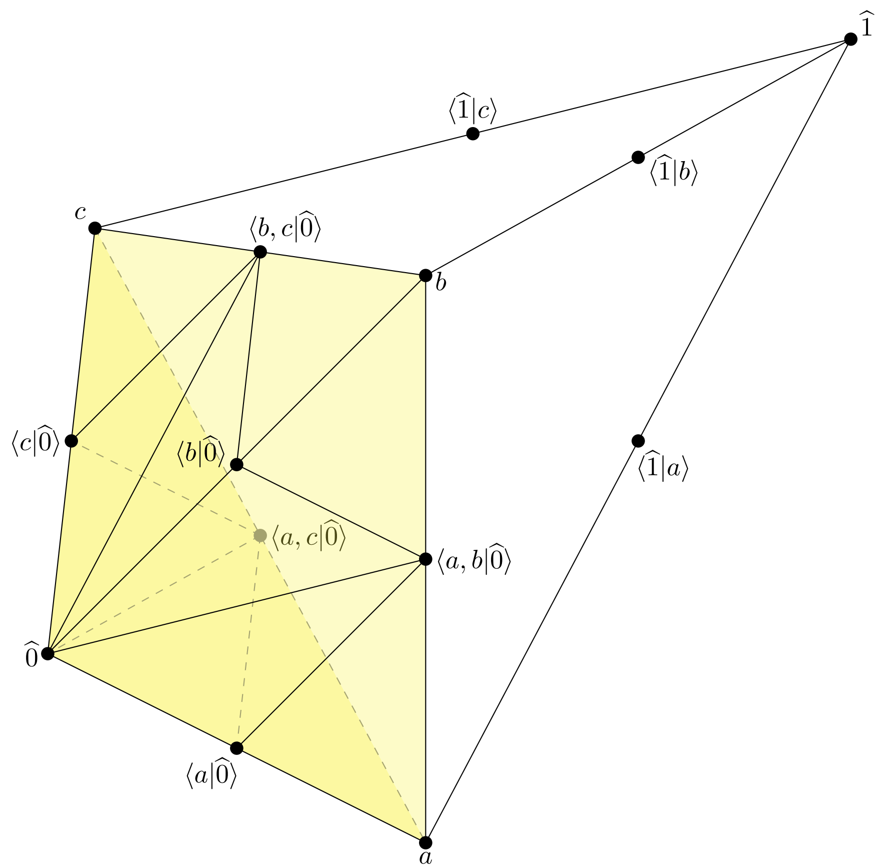


Figure 6.4: The order complex of a weak minor poset.

with n atoms admits a strong surjection onto the rank 2 lattice consisting of n atoms, a minimum $\widehat{0}$ and a maximum $\widehat{1}$. This is the lattice of flats of the uniform matroid of rank 2 with ground set $[n]$. Thus computing the **cd**-indices of uniform matroids would produce tight bounds for the **cd**-indices of minor posets of minimally generated geometric lattices.

Given a generator-enriched lattice (L, G) the **cd**-index of the minor poset $\mathrm{M}(L, G)$ only depends on the generator-enriched lattice (L, G) simply because the minor poset is computed from (L, G). Can this relationship be made more explicit, that is, is there a way to compute the **cd**-index directly from the generator-enriched lattice? A result along these lines is that the **cd**-index of a zonotope can be computed from the **ab**-index of the lattice of regions of the associated hyperplane arrangement [8, Theorem 3.1].

6.3 Weak minor posets

In Chapter 5 it was established a weak minor poset $\mathrm{WM}(L, G)$ is Cohen-Macaulay if and only if (L, G) has no parallels. A shelling was given for these weak minor posets, so the order complex $\Delta(\mathrm{WM}(L, G) \setminus \{\emptyset, (L, G)\})$ is homotopic to a wedge of spheres (all of the same dimension). Can more be said topologically about weak minor posets? For what generator-enriched lattices is the weak minor poset homeomorphic to a ball?

The homology of weak minor posets for which a shelling was given was shown to be the same as the original lattice's shifted up one dimension. Does the same result hold for more generator-enriched lattices? Is this homology result indicative of a stronger topological connection between the weak minor poset and the associated lattice? In general, the suspension operation takes a toplogical space to one whose homology is the same as the original space shifted up in one dimension. The weak minor poset is not generally homeomorphic to the suspension of the original lattice. For example, the order complex depicted in Figure 6.4 is not homeomorphic to the suspension over 3 points, which is the order complex of the associated lattice. This order complex is homotopic to the suspension though via collapsing the two-dimensional triangles all to one point. Are weak minor posets homotopic to a suspension over the associated lattice, perhaps via a sequence of simplicial collapses?

Appendices

Appendix A: Posets

In this appendix we list Hasse diagrams of various posets studied in this dissertation.

Permutation and Catalan posets

In this section we list a few posets studied in Chapter 2.

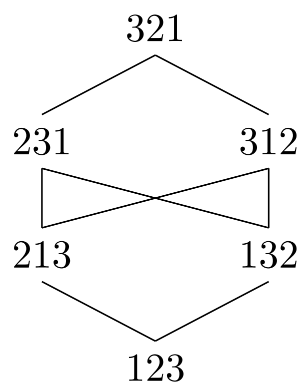


Figure A1: The Bruhat order on the symmetric group $\mathfrak{S}_3$.

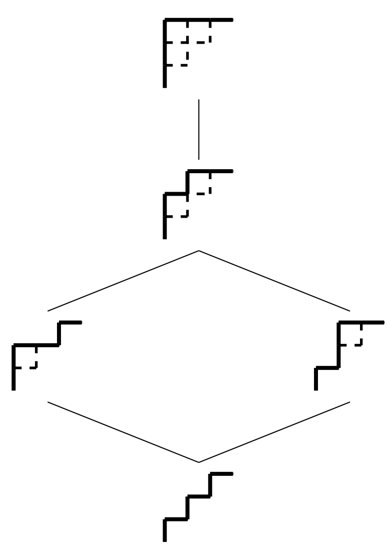


Figure A2: The dominance order on Dyck paths with 6 steps.

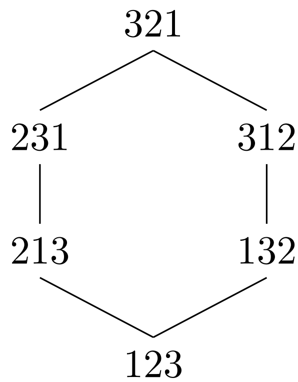


Figure A3: The weak order on the symmetric group $\mathfrak{S}_3$.

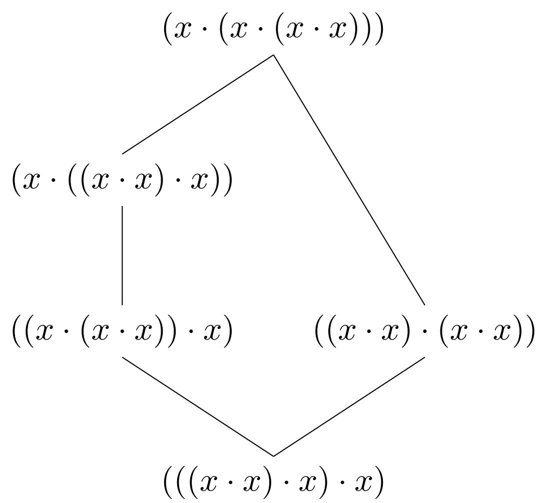


Figure A4: The Tamari lattice of binary paranthesizations of 4 symbols.

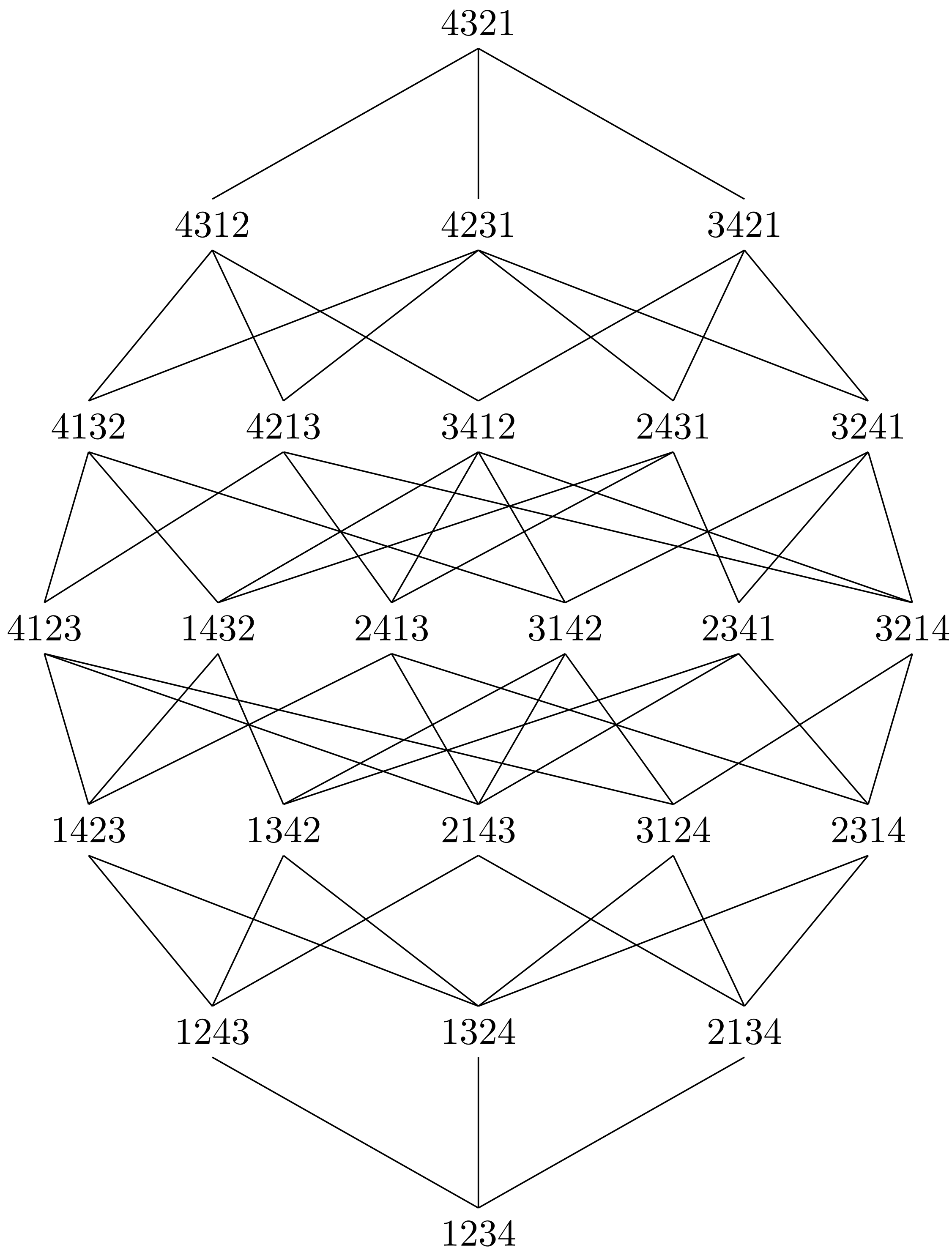


Figure A5: The Bruhat order on the symmetric group $\mathfrak{S}_4$.

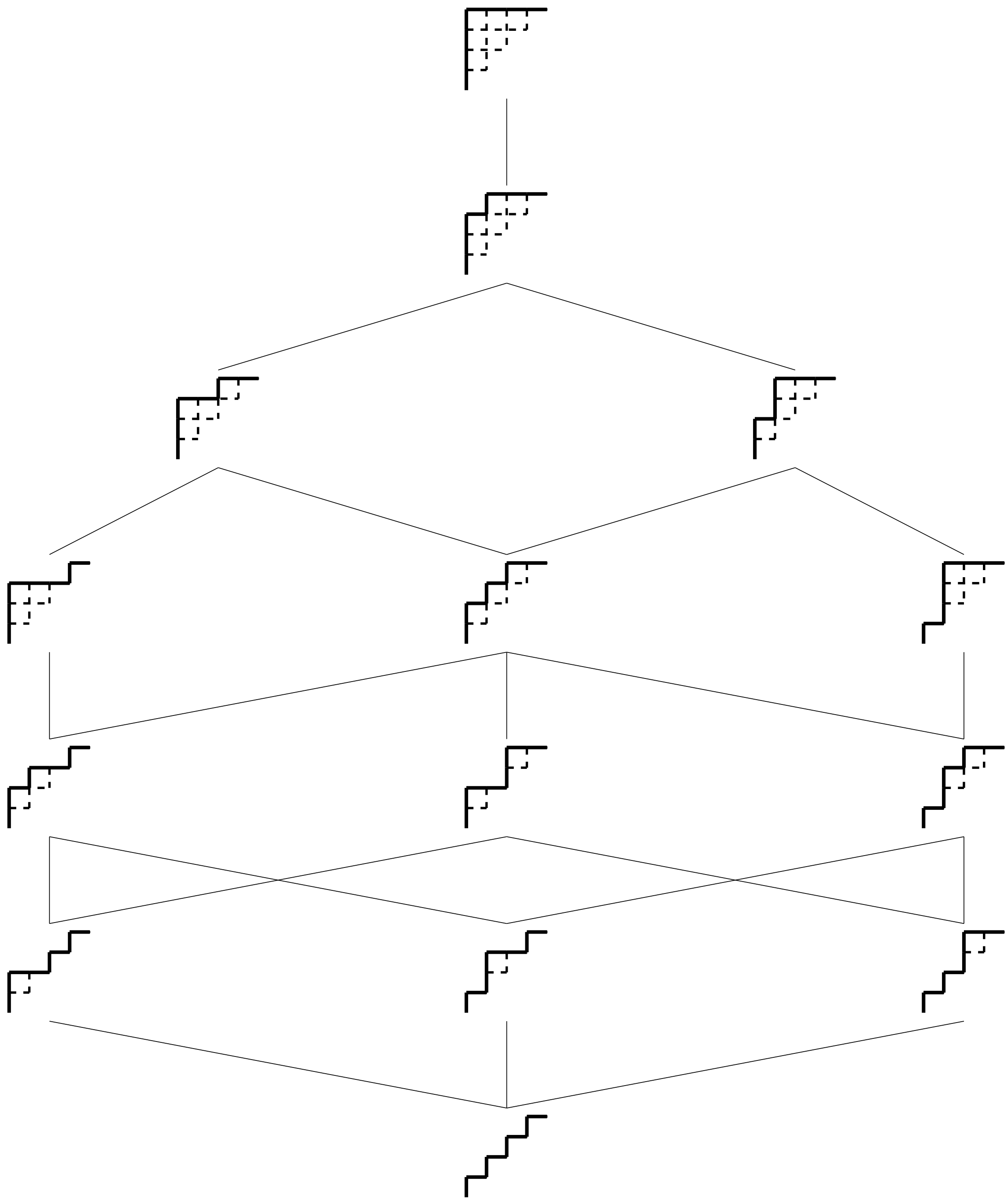


Figure A6: The dominance order on Dyck paths with 8 steps.

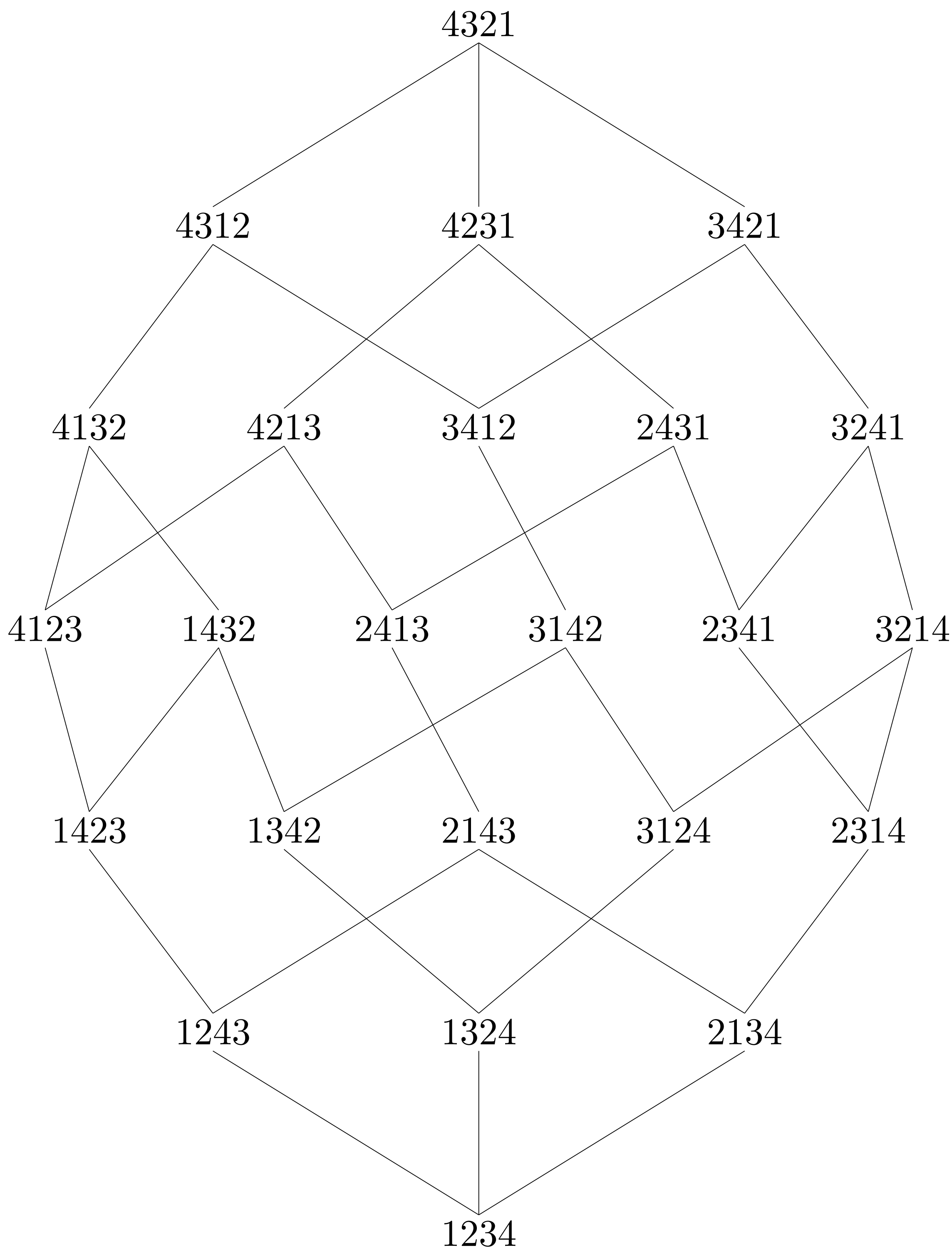


Figure A7: The weak order on the symmetric group $\mathfrak{S}_4$.

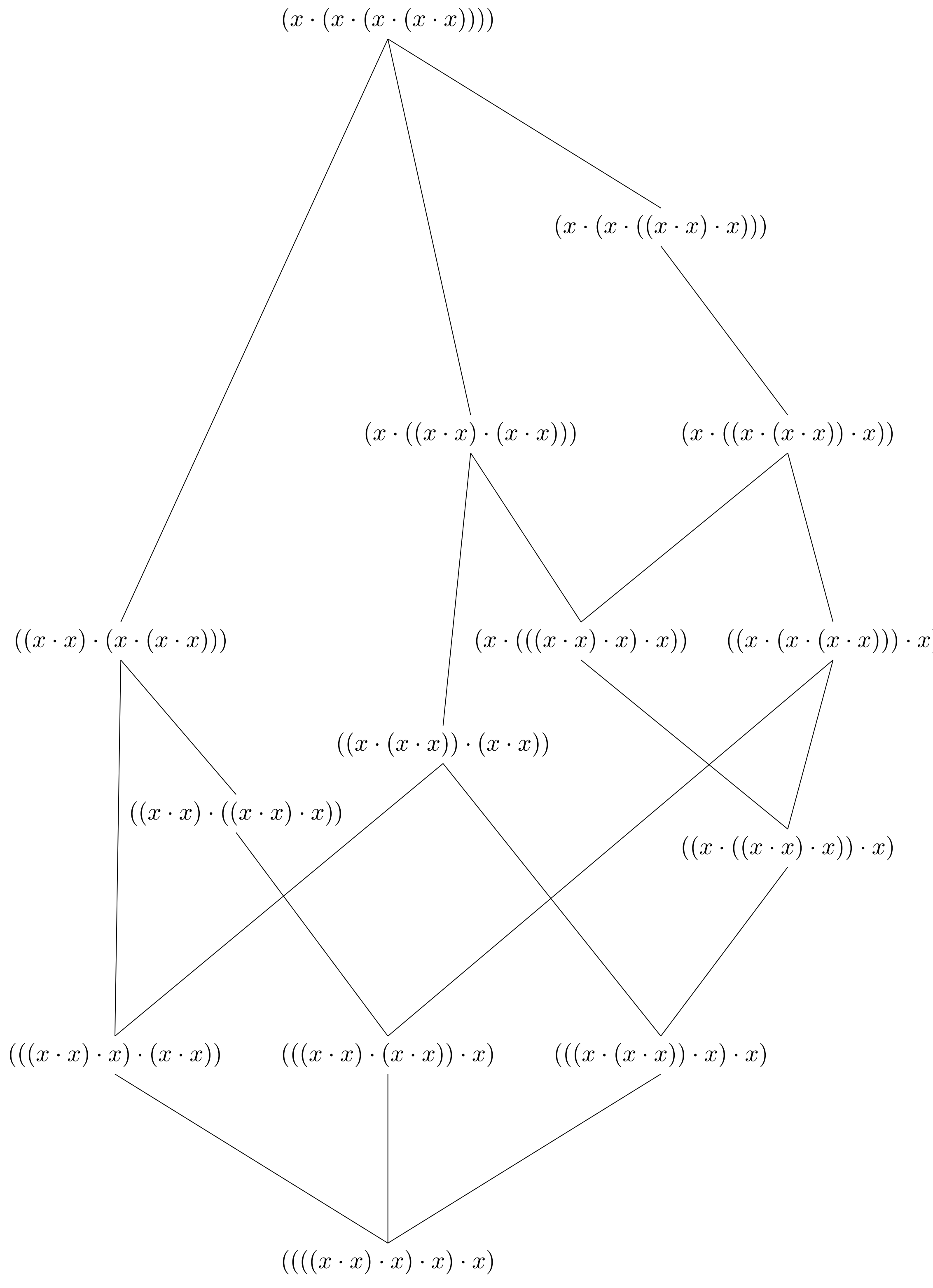


Figure A8: The Tamari lattice of binary paranthesizations of 5 symbols.

Uncrossing posets

In this section we list the uncrossing posets UC_n for $n = 2, 3, 4, 5$ and the decomposition into Bruhat intervals from Proposition 2.2.6. See Chapter 2 for details about the uncrossing posets.

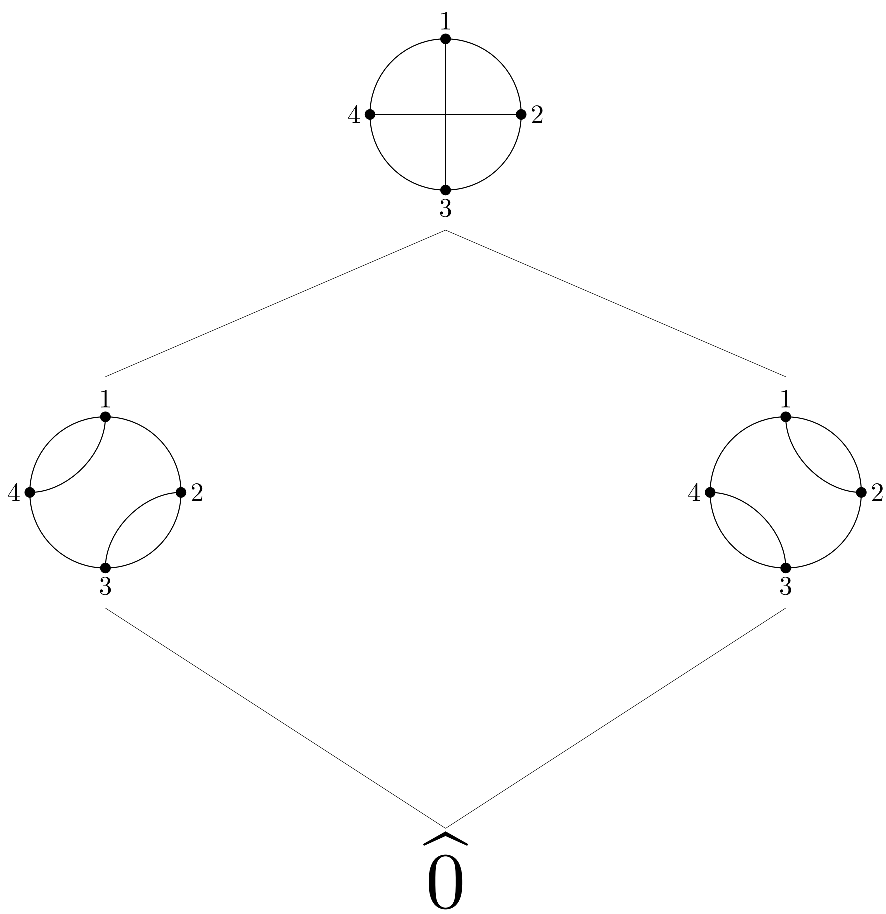


Figure A9: The uncrossing poset UC_2.

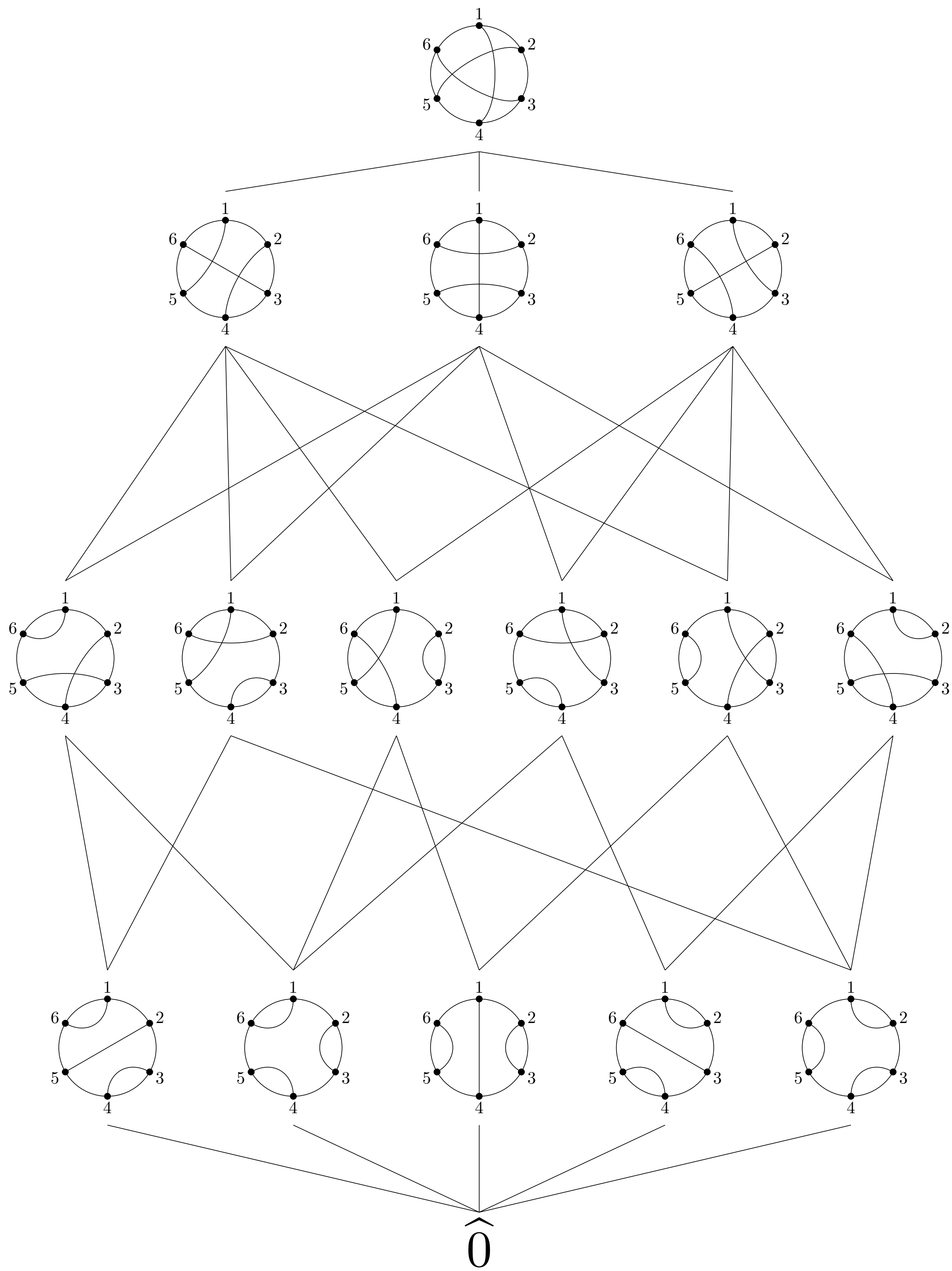


Figure A10: The uncrossing poset UC_3.

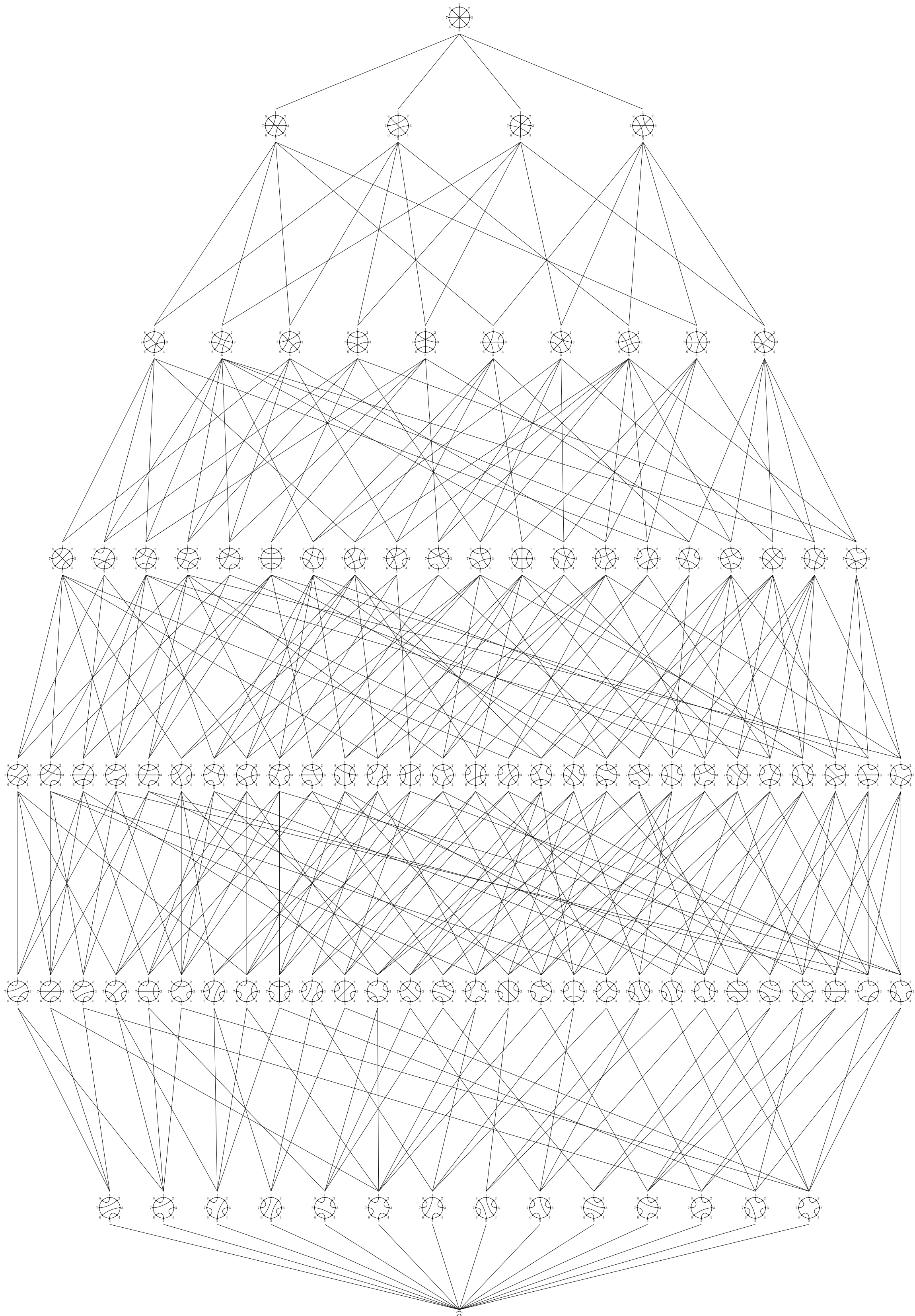


Figure A11: The uncrossing poset UC_4.

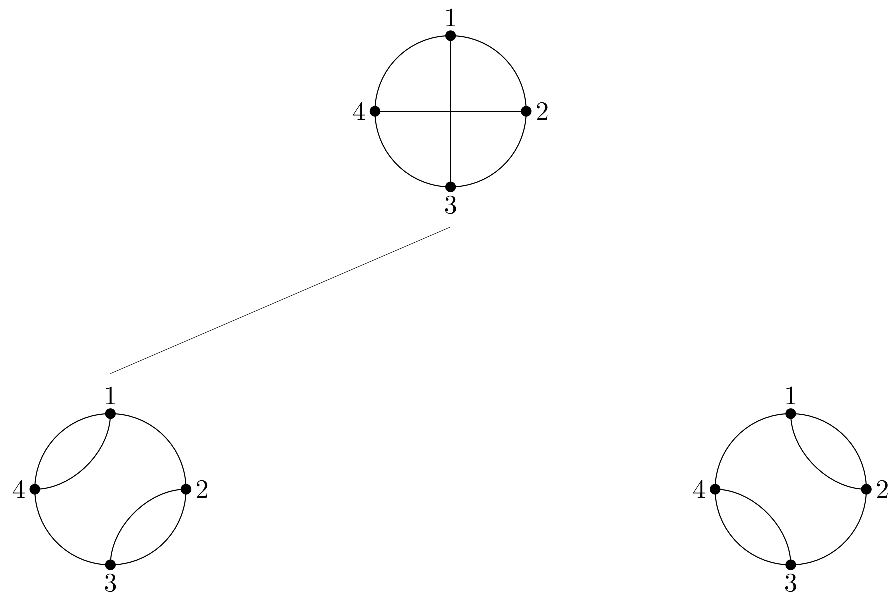


Figure A12: The Bruhat interval decomposition of the uncrossing poset UC_2.

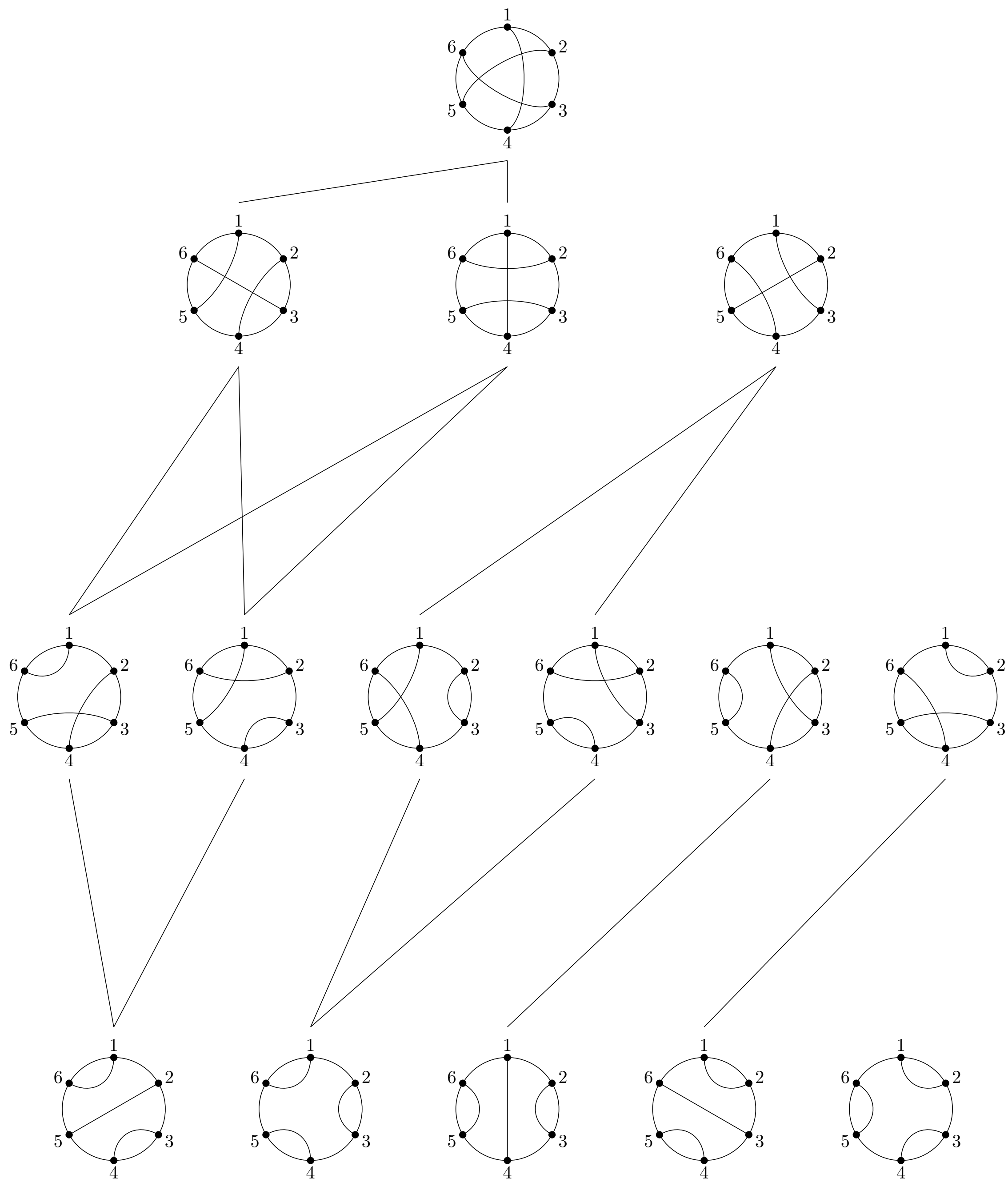


Figure A13: The Bruhat interval decomposition of the uncrossing poset UC_3.

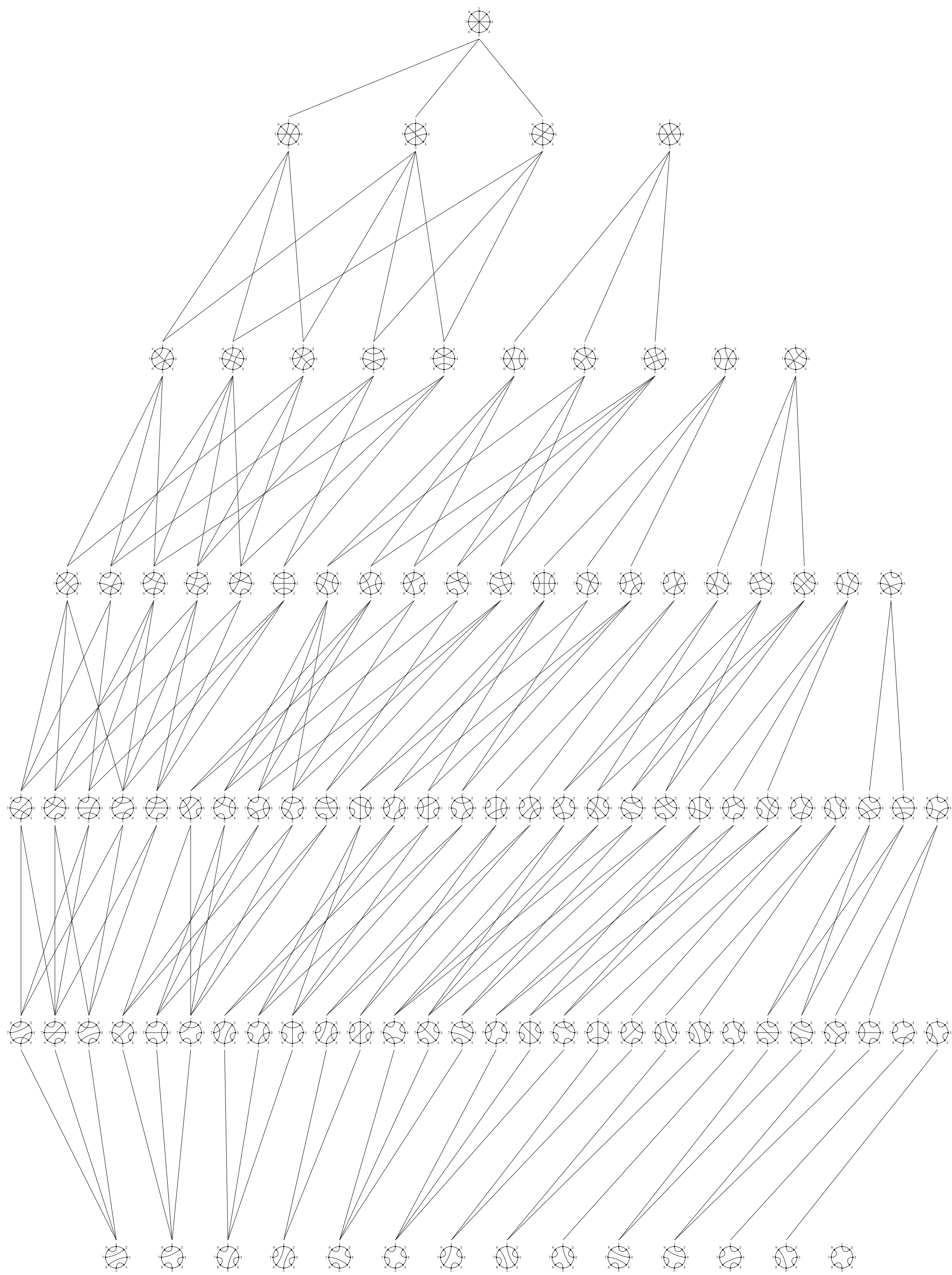


Figure A14: The Bruhat interval decomposition of the uncrossing poset UC_4.

Minor posets

In this section we list the 10 generator-enriched lattices with 3 generators along with their minor posets. See Chapter 4 for details about minor posets.

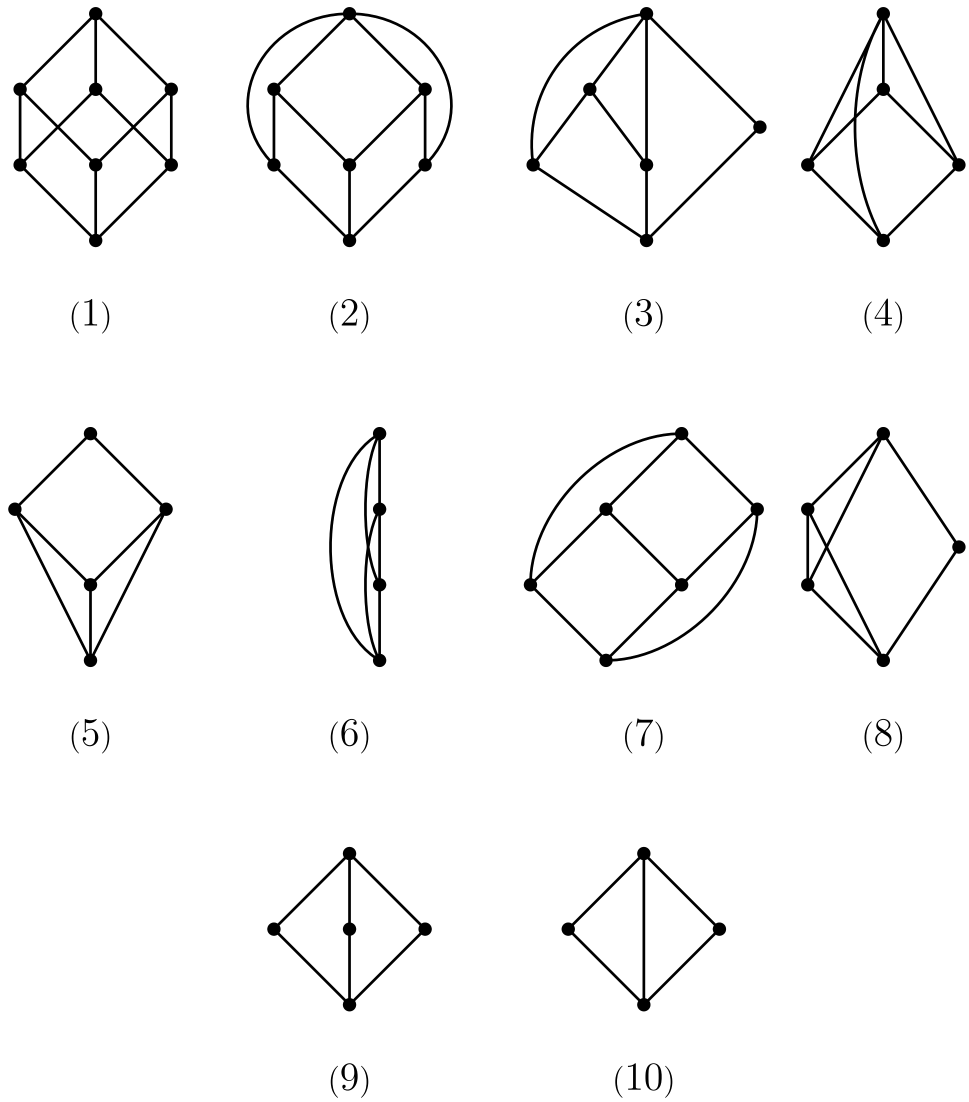


Figure A15: The 10 generator-enriched lattices with 3 generators

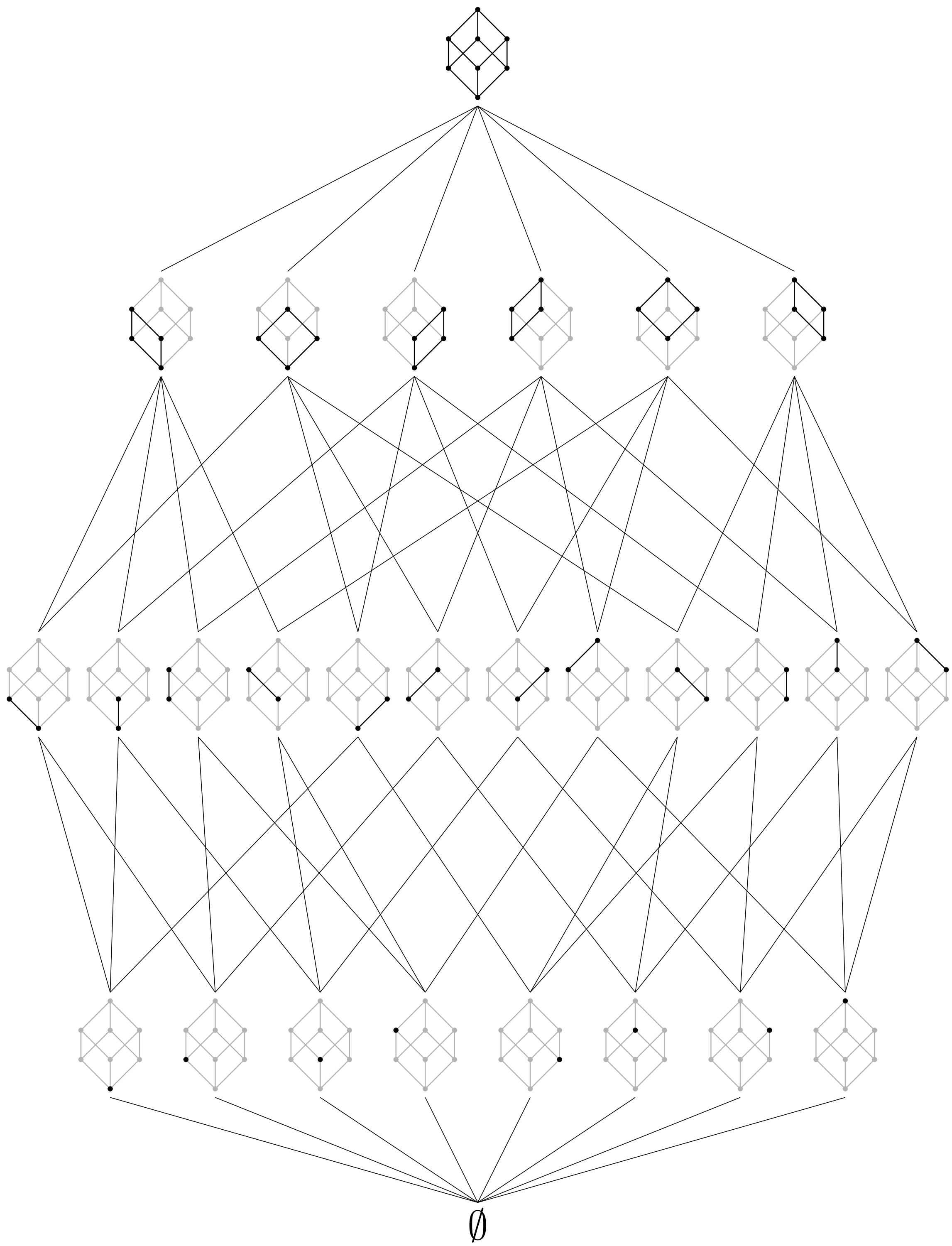


Figure A16: The minor poset of the generator-enriched lattice depicted in Figure A15 (1).

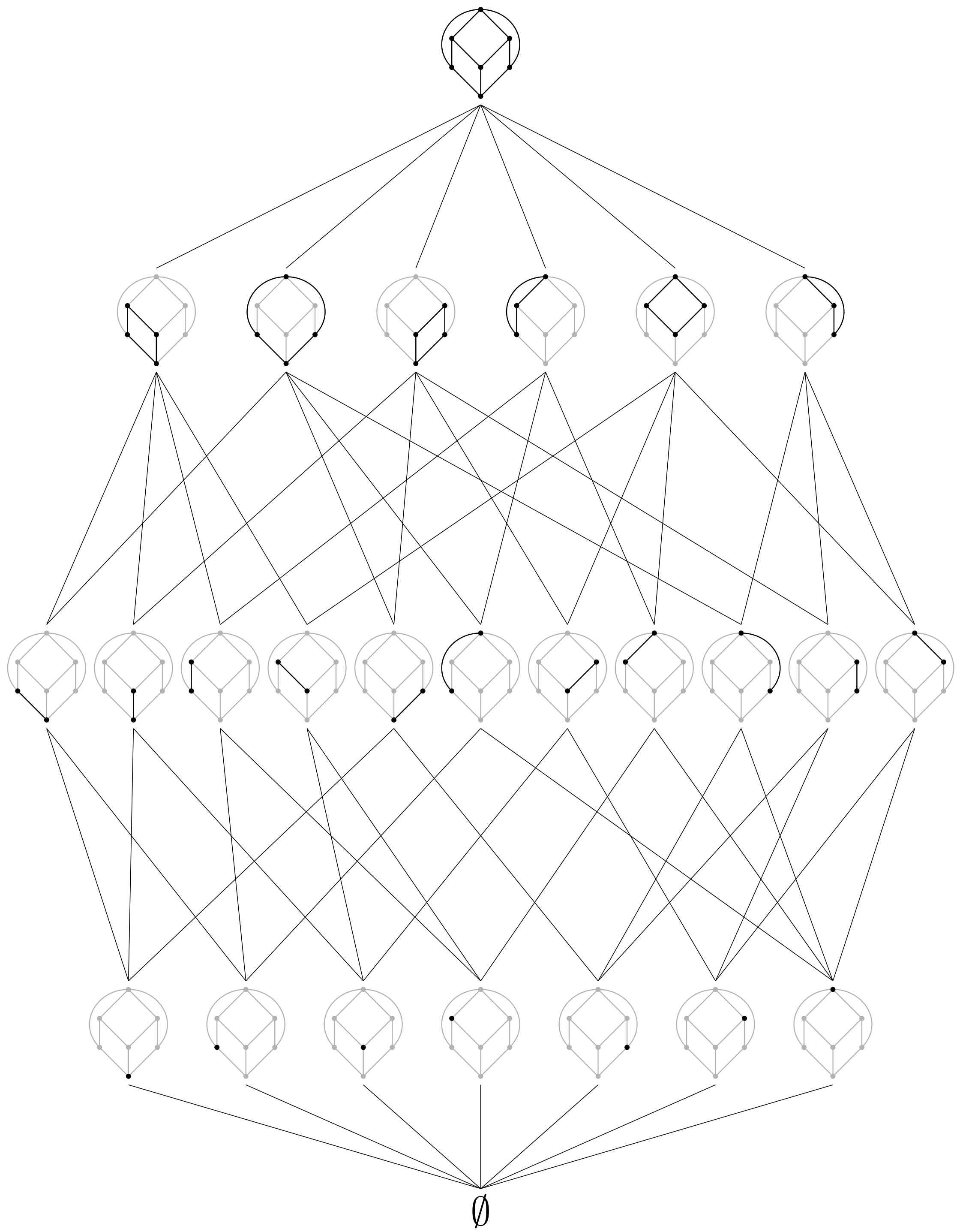


Figure A17: The minor poset of the generator-enriched lattice depicted in Figure A15 (2).

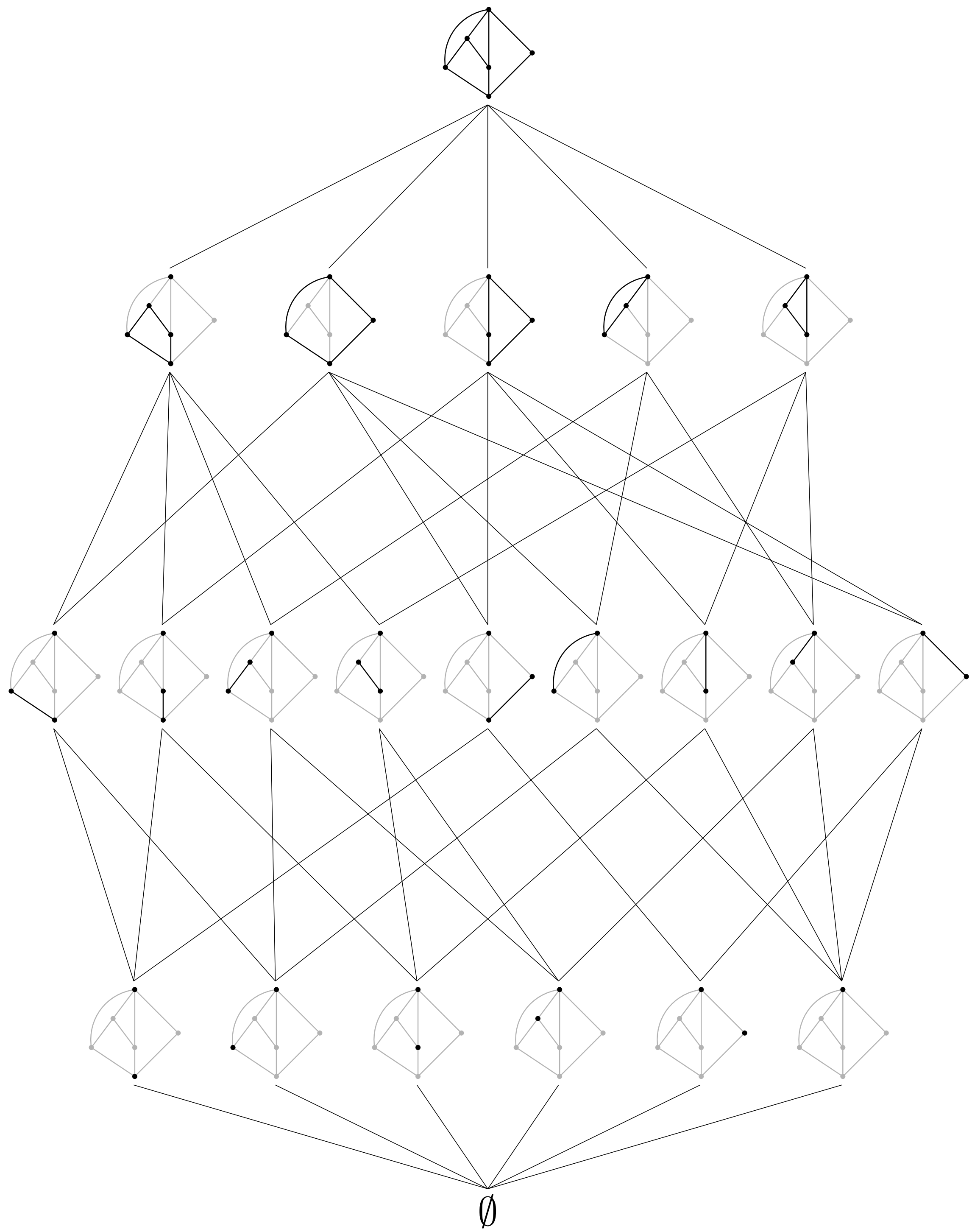


Figure A18: The minor poset of the generator-enriched lattice depicted in Figure A15 (3).

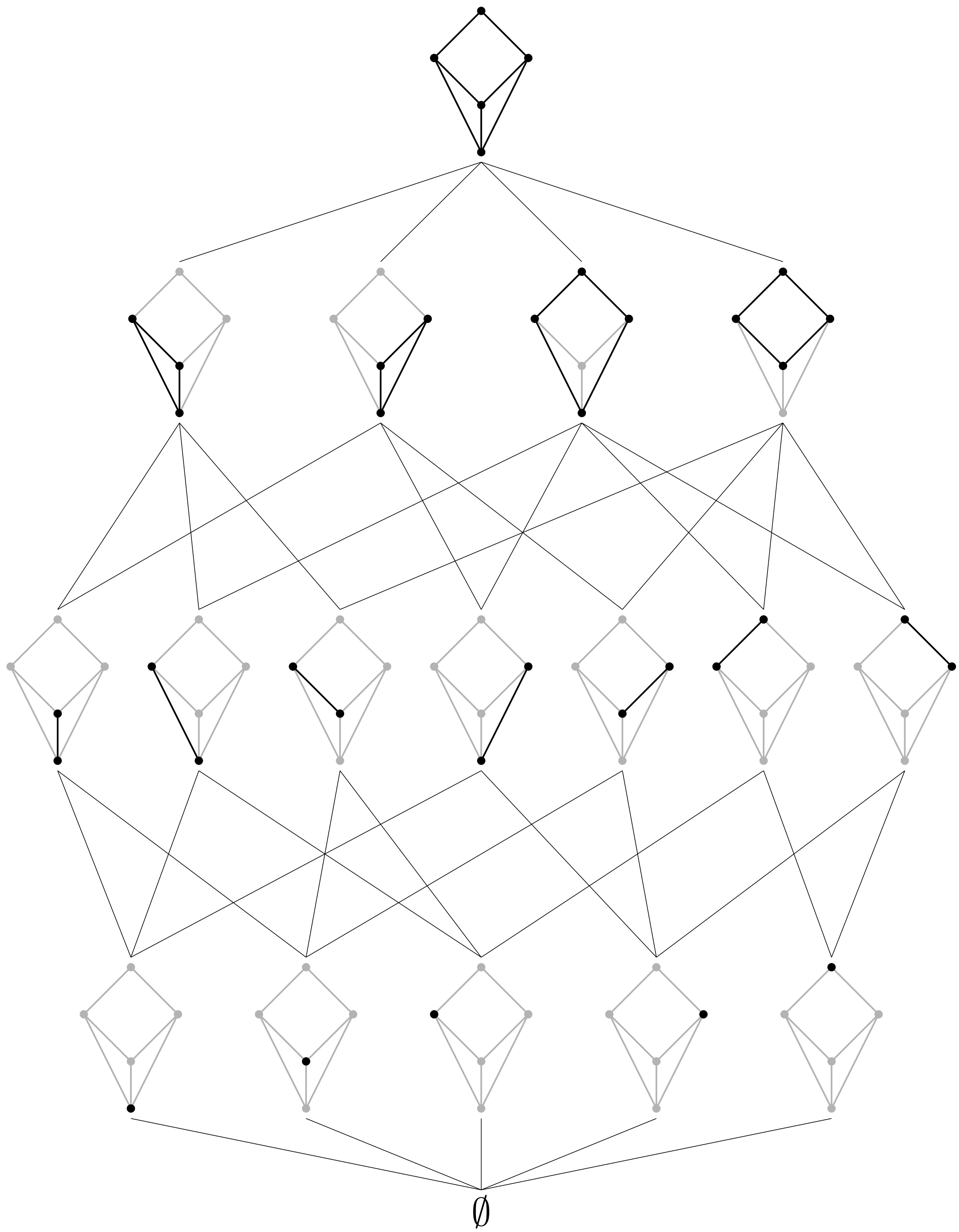


Figure A19: The minor poset of the generator-enriched lattice depicted in Figure A15 (4).

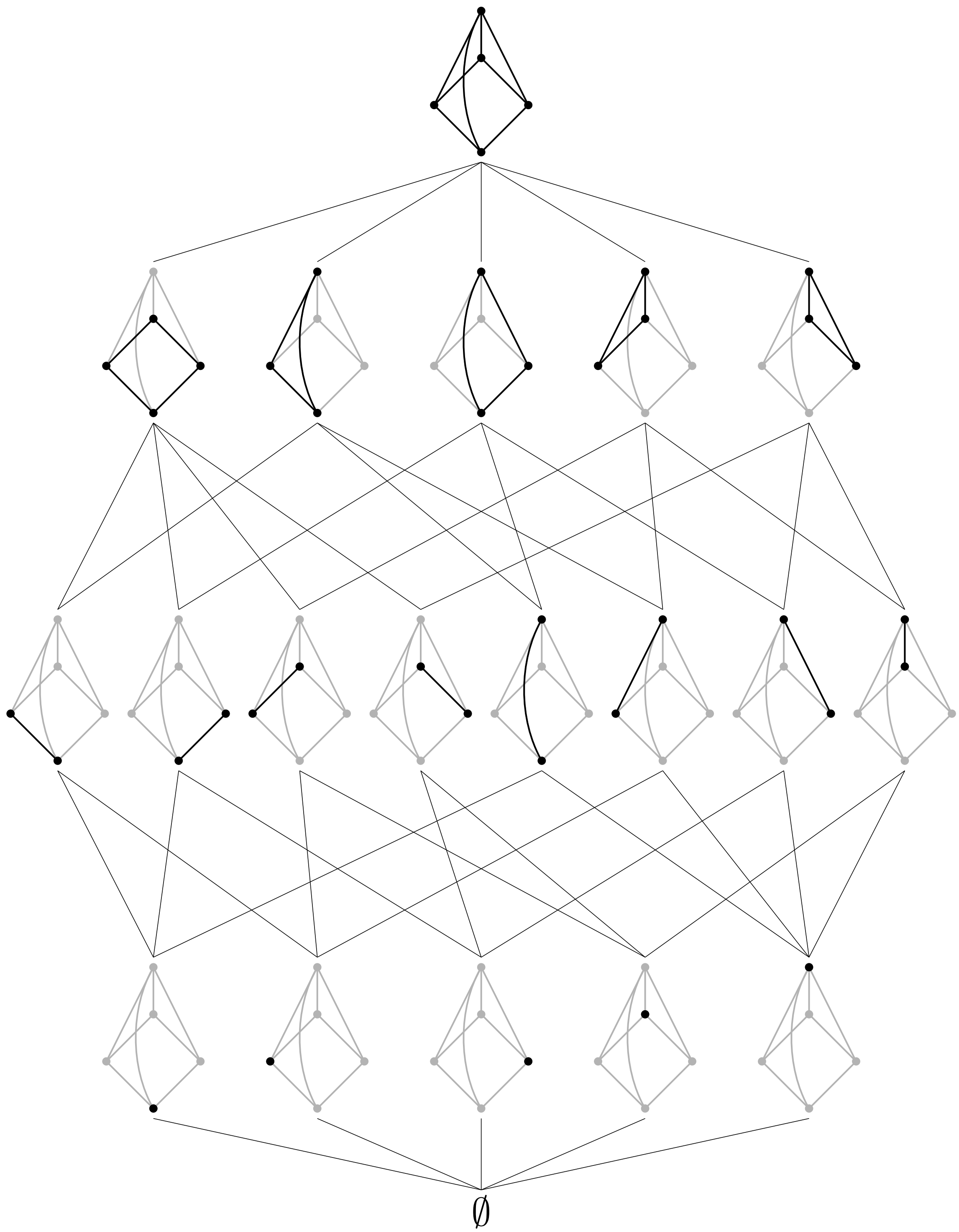


Figure A20: The minor poset of the generator-enriched lattice depicted in Figure A15 (5).

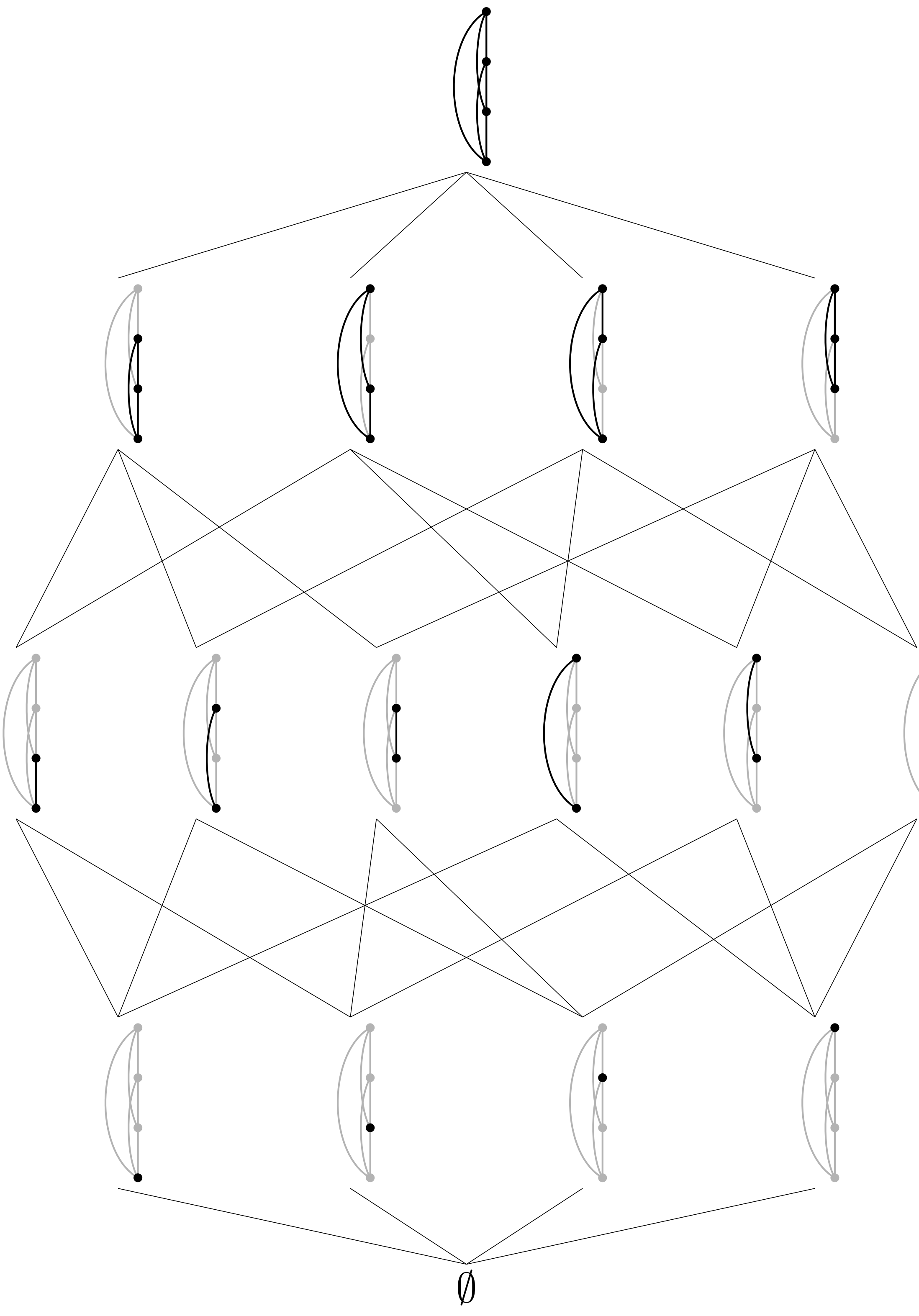


Figure A21: The minor poset of the generator-enriched lattice depicted in Figure A15 (6).

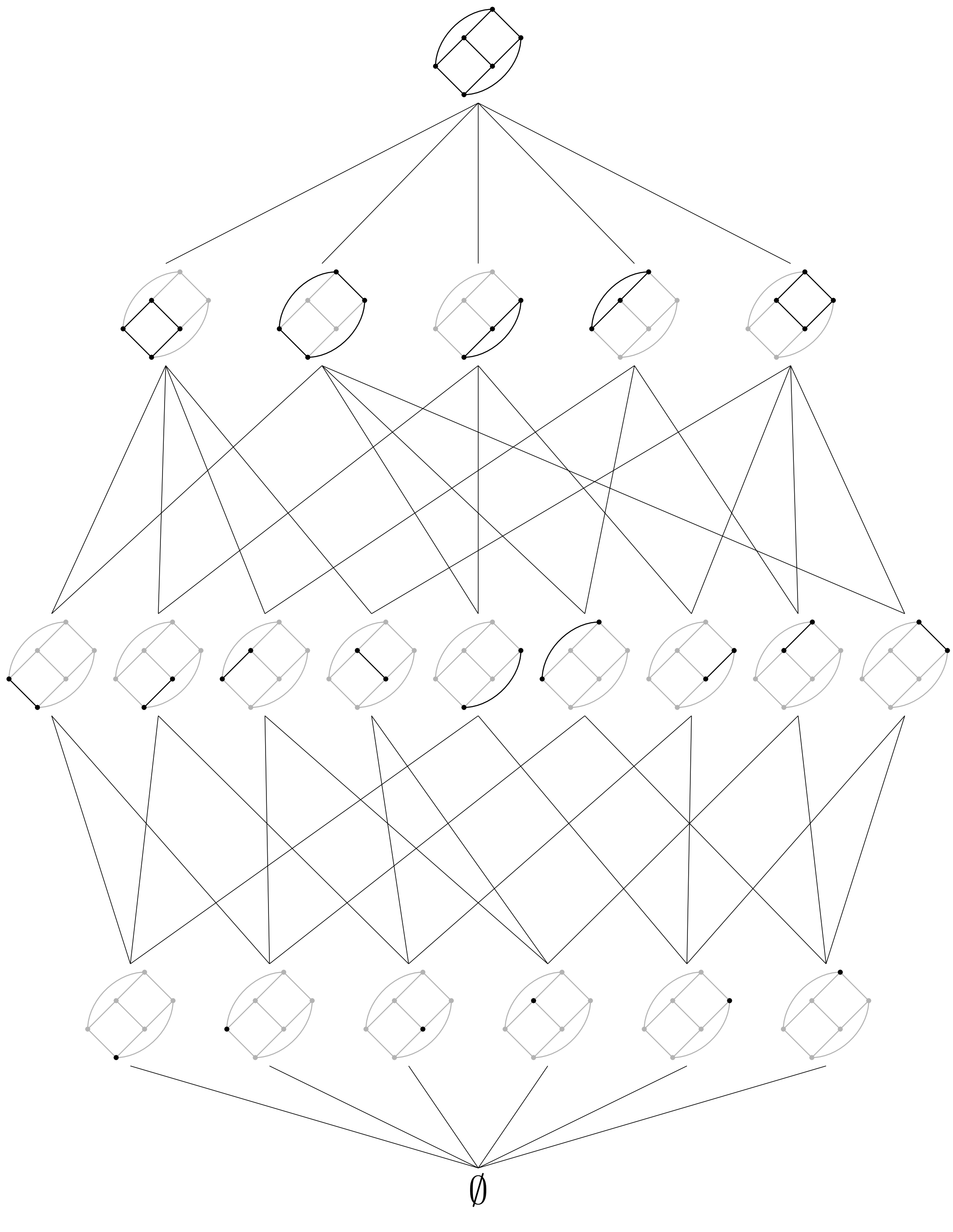


Figure A22: The minor poset of the generator-enriched lattice depicted in Figure A15 (7).

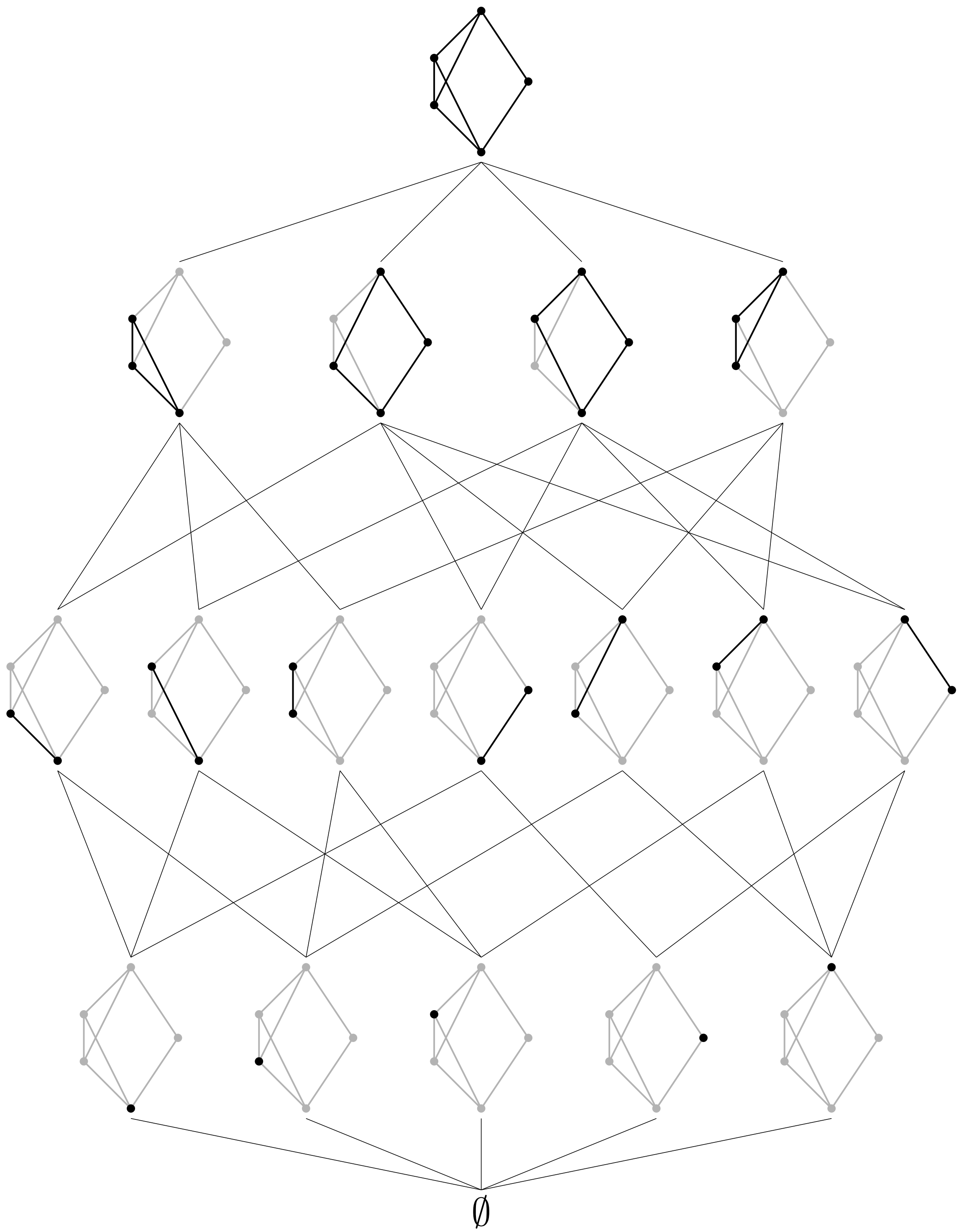


Figure A23: The minor poset of the generator-enriched lattice depicted in Figure A15 (8).

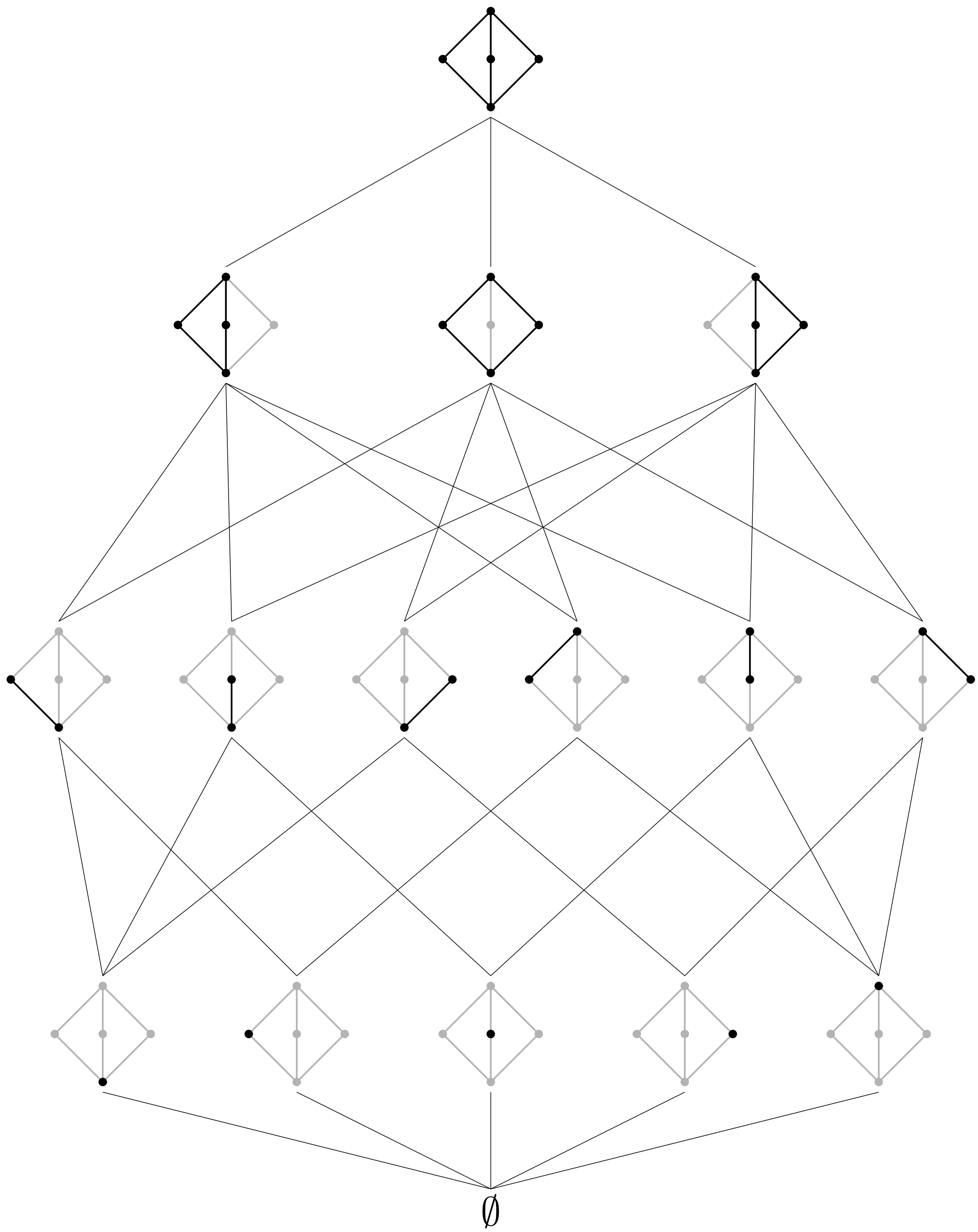


Figure A24: The minor poset of the generator-enriched lattice depicted in Figure A15 (9).

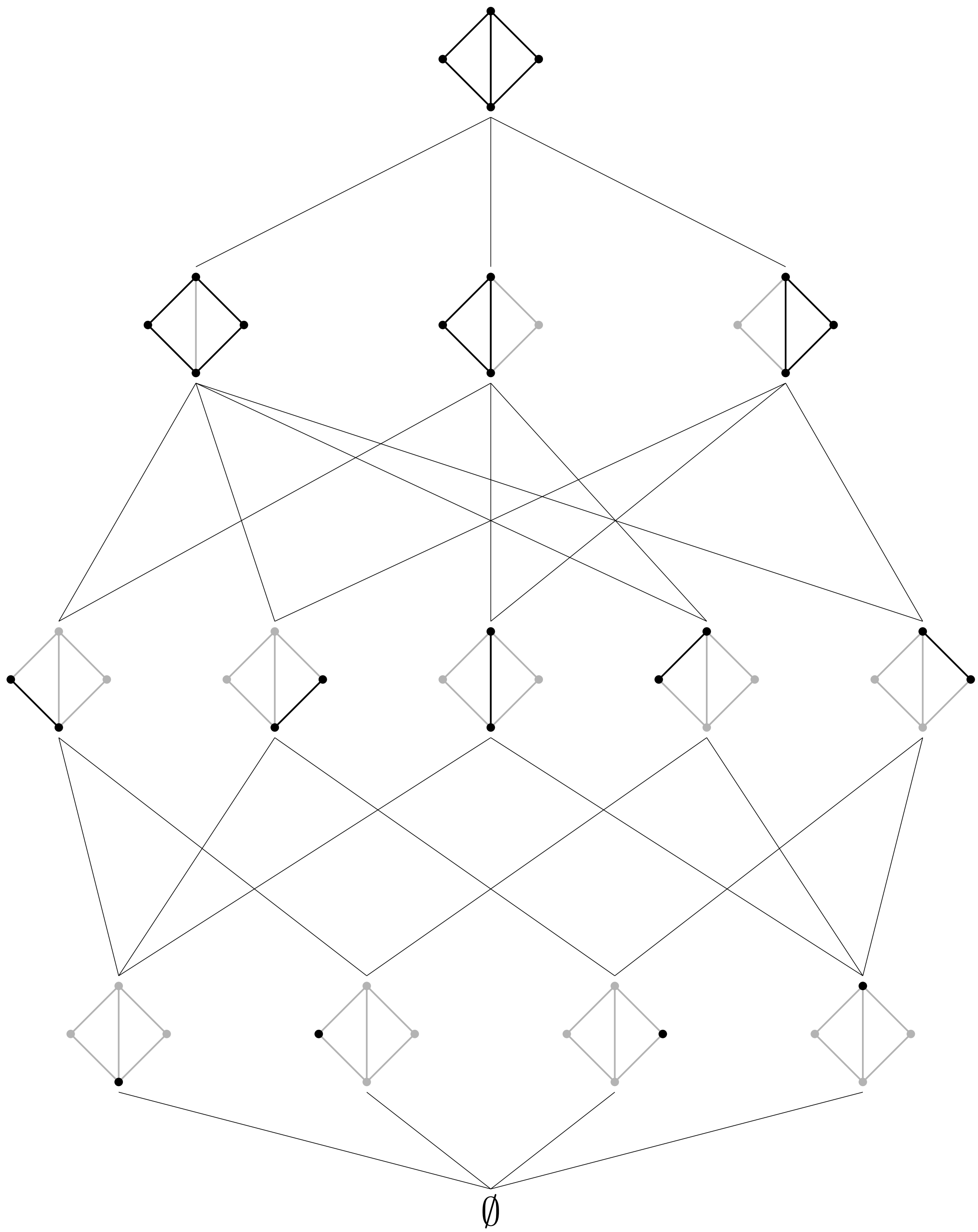


Figure A25: The minor poset of the generator-enriched lattice depicted in Figure A15 (10).

Weak minor posets

In this section we list the weak minor posets of all 10 generator-enriched lattices with 3 generators as well as the corresponding order complexes. See Chapter 5 for details about weak minor posets.

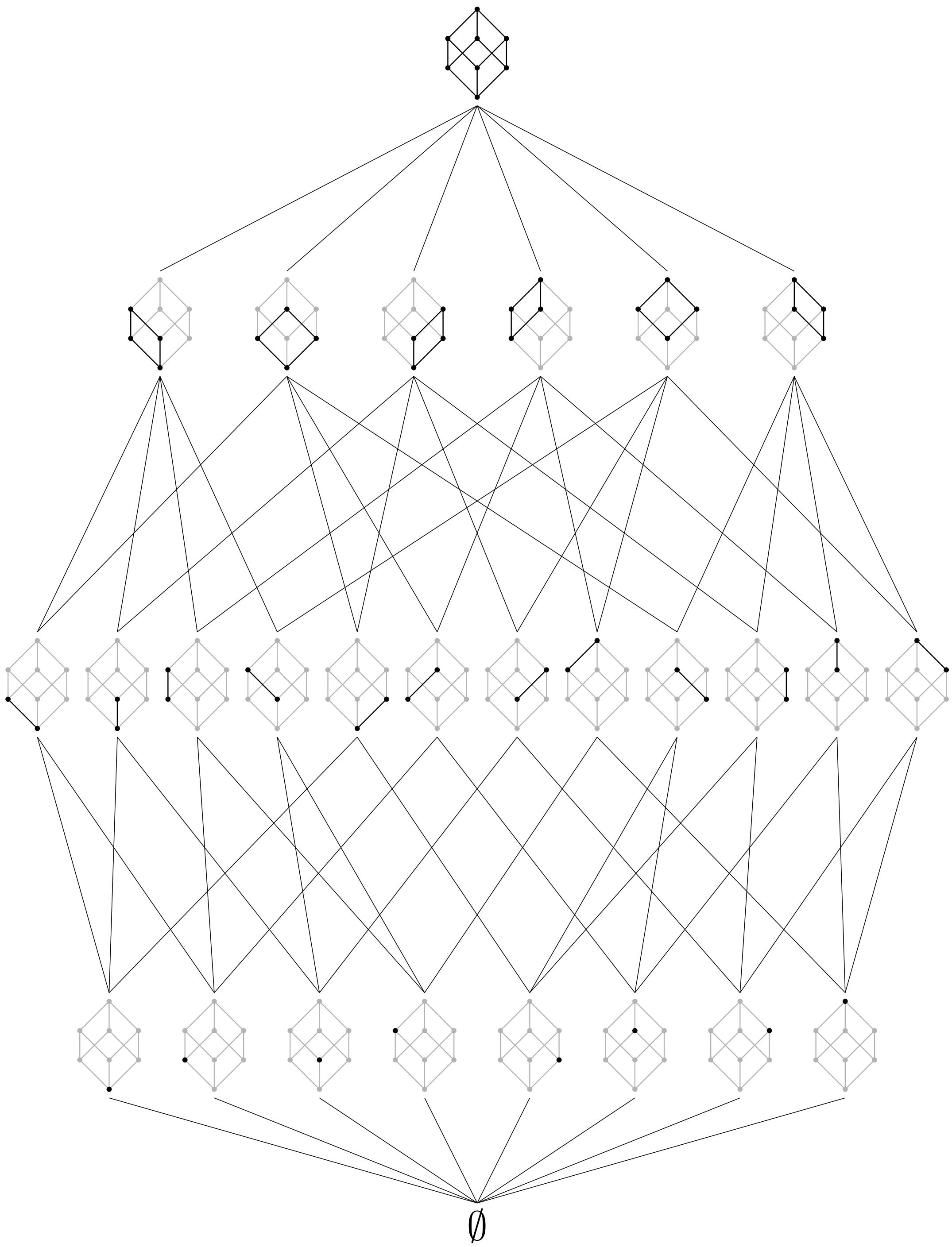


Figure A26: The weak minor poset of the generator-enriched lattice depicted in Figure A15 (1).

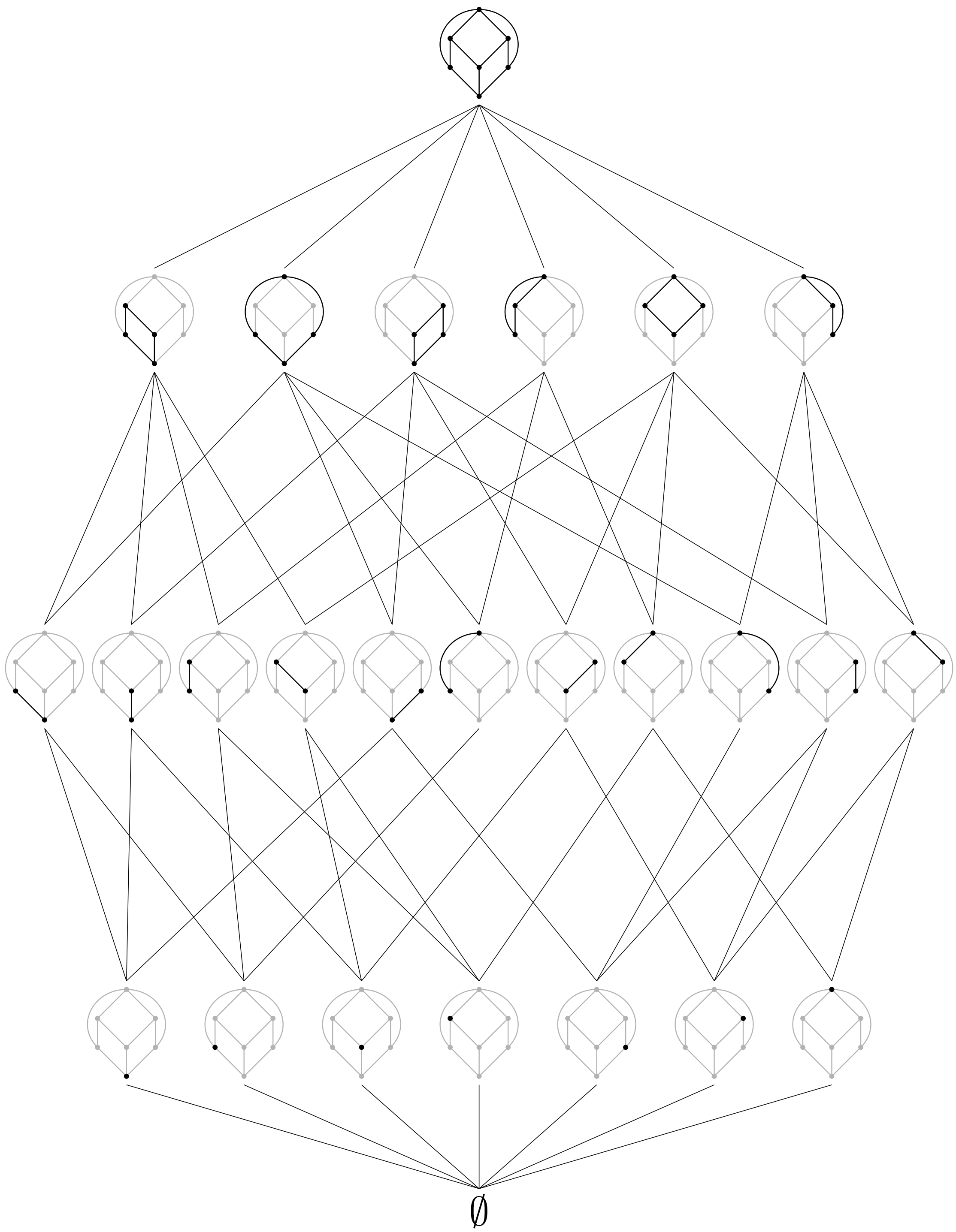


Figure A27: The weak minor poset of the generator-enriched lattice depicted in Figure A15 (2).

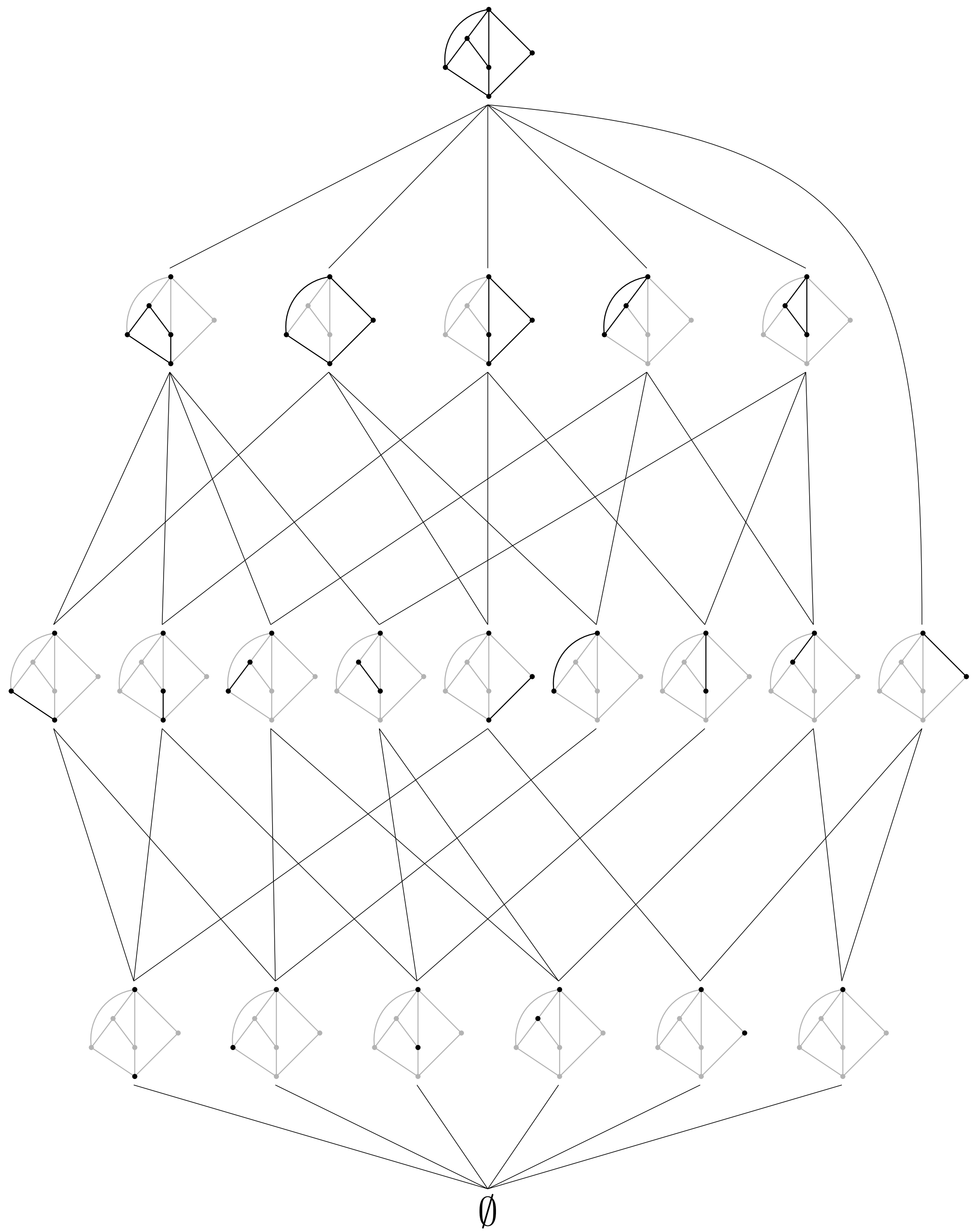


Figure A28: The weak minor poset of the generator-enriched lattice depicted in Figure A15 (3).

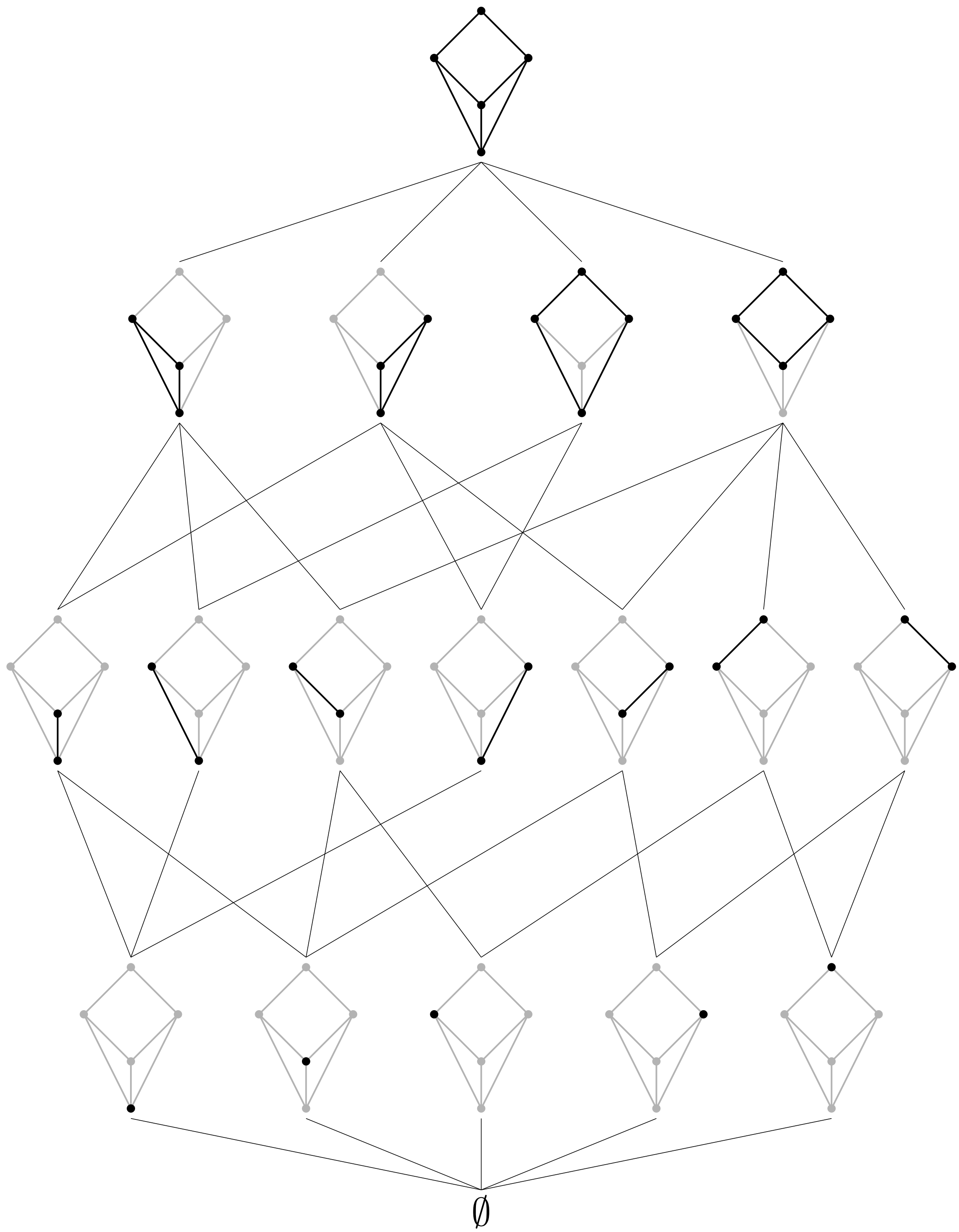


Figure A29: The weak minor poset of the generator-enriched lattice depicted in Figure A15 (4).

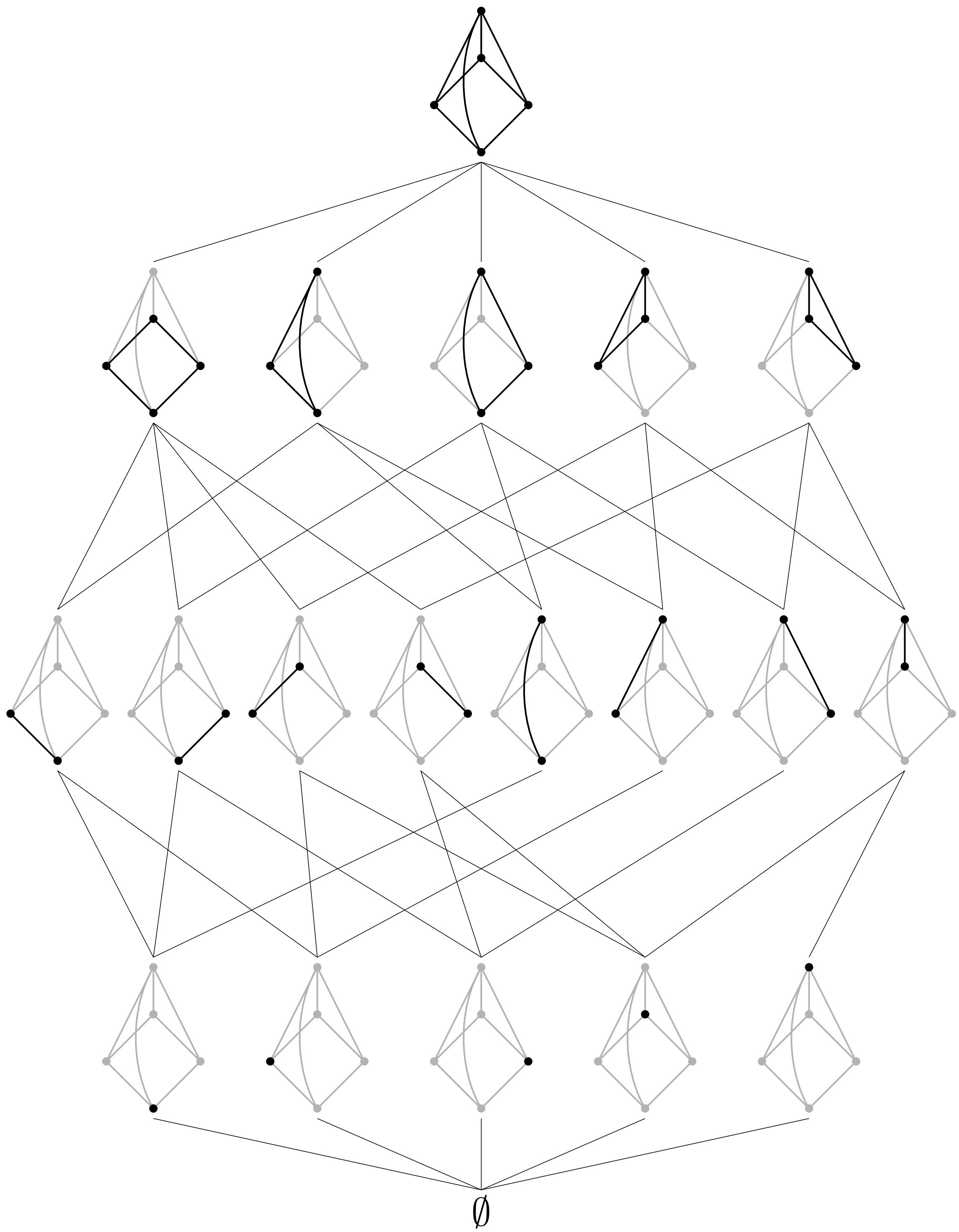


Figure A30: The weak minor poset of the generator-enriched lattice depicted in Figure A15 (5).

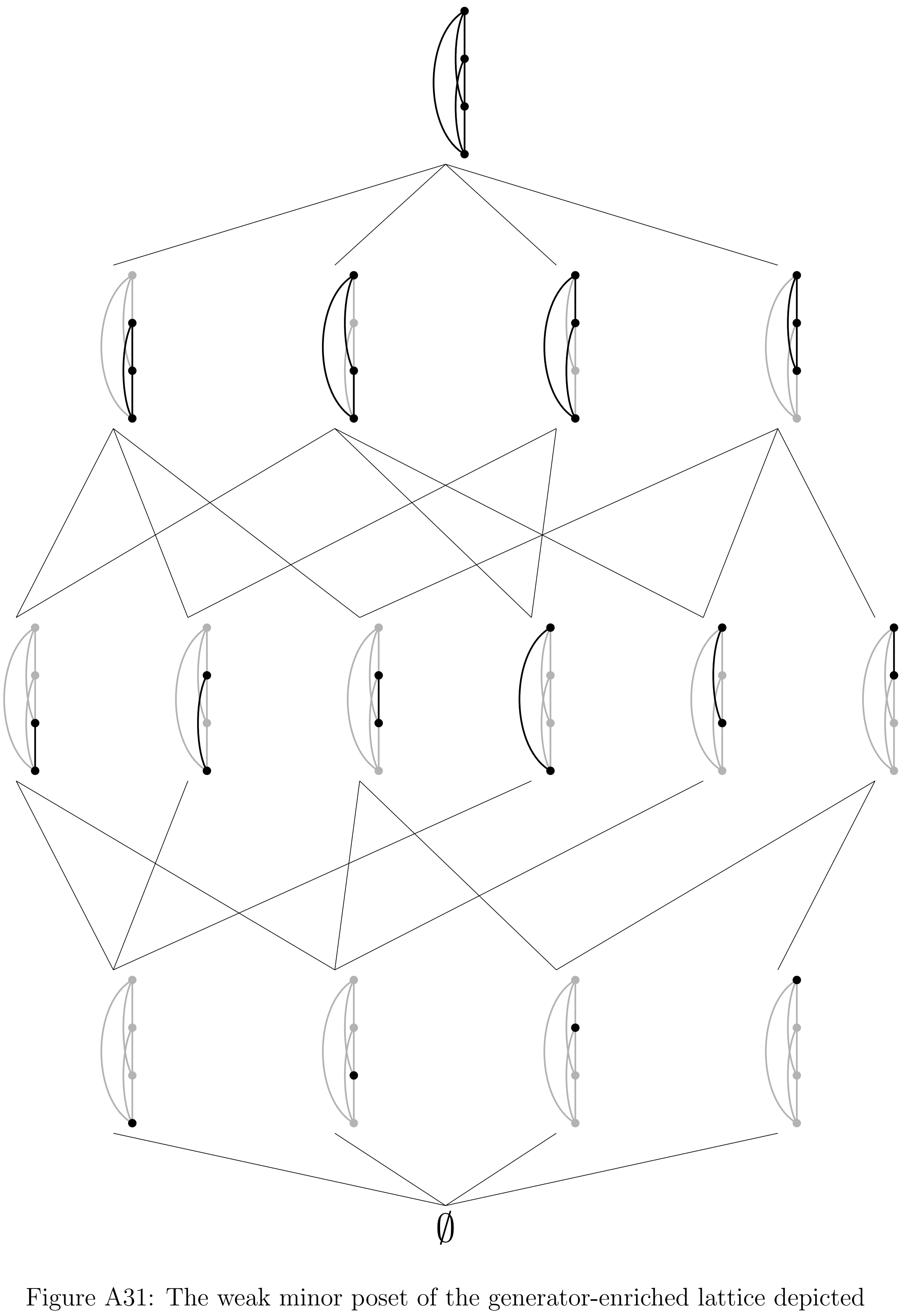


Figure A31: The weak minor poset of the generator-enriched lattice depicted in Figure A15 (6).

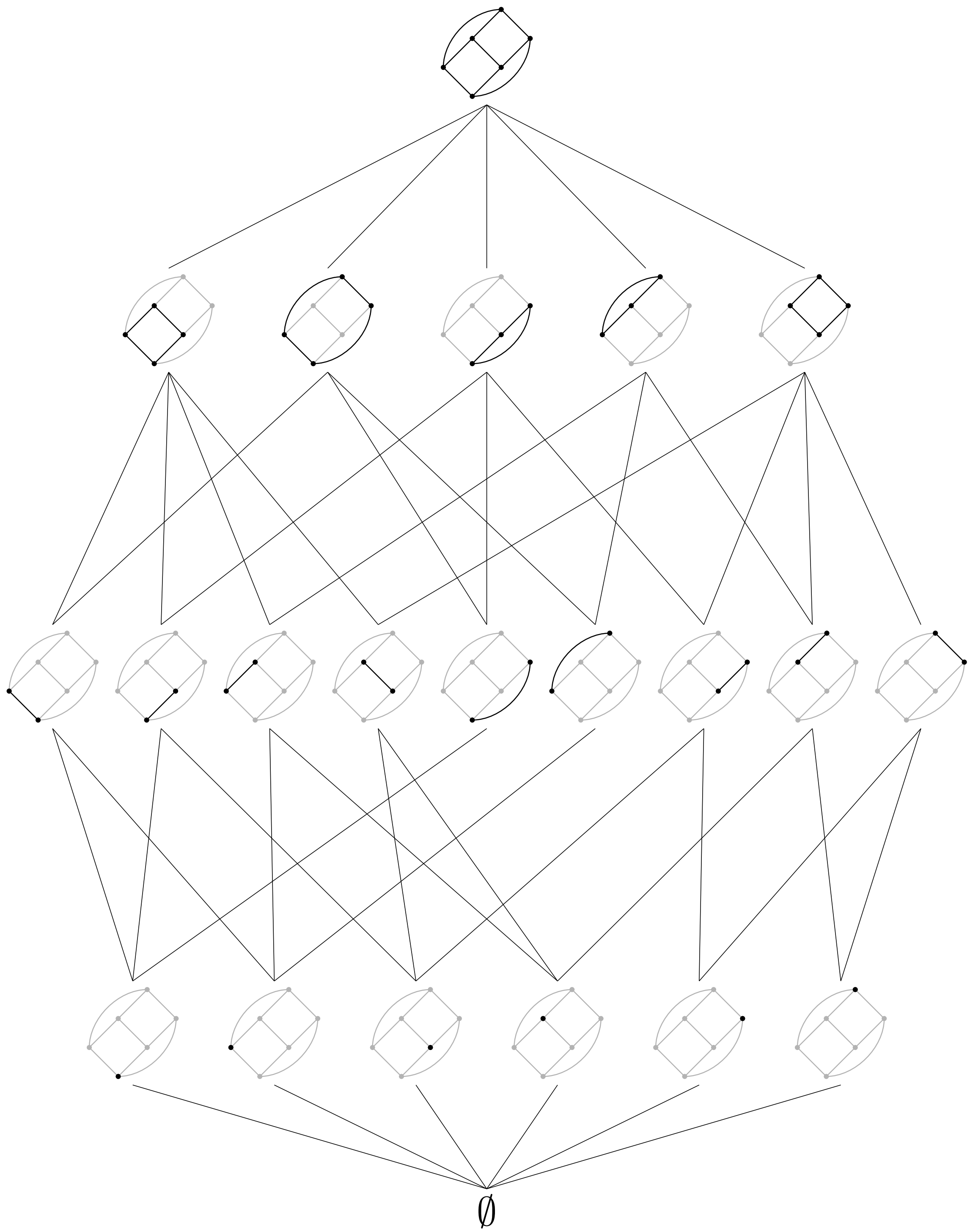


Figure A32: The weak minor poset of the generator-enriched lattice depicted in Figure A15 (7).

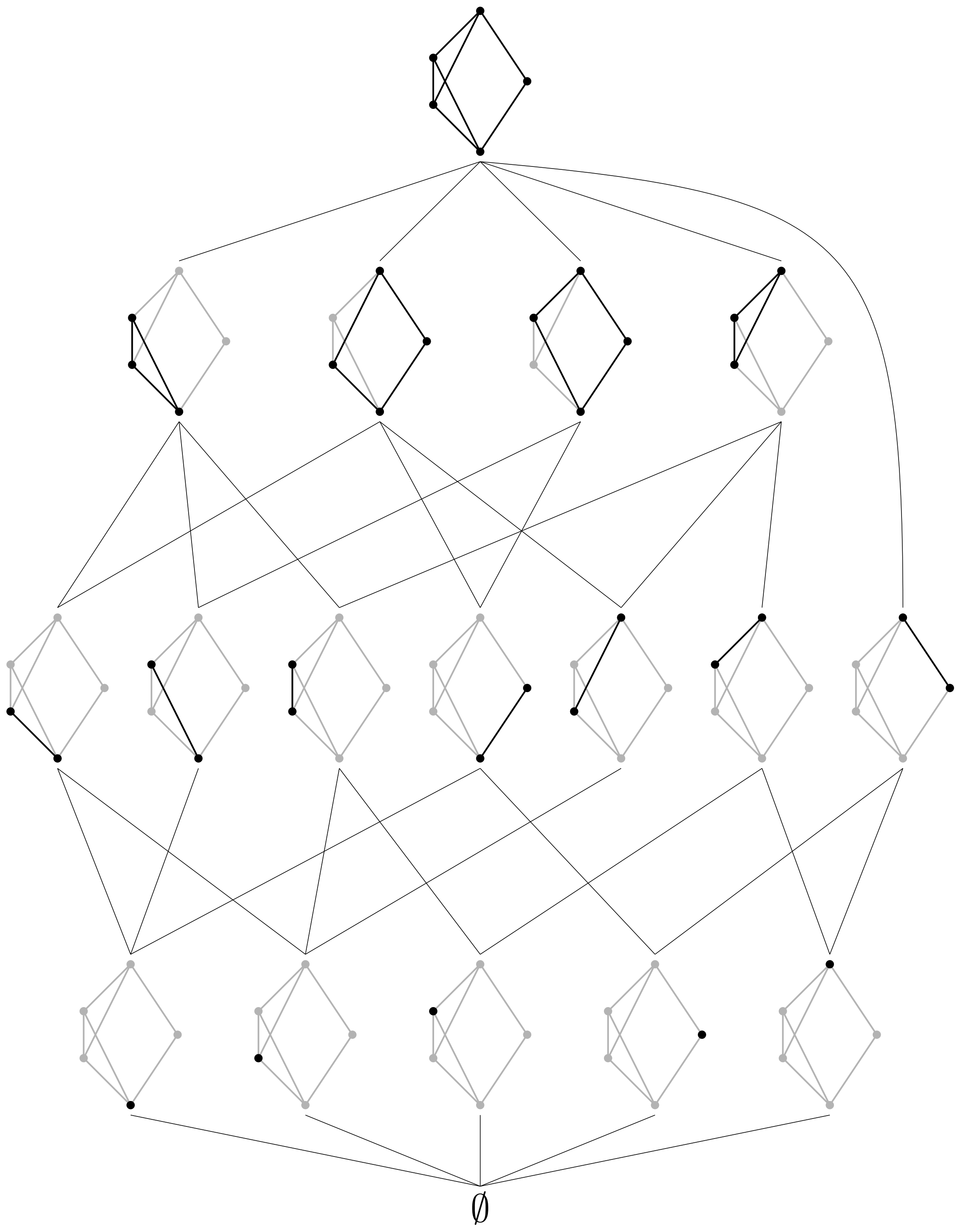


Figure A33: The weak minor poset of the generator-enriched lattice depicted in Figure A15 (8).

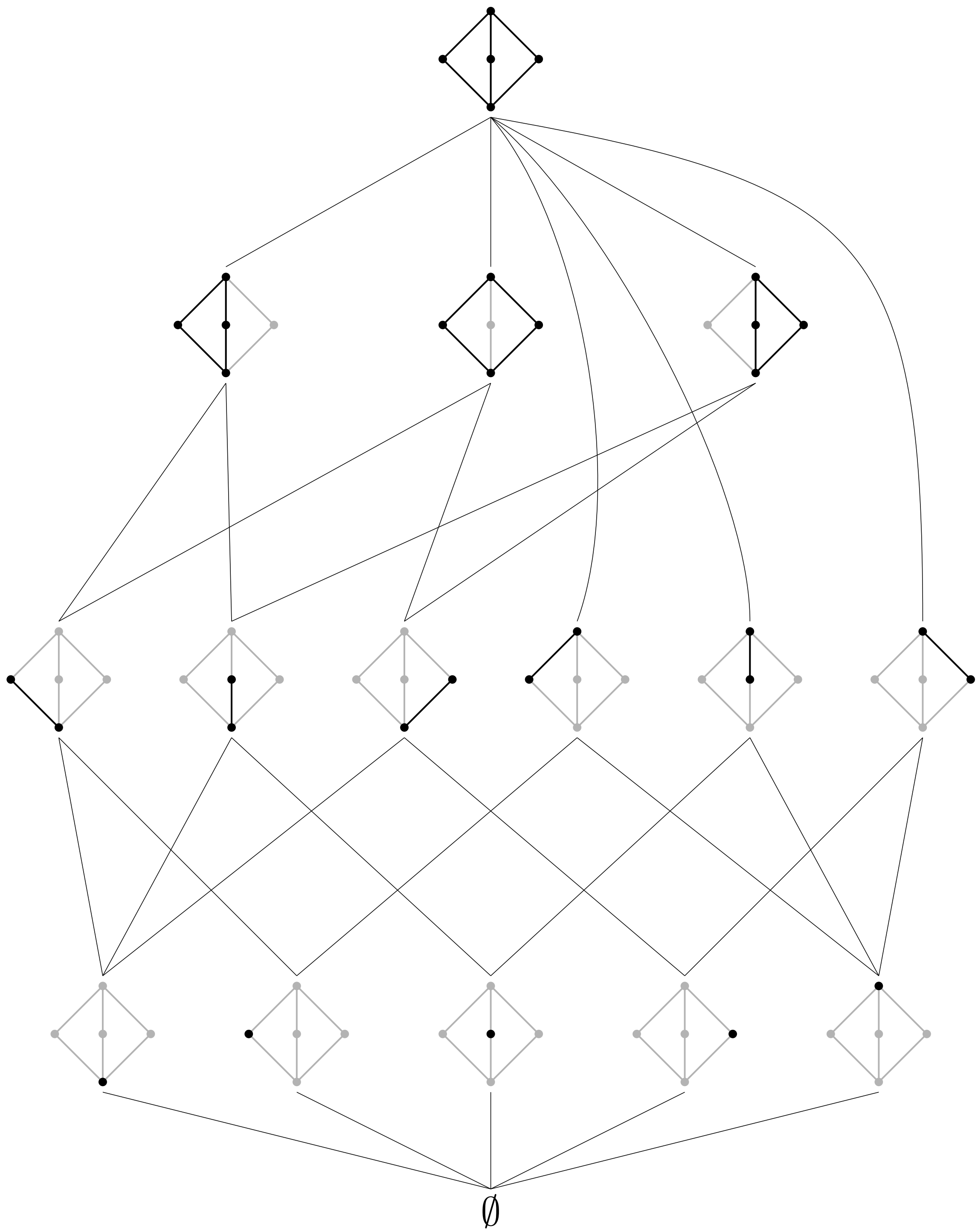


Figure A34: The weak minor poset of the generator-enriched lattice depicted in Figure A15 (9).

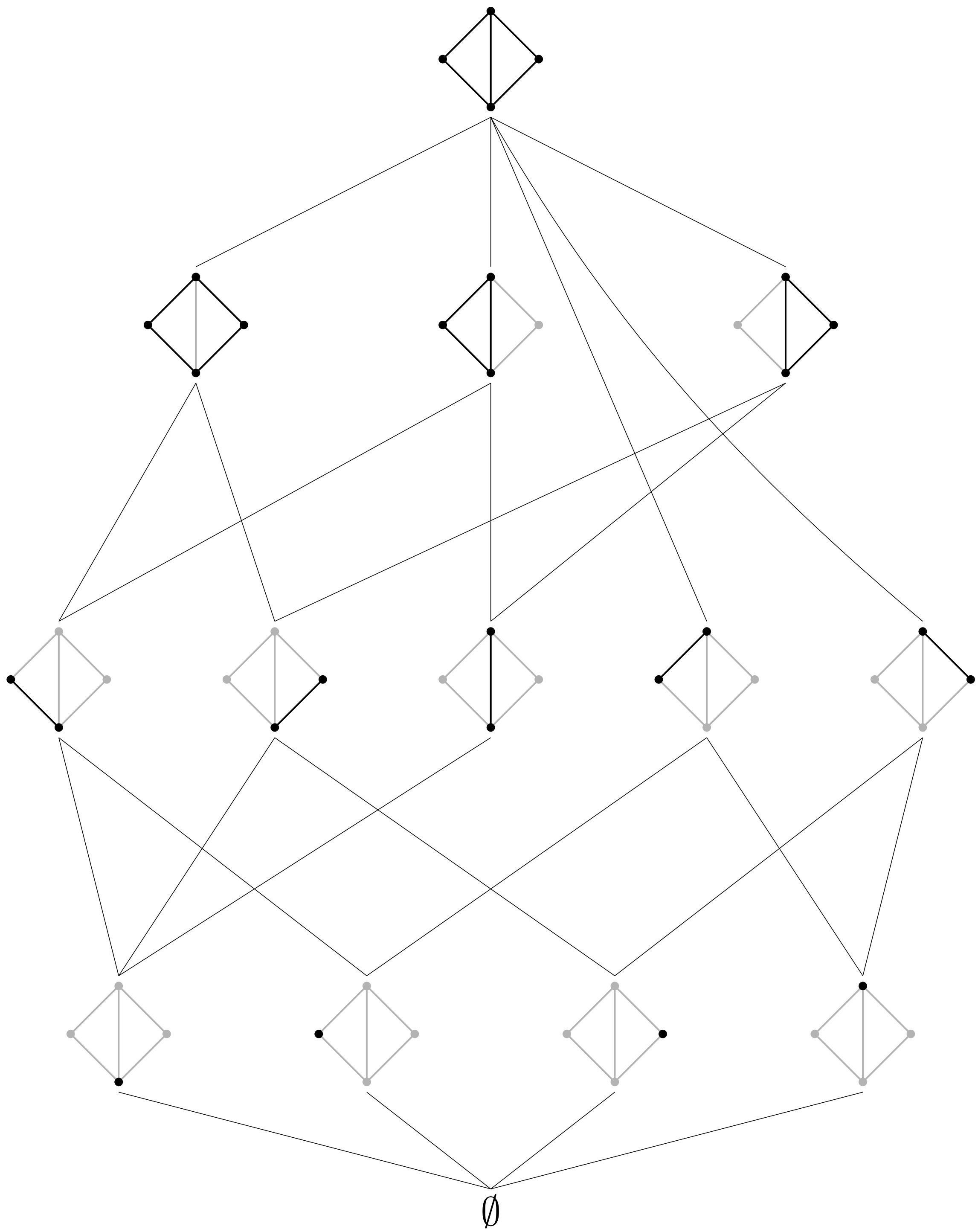
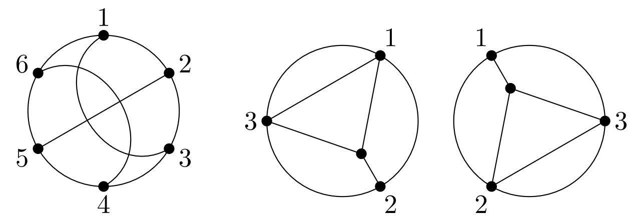


Figure A35: The weak minor poset of the generator-enriched lattice depicted in Figure A15 (10).

Appendix B: cd-index data

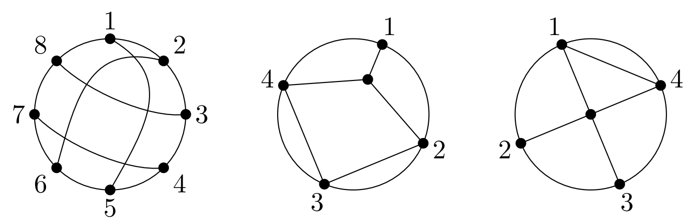
In this appendix we list some data in support of Conjecture 6.1.3. For brevity given a graph G we denote by $\mathrm{M}(G)$ the minor poset $\mathrm{M}(L, \mathrm{irr}(L))$ where L is the lattice of flats of G.

$M = M(\tau)$ $\qquad$ $G = G(M)$ $\qquad$ $G^* = G^*(M)$



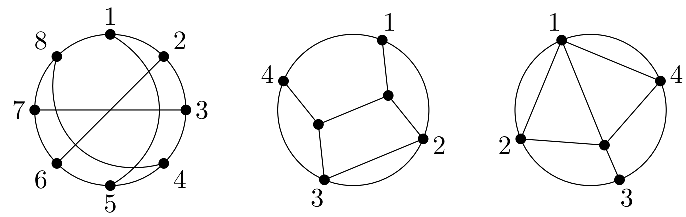
$$\Psi([\widehat{0}, \tau]) = \mathbf{c}^3 + \mathbf{cd} + 3\mathbf{dc}$$
$$\Psi(\mathrm{M}(G)) = \mathbf{c}^4 + 3\mathbf{c}^2\mathbf{d} + 7\mathbf{cdc} + 8\mathbf{dc}^2 + 8\mathbf{d}^2$$
$$\Psi(\mathrm{M}(G^*)) = \mathbf{c}^4 + 3\mathbf{c}^2\mathbf{d} + 7\mathbf{cdc} + 8\mathbf{dc}^2 + 8\mathbf{d}^2$$

$M = M(\tau)$ $\qquad$ $G = G(M)$ $\qquad$ $G^* = G^*(M)$



$$\Psi([\widehat{0}, \tau]) = \mathbf{c}^5 + 3\mathbf{c}^3\mathbf{d} + 9\mathbf{c}^2\mathbf{dc} + 12\mathbf{cdc}^2 + 14\mathbf{cd}^2 + 11\mathbf{dc}^3 + 20\mathbf{dcd} + 24\mathbf{d}^2\mathbf{c}$$
$$\Psi(\mathrm{M}(G)) = \mathbf{c}^5 + 8\mathbf{c}^3\mathbf{d} + 24\mathbf{c}^2\mathbf{dc} + 32\mathbf{cdc}^2 + 48\mathbf{cd}^2 + 22\mathbf{dc}^3 + 64\mathbf{dcd} + 72\mathbf{d}^2\mathbf{c}$$
$$\Psi(\mathrm{M}(G^*)) = \mathbf{c}^5 + 5\mathbf{c}^3\mathbf{d} + 15\mathbf{c}^2\mathbf{dc} + 24\mathbf{cdc}^2 + 28\mathbf{cd}^2 + 18\mathbf{dc}^3 + 38\mathbf{dcd} + 46\mathbf{d}^2\mathbf{c}$$

$M = M(\tau)$ $\qquad G = G(M)$ $\qquad G^* = G^*(M)$

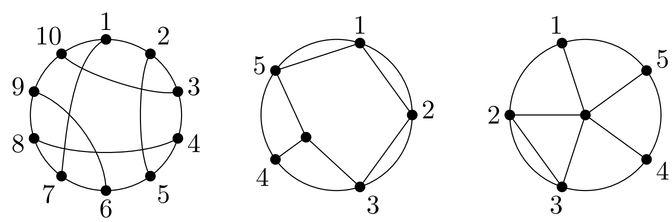


$$\Psi([\widehat{0},\tau]) = \mathbf{c}^6 + 2\mathbf{c}^4\mathbf{d} + 6\mathbf{c}^3\mathbf{dc} + 12\mathbf{c}^2\mathbf{dc}^2 + 12\mathbf{c}^2\mathbf{d}^2 + 14\mathbf{cdc}^3 + 20\mathbf{cdcd}$$
$$+ 28\mathbf{cd}^2\mathbf{c} + 12\mathbf{dc}^4 + 20\mathbf{dc}^2\mathbf{d} + 40\mathbf{dcdc} + 32\mathbf{d}^2\mathbf{c}^2 + 32\mathbf{d}^3$$

$$\Psi(\mathrm{M}(G)) = \mathbf{c}^6 + 10\mathbf{c}^4\mathbf{d} + 42\mathbf{c}^3\mathbf{dc} + 82\mathbf{c}^2\mathbf{dc}^2 + 112\mathbf{c}^2\mathbf{d}^2 + 88\mathbf{cdc}^3 + 240\mathbf{cdcd}$$
$$+ 280\mathbf{cd}^2\mathbf{c} + 46\mathbf{dc}^4 + 188\mathbf{dc}^2\mathbf{d} + 364\mathbf{dcdc} + 252\mathbf{d}^2\mathbf{c}^2 + 368\mathbf{d}^3$$

$$\Psi(\mathrm{M}(G^*)) = \mathbf{c}^6 + 5\mathbf{c}^4\mathbf{d} + 17\mathbf{c}^3\mathbf{dc} + 32\mathbf{c}^2\mathbf{dc}^2 + 38\mathbf{c}^2\mathbf{d}^2 + 37\mathbf{cdc}^3 + 78\mathbf{cdcd}$$
$$+ 94\mathbf{cd}^2\mathbf{c} + 24\mathbf{dc}^4 + 68\mathbf{dc}^2\mathbf{d} + 128\mathbf{dcdc} + 94\mathbf{d}^2\mathbf{c}^2 + 116\mathbf{d}^3$$

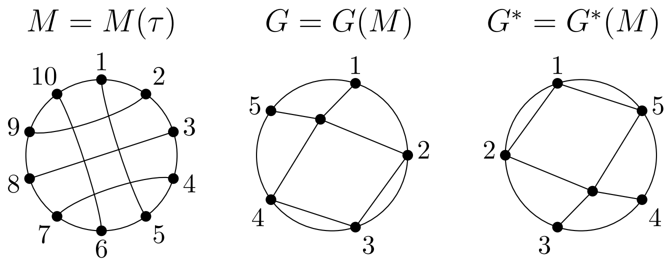
$M = M(\tau)$ $\qquad G = G(M)$ $\qquad G^* = G^*(M)$



$$\Psi([\widehat{0},\tau]) = \mathbf{c}^6 + 7\mathbf{c}^4\mathbf{d} + 24\mathbf{c}^3\mathbf{dc} + 46\mathbf{c}^2\mathbf{dc}^2 + 58\mathbf{c}^2\mathbf{d}^2 + 48\mathbf{cdc}^3 + 114\mathbf{cdcd}$$
$$+ 138\mathbf{cd}^2\mathbf{c} + 29\mathbf{dc}^4 + 98\mathbf{dc}^2\mathbf{d} + 186\mathbf{dcdc} + 134\mathbf{d}^2\mathbf{c}^2 + 176\mathbf{d}^3$$

$$\Psi(\mathrm{M}(G)) = \mathbf{c}^6 + 10\mathbf{c}^4\mathbf{d} + 48\mathbf{c}^3\mathbf{dc} + 100\mathbf{c}^2\mathbf{dc}^2 + 130\mathbf{c}^2\mathbf{d}^2 + 103\mathbf{cdc}^3 + 270\mathbf{cdcd}$$
$$+ 334\mathbf{cd}^2\mathbf{c} + 52\mathbf{dc}^4 + 206\mathbf{dc}^2\mathbf{d} + 430\mathbf{dcdc} + 306\mathbf{d}^2\mathbf{c}^2 + 428\mathbf{d}^3$$

$$\Psi(\mathrm{M}(G^*)) = \mathbf{c}^6 + 7\mathbf{c}^4\mathbf{d} + 27\mathbf{c}^3\mathbf{dc} + 56\mathbf{c}^2\mathbf{dc}^2 + 68\mathbf{c}^2\mathbf{d}^2 + 68\mathbf{cdc}^3 + 152\mathbf{cdcd}$$
$$+ 180\mathbf{cd}^2\mathbf{c} + 38\mathbf{dc}^4 + 122\mathbf{dc}^2\mathbf{d} + 234\mathbf{dcdc} + 176\mathbf{d}^2\mathbf{c}^2 + 224\mathbf{d}^3$$

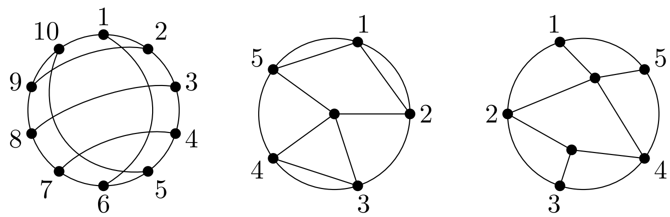


$$\Psi([\widehat{0},\tau]) = \mathbf{c}^6 + 10\mathbf{c}^4\mathbf{d} + 32\mathbf{c}^3\mathbf{dc} + 54\mathbf{c}^2\mathbf{dc}^2 + 84\mathbf{c}^2\mathbf{d}^2 + 53\mathbf{cdc}^3 + 166\mathbf{cdcd} + 180\mathbf{cd}^2\mathbf{c} + 31\mathbf{dc}^4 + 142\mathbf{dc}^2\mathbf{d} + 244\mathbf{dcdc} + 156\mathbf{d}^2\mathbf{c}^2 + 256\mathbf{d}^3$$

$$\Psi(\mathrm{M}(G)) = \mathbf{c}^6 + 10\mathbf{c}^4\mathbf{d} + 42\mathbf{c}^3\mathbf{dc} + 82\mathbf{c}^2\mathbf{dc}^2 + 112\mathbf{c}^2\mathbf{d}^2 + 88\mathbf{cdc}^3 + 240\mathbf{cdcd} + 280\mathbf{cd}^2\mathbf{c} + 46\mathbf{dc}^4 + 188\mathbf{dc}^2\mathbf{d} + 364\mathbf{dcdc} + 252\mathbf{d}^2\mathbf{c}^2 + 368\mathbf{d}^3$$

$$\Psi(\mathrm{M}(G^*)) = \mathbf{c}^6 + 10\mathbf{c}^4\mathbf{d} + 42\mathbf{c}^3\mathbf{dc} + 82\mathbf{c}^2\mathbf{dc}^2 + 112\mathbf{c}^2\mathbf{d}^2 + 88\mathbf{cdc}^3 + 240\mathbf{cdcd} + 280\mathbf{cd}^2\mathbf{c} + 46\mathbf{dc}^4 + 188\mathbf{dc}^2\mathbf{d} + 364\mathbf{dcdc} + 252\mathbf{d}^2\mathbf{c}^2 + 368\mathbf{d}^3$$

$M = M(\tau)$ $\qquad$ $G = G(M)$ $\qquad$ $G^* = G^*(M)$

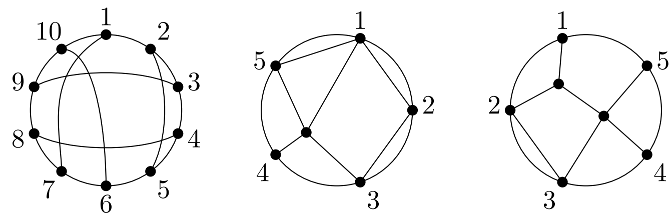


$$\begin{aligned}\Psi([\widehat{0},\tau]) = {} & \mathbf{c}^7 + 5\mathbf{c}^5\mathbf{d} + 19\mathbf{c}^4\mathbf{dc} + 41\mathbf{c}^3\mathbf{dc}^2 + 48\mathbf{c}^3\mathbf{d}^2 + 60\mathbf{c}^2\mathbf{dc}^3 + 120\mathbf{c}^2\mathbf{dcd} \\ & + 150\mathbf{c}^2\mathbf{d}^2\mathbf{c} + 56\mathbf{cdc}^4 + 159\mathbf{cdc}^2\mathbf{d} + 313\mathbf{cdcdc} + 225\mathbf{cd}^2\mathbf{c}^2 + 274\mathbf{cd}^3 \\ & + 32\mathbf{dc}^5 + 109\mathbf{dc}^3\mathbf{d} + 269\mathbf{dc}^2\mathbf{dc} + 307\mathbf{dcdc}^2 + 370\mathbf{dcd}^2 + 172\mathbf{d}^2\mathbf{c}^3 \\ & + 360\mathbf{d}^2\mathbf{cd} + 464\mathbf{d}^3\mathbf{c}\end{aligned}$$

$$\begin{aligned}\Psi(\mathrm{M}(G)) = {} & \mathbf{c}^7 + 9\mathbf{c}^5\mathbf{d} + 37\mathbf{c}^4\mathbf{dc} + 90\mathbf{c}^3\mathbf{dc}^2 + 114\mathbf{c}^3\mathbf{d}^2 + 134\mathbf{c}^2\mathbf{dc}^3 + 318\mathbf{c}^2\mathbf{dcd} \\ & + 368\mathbf{c}^2\mathbf{d}^2\mathbf{c} + 122\mathbf{cdc}^4 + 432\mathbf{cdc}^2\mathbf{d} + 812\mathbf{cdcdc} + 564\mathbf{cd}^2\mathbf{c}^2 + 768\mathbf{cd}^3 \\ & + 58\mathbf{dc}^5 + 266\mathbf{dc}^3\mathbf{d} + 650\mathbf{dc}^2\mathbf{dc} + 744\mathbf{dcdc}^2 + 996\mathbf{dcd}^2 + 400\mathbf{d}^2\mathbf{c}^3 \\ & + 1004\mathbf{d}^2\mathbf{cd} + 1208\mathbf{d}^3\mathbf{c}\end{aligned}$$

$$\begin{aligned}\Psi(\mathrm{M}(G^*)) = {} & \mathbf{c}^7 + 12\mathbf{c}^5\mathbf{d} + 64\mathbf{c}^4\mathbf{dc} + 168\mathbf{c}^3\mathbf{dc}^2 + 216\mathbf{c}^3\mathbf{d}^2 + 254\mathbf{c}^2\mathbf{dc}^3 + 648\mathbf{c}^2\mathbf{dcd} \\ & + 752\mathbf{c}^2\mathbf{d}^2\mathbf{c} + 224\mathbf{cdc}^4 + 864\mathbf{cdc}^2\mathbf{d} + 1664\mathbf{cdcdc} + 1152\mathbf{cd}^2\mathbf{c}^2 + 1632\mathbf{cd}^3 \\ & + 94\mathbf{dc}^5 + 488\mathbf{dc}^3\mathbf{d} + 1280\mathbf{dc}^2\mathbf{dc} + 1488\mathbf{dcdc}^2 + 2064\mathbf{dcd}^2 + 772\mathbf{d}^2\mathbf{c}^3 \\ & + 2096\mathbf{d}^2\mathbf{cd} + 2528\mathbf{d}^3\mathbf{c}\end{aligned}$$

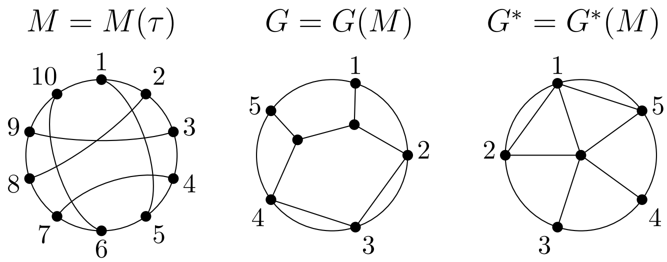
$M = M(\tau)$ $\quad G = G(M)$ $\quad G^* = G^*(M)$



$$\begin{aligned}\Psi([\widehat{0},\tau]) = {} & \mathbf{c}^7 + 7\mathbf{c}^5\mathbf{d} + 24\mathbf{c}^4\mathbf{dc} + 49\mathbf{c}^3\mathbf{dc}^2 + 62\mathbf{c}^3\mathbf{d}^2 + 68\mathbf{c}^2\mathbf{dc}^3 + 162\mathbf{c}^2\mathbf{dcd} \\ & + 188\mathbf{c}^2\mathbf{d}^2\mathbf{c} + 61\mathbf{cdc}^4 + 203\mathbf{cdc}^2\mathbf{d} + 373\mathbf{cdcdc} + 267\mathbf{cd}^2\mathbf{c}^2 + 354\mathbf{cd}^3 \\ & + 34\mathbf{dc}^5 + 141\mathbf{dc}^3\mathbf{d} + 325\mathbf{dc}^2\mathbf{dc} + 365\mathbf{dcdc}^2 + 478\mathbf{dcd}^2 + 194\mathbf{d}^2\mathbf{c}^3 \\ & + 476\mathbf{d}^2\mathbf{cd} + 568\mathbf{d}^3\mathbf{c}\end{aligned}$$

$$\begin{aligned}\Psi(\mathrm{M}(G)) = {} & \mathbf{c}^7 + 7\mathbf{c}^5\mathbf{d} + 28\mathbf{c}^4\mathbf{dc} + 56\mathbf{c}^3\mathbf{dc}^2 + 69\mathbf{c}^3\mathbf{d}^2 + 76\mathbf{c}^2\mathbf{dc}^3 + 183\mathbf{c}^2\mathbf{dcd} \\ & + 220\mathbf{c}^2\mathbf{d}^2\mathbf{c} + 66\mathbf{cdc}^4 + 221\mathbf{cdc}^2\mathbf{d} + 423\mathbf{cdcdc} + 305\mathbf{cd}^2\mathbf{c}^2 + 400\mathbf{cd}^3 \\ & + 36\mathbf{dc}^5 + 150\mathbf{dc}^3\mathbf{d} + 371\mathbf{dc}^2\mathbf{dc} + 417\mathbf{dcdc}^2 + 538\mathbf{dcd}^2 + 216\mathbf{d}^2\mathbf{c}^3 \\ & + 536\mathbf{d}^2\mathbf{cd} + 656\mathbf{d}^3\mathbf{c}\end{aligned}$$

$$\begin{aligned}\Psi(\mathrm{M}(G^*)) = {} & \mathbf{c}^7 + 12\mathbf{c}^5\mathbf{d} + 64\mathbf{c}^4\mathbf{dc} + 168\mathbf{c}^3\mathbf{dc}^2 + 216\mathbf{c}^3\mathbf{d}^2 + 254\mathbf{c}^2\mathbf{dc}^3 + 648\mathbf{c}^2\mathbf{dcd} \\ & + 752\mathbf{c}^2\mathbf{d}^2\mathbf{c} + 224\mathbf{cdc}^4 + 864\mathbf{cdc}^2\mathbf{d} + 1664\mathbf{cdcdc} + 1152\mathbf{cd}^2\mathbf{c}^2 + 1632\mathbf{cd}^3 \\ & + 94\mathbf{dc}^5 + 488\mathbf{dc}^3\mathbf{d} + 1280\mathbf{dc}^2\mathbf{dc} + 1488\mathbf{dcdc}^2 + 2064\mathbf{dcd}^2 + 772\mathbf{d}^2\mathbf{c}^3 \\ & + 2096\mathbf{d}^2\mathbf{cd} + 2528\mathbf{d}^3\mathbf{c}\end{aligned}$$

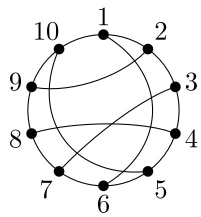


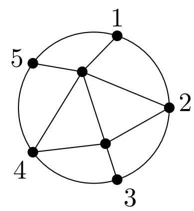
$$\Psi([\widehat{0},\tau]) = \mathbf{c}^7 + 7\mathbf{c}^5\mathbf{d} + 24\mathbf{c}^4\mathbf{dc} + 49\mathbf{c}^3\mathbf{dc}^2 + 62\mathbf{c}^3\mathbf{d}^2 + 68\mathbf{c}^2\mathbf{dc}^3 + 162\mathbf{c}^2\mathbf{dcd} + 188\mathbf{c}^2\mathbf{d}^2\mathbf{c} + 61\mathbf{cdc}^4 + 203\mathbf{cdc}^2\mathbf{d} + 373\mathbf{cdcdc} + 267\mathbf{cd}^2\mathbf{c}^2 + 354\mathbf{cd}^3 + 34\mathbf{dc}^5 + 141\mathbf{dc}^3\mathbf{d} + 325\mathbf{dc}^2\mathbf{dc} + 365\mathbf{dcdc}^2 + 478\mathbf{dcd}^2 + 194\mathbf{d}^2\mathbf{c}^3 + 476\mathbf{d}^2\mathbf{cd} + 568\mathbf{d}^3\mathbf{c}$$

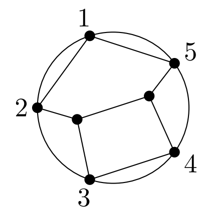
$$\Psi(\mathrm{M}(G)) = \mathbf{c}^7 + 12\mathbf{c}^5\mathbf{d} + 70\mathbf{c}^4\mathbf{dc} + 198\mathbf{c}^3\mathbf{dc}^2 + 246\mathbf{c}^3\mathbf{d}^2 + 305\mathbf{c}^2\mathbf{dc}^3 + 750\mathbf{c}^2\mathbf{dcd} + 890\mathbf{c}^2\mathbf{d}^2\mathbf{c} + 260\mathbf{cdc}^4 + 972\mathbf{cdc}^2\mathbf{d} + 1940\mathbf{cdcdc} + 1380\mathbf{cd}^2\mathbf{c}^2 + 1896\mathbf{cd}^3 + 106\mathbf{dc}^5 + 536\mathbf{dc}^3\mathbf{d} + 1472\mathbf{dc}^2\mathbf{dc} + 1776\mathbf{dcdc}^2 + 2388\mathbf{dcd}^2 + 922\mathbf{d}^2\mathbf{c}^3 + 2420\mathbf{d}^2\mathbf{cd} + 2996\mathbf{d}^3\mathbf{c}$$

$$\Psi(\mathrm{M}(G^*)) = \mathbf{c}^7 + 7\mathbf{c}^5\mathbf{d} + 29\mathbf{c}^4\mathbf{dc} + 68\mathbf{c}^3\mathbf{dc}^2 + 82\mathbf{c}^3\mathbf{d}^2 + 103\mathbf{c}^2\mathbf{dc}^3 + 228\mathbf{c}^2\mathbf{dcd} + 268\mathbf{c}^2\mathbf{d}^2\mathbf{c} + 100\mathbf{cdc}^4 + 308\mathbf{cdc}^2\mathbf{d} + 580\mathbf{cdcdc} + 420\mathbf{cd}^2\mathbf{c}^2 + 536\mathbf{cd}^3 + 50\mathbf{dc}^5 + 194\mathbf{dc}^3\mathbf{d} + 474\mathbf{dc}^2\mathbf{dc} + 556\mathbf{dcdc}^2 + 700\mathbf{dcd}^2 + 310\mathbf{d}^2\mathbf{c}^3 + 712\mathbf{d}^2\mathbf{cd} + 864\mathbf{d}^3\mathbf{c}$$

$M = M(\tau)$ $\quad G = G(M)$ $\quad G^* = G^*(M)$





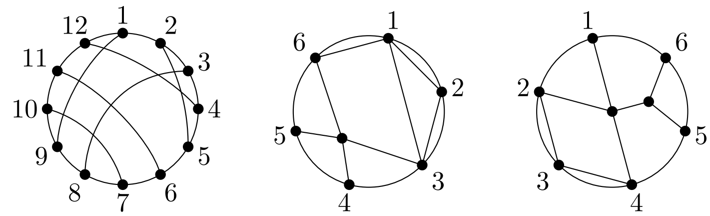


$$\begin{aligned}\Psi([\widehat{0},\tau]) = {} & \mathbf{c}^8 + 4\mathbf{c}^6\mathbf{d} + 15\mathbf{c}^5\mathbf{dc} + 34\mathbf{c}^4\mathbf{dc}^2 + 37\mathbf{c}^4\mathbf{d}^2 + 58\mathbf{c}^3\mathbf{dc}^3 + 108\mathbf{c}^3\mathbf{dcd} \\ & + 132\mathbf{c}^3\mathbf{d}^2\mathbf{c} + 74\mathbf{c}^2\mathbf{dc}^4 + 180\mathbf{c}^2\mathbf{dc}^2\mathbf{d} + 342\mathbf{c}^2\mathbf{dcdc} + 260\mathbf{c}^2\mathbf{d}^2\mathbf{c}^2 \\ & + 294\mathbf{c}^2\mathbf{d}^3 + 64\mathbf{cdc}^5 + 185\mathbf{cdc}^3\mathbf{d} + 440\mathbf{cdc}^2\mathbf{dc} + 530\mathbf{cdcdc}^2 \\ & + 592\mathbf{cdcd}^2 + 312\mathbf{cd}^2\mathbf{c}^3 + 598\mathbf{cd}^2\mathbf{cd} + 752\mathbf{cd}^3\mathbf{c} + 35\mathbf{dc}^6 \\ & + 112\mathbf{dc}^4\mathbf{d} + 306\mathbf{dc}^3\mathbf{dc} + 462\mathbf{dc}^2\mathbf{dc}^2 + 512\mathbf{dc}^2\mathbf{d}^2 + 428\mathbf{dcdc}^3 \\ & + 814\mathbf{dcdcd} + 1016\mathbf{dcd}^2\mathbf{c} + 210\mathbf{d}^2\mathbf{c}^4 + 522\mathbf{d}^2\mathbf{c}^2\mathbf{d} + 1012\mathbf{d}^2\mathbf{cdc} \\ & + 792\mathbf{d}^3\mathbf{c}^2 + 896\mathbf{d}^4\end{aligned}$$

$$\begin{aligned}\Psi(\mathrm{M}(G)) = {} & \mathbf{c}^8 + 9\mathbf{c}^6\mathbf{d} + 45\mathbf{c}^5\mathbf{dc} + 128\mathbf{c}^4\mathbf{dc}^2 + 154\mathbf{c}^4\mathbf{d}^2 + 241\mathbf{c}^3\mathbf{dc}^3 + 542\mathbf{c}^3\mathbf{dcd} \\ & + 626\mathbf{c}^3\mathbf{d}^2\mathbf{c} + 308\mathbf{c}^2\mathbf{dc}^4 + 984\mathbf{c}^2\mathbf{dc}^2\mathbf{d} + 1836\mathbf{c}^2\mathbf{dcdc} + 1298\mathbf{c}^2\mathbf{d}^2\mathbf{c}^2 \\ & + 1692\mathbf{c}^2\mathbf{d}^3 + 252\mathbf{cdc}^5 + 1024\mathbf{cdc}^3\mathbf{d} + 2508\mathbf{cdc}^2\mathbf{dc} + 2892\mathbf{cdcdc}^2 \\ & + 3712\mathbf{cdcd}^2 + 1556\mathbf{cd}^2\mathbf{c}^3 + 3696\mathbf{cd}^2\mathbf{cd} + 4472\mathbf{cd}^3\mathbf{c} + 102\mathbf{dc}^6 \\ & + 502\mathbf{dc}^4\mathbf{d} + 1494\mathbf{dc}^3\mathbf{dc} + 2296\mathbf{dc}^2\mathbf{dc}^2 + 2900\mathbf{dc}^2\mathbf{d}^2 + 2038\mathbf{dcdc}^3 \\ & + 4780\mathbf{dcdcd} + 5724\mathbf{dcd}^2\mathbf{c} + 920\mathbf{d}^2\mathbf{c}^4 + 3048\mathbf{d}^2\mathbf{c}^2\mathbf{d} + 5880\mathbf{d}^2\mathbf{cdc} \\ & + 4300\mathbf{d}^3\mathbf{c}^2 + 5624\mathbf{d}^4\end{aligned}$$

$$\begin{aligned}\Psi(\mathrm{M}(G^*)) = {} & \mathbf{c}^8 + 14\mathbf{c}^6\mathbf{d} + 90\mathbf{c}^5\mathbf{dc} + 288\mathbf{c}^4\mathbf{dc}^2 + 358\mathbf{c}^4\mathbf{d}^2 + 545\mathbf{c}^3\mathbf{dc}^3 + 1346\mathbf{c}^3\mathbf{dcd} \\ & + 1554\mathbf{c}^3\mathbf{d}^2\mathbf{c} + 644\mathbf{c}^2\mathbf{dc}^4 + 2404\mathbf{c}^2\mathbf{dc}^2\mathbf{d} + 4614\mathbf{c}^2\mathbf{dcdc} + 3170\mathbf{c}^2\mathbf{d}^2\mathbf{c}^2 \\ & + 4426\mathbf{c}^2\mathbf{d}^3 + 453\mathbf{cdc}^5 + 2273\mathbf{cdc}^3\mathbf{d} + 5967\mathbf{cdc}^2\mathbf{dc} + 6901\mathbf{cdcdc}^2 \\ & + 9438\mathbf{cdcd}^2 + 3501\mathbf{cd}^2\mathbf{c}^3 + 9342\mathbf{cd}^2\mathbf{cd} + 11376\mathbf{cd}^3\mathbf{c} + 163\mathbf{dc}^6 \\ & + 1034\mathbf{dc}^4\mathbf{d} + 3418\mathbf{dc}^3\mathbf{dc} + 5405\mathbf{dc}^2\mathbf{dc}^2 + 7222\mathbf{dc}^2\mathbf{d}^2 + 4591\mathbf{dcdc}^3 \\ & + 12034\mathbf{dcdcd} + 14484\mathbf{dcd}^2\mathbf{c} + 1872\mathbf{d}^2\mathbf{c}^4 + 7342\mathbf{d}^2\mathbf{c}^2\mathbf{d} + 14686\mathbf{d}^2\mathbf{cdc} \\ & + 10484\mathbf{d}^3\mathbf{c}^2 + 14716\mathbf{d}^4\end{aligned}$$

$M = M(\tau)$ $\quad G = G(M)$ $\quad G^* = G^*(M)$



$$\begin{aligned}\Psi([\widehat{0},\tau]) = {} & \mathbf{c}^8 + 11\mathbf{c}^6\mathbf{d} + 52\mathbf{c}^5\mathbf{dc} + 138\mathbf{c}^4\mathbf{dc}^2 + 178\mathbf{c}^4\mathbf{d}^2 + 235\mathbf{c}^3\mathbf{dc}^3 + 585\mathbf{c}^3\mathbf{dcd} \\ & + 670\mathbf{c}^3\mathbf{d}^2\mathbf{c} + 269\mathbf{c}^2\mathbf{dc}^4 + 991\mathbf{c}^2\mathbf{dc}^2\mathbf{d} + 1857\mathbf{c}^2\mathbf{dcdc} + 1271\mathbf{c}^2\mathbf{d}^2\mathbf{c}^2 \\ & + 1774\mathbf{c}^2\mathbf{d}^3 + 200\mathbf{cdc}^5 + 982\mathbf{cdc}^3\mathbf{d} + 2443\mathbf{cdc}^2\mathbf{dc} + 2731\mathbf{cdcdc}^2 \\ & + 3762\mathbf{cdcd}^2 + 1384\mathbf{cd}^2\mathbf{c}^3 + 3672\mathbf{cd}^2\mathbf{cd} + 4420\mathbf{cd}^3\mathbf{c} + 84\mathbf{dc}^6 \\ & + 505\mathbf{dc}^4\mathbf{d} + 1512\mathbf{dc}^3\mathbf{dc} + 2238\mathbf{dc}^2\mathbf{dc}^2 + 3030\mathbf{dc}^2\mathbf{d}^2 + 1854\mathbf{dcdc}^3 \\ & + 4842\mathbf{dcdcd} + 5760\mathbf{dcd}^2\mathbf{c} + 774\mathbf{d}^2\mathbf{c}^4 + 2980\mathbf{d}^2\mathbf{c}^2\mathbf{d} + 5784\mathbf{d}^2\mathbf{cdc} \\ & + 4080\mathbf{d}^3\mathbf{c}^2 + 5724\mathbf{d}^4\end{aligned}$$

$$\begin{aligned}\Psi(\mathrm{M}(G)) = {} & \mathbf{c}^8 + 11\mathbf{c}^6\mathbf{d} + 60\mathbf{c}^5\mathbf{dc} + 178\mathbf{c}^4\mathbf{dc}^2 + 218\mathbf{c}^4\mathbf{d}^2 + 336\mathbf{c}^3\mathbf{dc}^3 + 786\mathbf{c}^3\mathbf{dcd} \\ & + 906\mathbf{c}^3\mathbf{d}^2\mathbf{c} + 413\mathbf{c}^2\mathbf{dc}^4 + 1410\mathbf{c}^2\mathbf{dc}^2\mathbf{d} + 2658\mathbf{c}^2\mathbf{dcdc} + 1862\mathbf{c}^2\mathbf{d}^2\mathbf{c}^2 \\ & + 2496\mathbf{c}^2\mathbf{d}^3 + 316\mathbf{cdc}^5 + 1412\mathbf{cdc}^3\mathbf{d} + 3548\mathbf{cdc}^2\mathbf{dc} + 4100\mathbf{cdcdc}^2 \\ & + 5408\mathbf{cdcd}^2 + 2156\mathbf{cd}^2\mathbf{c}^3 + 5376\mathbf{cd}^2\mathbf{cd} + 6512\mathbf{cd}^3\mathbf{c} + 122\mathbf{dc}^6 \\ & + 674\mathbf{dc}^4\mathbf{d} + 2084\mathbf{dc}^3\mathbf{dc} + 3240\mathbf{dc}^2\mathbf{dc}^2 + 4196\mathbf{dc}^2\mathbf{d}^2 + 2828\mathbf{dcdc}^3 \\ & + 6948\mathbf{dcdcd} + 8324\mathbf{dcd}^2\mathbf{c} + 1218\mathbf{d}^2\mathbf{c}^4 + 4340\mathbf{d}^2\mathbf{c}^2\mathbf{d} + 8484\mathbf{d}^2\mathbf{cdc} \\ & + 6156\mathbf{d}^3\mathbf{c}^2 + 8288\mathbf{d}^4\end{aligned}$$

$$\begin{aligned}\Psi(\mathrm{M}(G^*)) = {} & \mathbf{c}^8 + 14\mathbf{c}^6\mathbf{d} + 90\mathbf{c}^5\mathbf{dc} + 298\mathbf{c}^4\mathbf{dc}^2 + 368\mathbf{c}^4\mathbf{d}^2 + 592\mathbf{c}^3\mathbf{dc}^3 + 1440\mathbf{c}^3\mathbf{dcd} \\ & + 1648\mathbf{c}^3\mathbf{d}^2\mathbf{c} + 734\mathbf{c}^2\mathbf{dc}^4 + 2692\mathbf{c}^2\mathbf{dc}^2\mathbf{d} + 5100\mathbf{c}^2\mathbf{dcdc} + 3500\mathbf{c}^2\mathbf{d}^2\mathbf{c}^2 \\ & + 4864\mathbf{c}^2\mathbf{d}^3 + 544\mathbf{cdc}^5 + 2688\mathbf{cdc}^3\mathbf{d} + 6912\mathbf{cdc}^2\mathbf{dc} + 7904\mathbf{cdcdc}^2 \\ & + 10816\mathbf{cdcd}^2 + 4032\mathbf{cd}^2\mathbf{c}^3 + 10688\mathbf{cd}^2\mathbf{cd} + 12928\mathbf{cd}^3\mathbf{c} + 190\mathbf{dc}^6 \\ & + 1188\mathbf{dc}^4\mathbf{d} + 3852\mathbf{dc}^3\mathbf{dc} + 6060\mathbf{dc}^2\mathbf{dc}^2 + 8096\mathbf{dc}^2\mathbf{d}^2 + 5216\mathbf{dcdc}^3 \\ & + 13568\mathbf{dcdcd} + 16224\mathbf{dcd}^2\mathbf{c} + 2180\mathbf{d}^2\mathbf{c}^4 + 8440\mathbf{d}^2\mathbf{c}^2\mathbf{d} + 16680\mathbf{d}^2\mathbf{cdc} \\ & + 11880\mathbf{d}^3\mathbf{c}^2 + 16640\mathbf{d}^4\end{aligned}$$

Appendix C: cd-index programs

In this appendix we describe four programs, `cdIndex.py`, `UC_lowerInterval.py`, `latticeOfFlats.py` and `minorPoset.py`, which were used to compute the data presented in Appendix B. Each of the programs is written in the Python language version 2.7.

cdIndex.py

This program is a collection of functions used to calculate the **cd**-index of a given poset, it is imported as a module by the programs described in `UC_lowerInterval.py` and `minorPoset.py`. The **cd**-index and the algorithm used in this module to compute it are described in Section 1.5.

In this module graded posets are encoded by a relation matrix and a rank list. Given a poset P the *relation matrix* is a square matrix M indexed by P with entries

$$M_{ij} = \begin{cases} 1 & \text{if } i < j, \\ -1 & \text{if } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

The ith entry of the rank list contains all elements of rank i. The main function in this module is `calcCdIndex`, which returns the **cd**-index of a given poset. The rest of the module consists of helper functions.

Various monomials in noncommutative variables **a**,**b**,**c**,**d**, and **e** are used. A monomial is encoded as a list whose first element is the coefficient and the second element is the product of the variables as a string. Polynomials are encoded as lists of monomials. For example, the polynomial $\mathbf{c}^2 - \mathbf{d}$ is encoded as `[[1,'cc'],[-1,'d']]`.

A function `transClose`, which computes the transitive closure of a matrix encoding a poset, is also provided. This is used in `UC_lowerInterval.py` which does not directly compute the relation matrix but starts by computing a similar matrix encoding the cover relations of the poset. This function is also used in `minorPoset.py` for a similar reason.

```
def multPolys(p,q):
    r=[[x[0]*y[0],x[1]+y[1]] for x in p for y in q]
       #this is the multiplication, we still need
       to collect like terms
    ret=[]
    for x in r:
        monoms=[y[1] for y in ret]
        if x[1] not in monoms:
            ret.append(x)
            continue
        ret[monoms.index(x[1])][0]+=x[0]
    return ret

def addPolys(p,q):
    ret=[x for x in p]
    for x in q:
        temp=[y[1] for y in ret]
        if x[1] in temp:
            ret[temp.index(x[1])][0]+=x[0]
        else:
```

```
            ret.append(x)
    return [x for x in ret if x[0]!=0]

#substitutes the polynomial p in place of occurrences of the monomial m in the polynomial x
def subPolyForMonom(x,p,m):
    X=[[y[0],y[1].replace(m,'*')] for y in x] #'*' is a placeholder for p (* must not be a variable in x)
    ret=[]
    for y in X:
        q=[[y[0],'']]
        for i in range(0,len(y[1])):
            if y[1][i]=='*':
                q=multPolys(q,p)
            else:
                for j in range(0,len(q)):
                    q[j][1]+=y[1][i]
        ret=addPolys(ret,q)
    return ret

#converts a polynomial in a and b to a polynomial in c and d, assuming this can be done.
#If this cannot be done the ab-index is returned
def abToCd(ab):
    if len(ab)==0: return ab
    #substitute a↦c+e and b↦c−e
    #where e=b−a
    #we pick up a factor of 2^n which we correct for before returning
    ce=subPolyForMonom(subPolyForMonom(ab,[[1,'c'],[1,'e']],'a'),[[1,'c'],[-1,'e']],'b')

    #If cd-index exists we get it by substituting e^2↦c^2−2d
    cd=subPolyForMonom(ce,[[1,'cc'],[-2,'d']],'ee')
    for m in cd:
        if 'e' in m[1]: #could not convert to c and d
            return ab

    cd.sort(key=lambda x:x[1])
    #return cd with coefficients divided by the extra factor
```

```
    return [[x[0]>>sum([2 if cd[0][1][i]=='d' else 1 for i in range(0,len(cd[0][1]))]),x[1]] for x in cd]

#converts a cover matrix into an incidence matrix
def transClose(M):
    for i in range(0,len(M)):
        uoi = [x for x in range(0,len(M)) if M[i][x] == 1]
        while True:
            next = [x for x in uoi]
            for x in uoi:
                for y in range(0,len(M)):
                    if M[x][y] == 1 and y not in next: next.append(y)
            if uoi == next: break
            uoi = next

        for x in uoi:
            M[i][x] = 1
            M[x][i] = -1

#helper function that calculates the entry of the flag f-vector indexed by the set S
#for the poset encoded by M and ranks
def fVectorCalc(ranks,S,M, i, count):
    newCount = count
    if S == []: return 1
    for j in ranks[S[0]]:
        if M[i][j] == 1:
            newCount += fVectorCalc(ranks, S[1:], M, j, count)
    return newCount

def makeFlagVectorsTable(M,ranks):
    table = [[[],1,1]]

    if len(ranks)<=2: return table

    #compute the flag f-vector
    for i in range(1,1<<(len(ranks)-1)-1): #iterate over all subsets of the ranks
        #construct the corresponding set
        pad = 1
        elem = 1
```

```
        S = []
        while pad <= i:
            if pad&i:
                S.append(elem)

            pad <<= 1
            elem += 1
        table.append([S,fVectorCalc(ranks,S,M, ranks[0][0], 0),0])


    #now we compute the flag h-vector using Möbius inversion.
    for i in range(1,len(table)):
        sign = (2*(len(table[i][0])%2)) - 1 #-1 if even number of elements
        for j in range(0,i+1):
            if set(table[j][0]).issubset(table[i][0]):
                table[i][2] += sign*(2*(len(table[j][0])%2)-1)*table[j][1]
    return table

#returns the ab-index of a poset given the flag vectors table and rank
def abIndex(table, rank):
    abIndex = []
    for x in table:
        u = ['a']*(rank-1)
        for s in x[0]: u[s-1] = 'b'
        abIndex.append([x[2],''.join(u)])

    return abIndex

#returns the cd-index of a poset encoded in M and ranks
def calcCdIndex(M, ranks, table=None, ab=None):
    if table == None: table = []
    if ab == None: ab = []

    table += makeFlagVectorsTable(M,ranks)

    ab += abIndex(table, len(ranks)-1)

    cdIndex=abToCd(ab)
```

```
    cdIndex.sort(key=lambda x:x[1])
    return cdIndex

#formats output of calcCdIndex
def cdIndexLatex(cdindex):
    s = ""
    for i in range(0,len(cdindex)):
        if cdindex[i][0] == 0: continue
        if cdindex[i][0] == -1: s+= '-'
        elif cdindex[i][0] != 1: s += str(cdindex[i][0])
        current = ''
        power = 0
        for c in cdindex[i][1]:
            if current == '':
                current = c
                power = 1
                continue
            if c == current:
                power += 1
                continue
            s += current
            if power != 1: s += '^{' + str(power) + '}'
            current = c
            power = 1
        s += current
        if power != 1 and power != 0: s += '^{' + str(power) + '}'
        if power == 0 and current == "": s += '1'

        if i != len(cdindex)-1:
            if cdindex[i+1][0] >= 0: s += "+"
    if s == '': return '0'
    return s

#computes the join of two elements given the incidence matrix of a poset
#this is used by minorPoset.py
def join(i,j,M):
    if i==j: return i
    if M[i][j] == -1: return i
    if M[i][j] == 1: return j
    m = [x for x in range(0,len(M)) if M[i][x] == 1 and M[j][x] == 1] #the upper bounds for i
```

```
        and j
    for x in range(0,len(m)):
        isJoin = True
        for y in range(0,len(m)):
            if x!=y and M[m[x]][m[y]] != 1:
                isJoin = False
                break
        if isJoin: return m[x]
    return None
```

UC_lowerInterval.py

This program computes the **cd**-index of a given lower interval in the uncrossing poset using functions from cdIndex.py. The top element τ of the lower interval is specified as a comma separated list in the form $i_1, \tau(i_1), i_2, \tau(i_2), \ldots, i_n, \tau(i_n)$.

Within the program a pairing σ is encoded as a list of arcs, with an arc $(i, \sigma(i))$ encoded as the number $2^{i-1} + 2^{\sigma(i)-1}$.

```
#!/usr/bin/env python2
from cdIndex import *
import sys

#program entrypoint is on line 83

#converts the pairing given in the input into the internal format described above
def readPairing(input):
    input = input.split(',')
    t = []
    for i in range(0,len(input)/2):
        t.append(1<<(int(input[i<<1])-1)|1<<(int(input[(i<<1)+1])-1))
    return sorted(t)

#If p represents a pairing τ then the output represents the pairing (ij)τ(ij)
def swap(p,i,j):
    return sorted([(x^((((x&(1<<i))>>i)^((x&(1<<j))>>j))<<i)^((((x&(1<<i))>>i)^((x&(1<<j))>>j))<<j)) for x in p])

#returns the crossing number for p
def c(p):
    ret = 0
    for i in range(0,len(p)):
```

```
        xi = bin(p[i])[::-1]
        Ni = xi.find('1')
        Ei = xi.rfind('1')
        for j in range(i+1,len(p)):
            xj = bin(p[j])[::-1]
            Nj = xj.find('1')
            Ej = xj.rfind('1')
            if (Ni - Nj > 0) == (Ei - Ej > 0) == (Nj - Ei > 0): ret += 1

    return ret

#computes the lower interval generated by the given pairing via Lemma 2.2.1
#returns a tuple (P,ranks,M) which is the list of elements, the rank list and the cover matrix
def lowerOrderIdeal(t):
    if c(t)==0: return [t],[[1],[0]],[[0,-1],[1,0]]

    P=[t]
    ranks = [[0]] #this is built up backwards for convenience and reversed before returning
    M=[[0]]

    num = 1 #index in to P of next element to add
    level = [t] #list of current rank to expand in next step
    leveli = [0] #indices in to P of the elements of level
    newLevel = [] #we build level for the next step during the current step here
    newLeveli = [] #indices in to P for the next step
    newRank = [] #the new rank indices to add
    while len(level) > 0:
        for i in range(0,(len(t)<<1)-1): #iterate over all pairs we can uncross
            for j in range(i+1,len(t)<<1):
                for k in range(0,len(level)): #do the uncross
                    temp = swap(level[k],i,j)
                    c_temp = c(temp)
                    if c_temp != c(level[k])-1: continue
                    if temp in P:
```

```
                        M[P.index(temp)][leveli[k]]=1
                        continue
                    P.append(temp)
                    newRank.append(num)
                    if c_temp > 0: #if not minimal continue uncrossing
                        newLevel.append(temp)
                        newLeveli.append(num)
                    num+= 1

                    for x in M: x.append(0)
                    M.append([0 for x in range(0,len(M[0]))])
                    M[-1][leveli[k]]=1

        level = newLevel
        newLevel = []
        leveli = newLeveli
        newLeveli = []
        ranks.append(newRank)
        newRank = []

    ranks.reverse()
    ranks=[[len(M)]]+ranks
    for r in M: r.append(-1)
    M.append([0]+[1 for i in range(0,len(P))])
    return P,ranks,M

input=sys.argv[1]
t = readPairing(input)
P,ranks,M = lowerOrderIdeal(t)
transClose(M)
print cdIndexLatex(calcCdIndex(M,ranks))
```

minorPoset.py

This program computes the minor poset of a given generator-enriched lattice as defined in Chapter 4 and outputs the **cd**-index. The input lattice should be given as a semicolon and comma separated list specifying the downcovers for each element except the minimal element $\widehat{0}$. The semicolons separate the downcover lists for different elements and the commas separate the elements within a downcover list. For example, consider the Boolean algebra B_3 labeled 0,…,7 via the ordering

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}.$$

This lattice is specified as "0;0;0;1,2;1,3;2,3;4,5,6;". Optionally a set of generators to add may be specified as a comma separated list with the flag -g. When provided the specified elements are added to the generating set in addition to the join irreducibles. Specifying no generators indicates a minimally generated lattice. The minimal element should not be specified as a generator.

```
#!/usr/bin/env python2
import sys
from cdIndex import * #imports methods from cdIndex.py

input=sys.argv[1] #the input describing the lattice

L_pre=[[]]+[[int(i) for i in x.split(',')] for x in input.split(';')[:-1]]

irr = [i for i in range(0,len(L_pre)) if len(L_pre[i])==1]
genL = sorted(list(set(irr + ([int(x) for x in ([] if '-g' not in sys.argv else sys.argv[sys.argv.index('-g')+1].split(','))]))))


L=[[1 if i in L_pre[j] else 0 for j in range(0,len(L_pre))] for i in range(0,len(L_pre))] #This matrix encodes covers of the input lattice. L[x][y] is 1 when x is covered by y and otherwise 0

transClose(L) #computes the transitive closure of L, now L[x][y] is 1 if x<y, is -1 if x>y and otherwise is 0.
joins = [[0 for i in range(0,len(L))]for j in range(0,len(L))] #a table of all the joins of two elements of the lattice
for i in range(0,len(L)):
    for j in range(i,len(L)):
        k = join(i,j,L)
        if k == None: #The input did not describe a lattice
            exit()
        joins[i][j]=k
        joins[j][i]=k
minors = [[0,genL]] #This will be a list of all minors of the lattice. A minor is encoded as a tuple [z,H] where z is the minimal element and
```

```
  H is a sorted list of the generators
minors_M = [[0]] #relation matrix for the minor poset
minors_ranks = [[] for i in range(0,len(genL)+1)]
minors_ranks[len(genL)].append(1) #we append 1 because a minimal element is later added as the first element
new = [[0,genL]] #the newly added minors which in the next iteration we perform deletions and contractions on

#We now iterate performing all deletions and contractions by a single generator until we get no more new minors in this way.
while len(new)>0:
    old = new
    new = []
    for l in old:
        r = minors.index(l)
        for i in range(0,len(l[1])):
            minor=[l[0],l[1][:i]+l[1][i+1:]] #delete i from l
            if minor in minors:
                s = minors.index(minor)
            else: #add new minor to minors and to minors_M
                s = len(minors_M)
                minors_ranks[len(minor[1])].append(s+1)
                minors.append(minor)
                for x in minors_M: x.append(0)
                minors_M.append([0 for x in range(0,s+1)])
            #add the new relation from the deletion
            minors_M[r][s] = -1
            minors_M[s][r] = 1

            if minor not in new: new.append(minor)

            #compute the contraction by i
            temp = set([joins[l[1][i]][j] for j in l[1]])
            temp.remove(l[1][i])
            minor=[l[1][i],sorted(list(temp))]
            if minor in minors:
```

```
                s = minors.index(minor)
            else: #add the new minor to minors and minors_M
                s = len(minors_M)
                minors_ranks[len(minor[1])].append(s+1)
                minors.append(minor)
                for x in minors_M: x.append(0)
                minors_M.append([0 for x in range(-1,s)])
            #add the new relation from the contraction
            minors_M[r][s] = -1
            minors_M[s][r] = 1

            if minor not in new: new.append(minor)

minors_ranks = [[0]]+minors_ranks #prepend a new minimum and add the relations for it
for i in range(0,len(minors_M)):
    minors_M[i] = [-1]+minors_M[i]
minors_M = [[0]+[1 for i in range(0,len(minors_M))]]+minors_M
transClose(minors_M)
print cdIndexLatex(calcCdIndex(minors_M,minors_ranks))
```

latticeOfFlats.py

This program computes the lattice of flats of a given graph. The output can be fed into `minorPoset.py`. The graph is input as a comma and space separated list of edges. The input $i_1,j_1\ i_2,j_2\ \ldots\ i_n,j_n$ represents a graph with edges between vertices i_k and j_k for $k=1,\ldots,n$.

A flat of the graph is viewed as a partition of the vertices $0,\ldots,n$. Partitions are encoded as lists whose ith element is the block containing i. A subset S of $0,\ldots,n$ is encoded as the number $\sum_{s\in S}2^s$.

```
#!/usr/bin/env python2
from sys import argv

E=[[int(x) for x in a.split(',')] for a in argv[1:]] #edge set
n=max([max(e) for e in E])

flats=[]
```

```
#here we iterate over all subsets of edges,
#compute the corresponding partition and add it to flats
for S in range(0,1<<len(E)):
    F=[1<<i for i in range(0,n+1)]
    for i in [i for i in range(0,len(E)) if (1<<i)&S !=0]: #iterates over elements of S
        b1=F[E[i][0]]
        b2=F[E[i][1]]

        for j in range(0,n+1):
            if (1<<j)&b1!=0: F[j]|=b2 #if j is in the block b1 add the block b2 to the block containing j
            if (1<<j)&b2!=0: F[j]|=b1 #likewise with b1 and b2 exchanged

    if F not in flats: flats.append(F)

#This outputs the downcovers of each element except for the minimal element \hat{0} as a comma seperated list, with semicolons delimiting the downcover lists
print ';'.join([','.join([str(i) for i in range(0,len(flats)) if len(set(flats[i]))-len(set(F))==1 and all([F[j]&flats[i][j]==flats[i][j] for j in range(0,len(F))])]) for F in flats[1:]])+';'
```

Bibliography

[1] S. P. Avann. Application of the join-irreducible excess function to semi-modular lattices. *Math. Ann.*, 142:345–354, 1960/61.

[2] E. Barcucci, A. Bernini, L. Ferrari, and M. Poneti. A distributive lattice structure connecting Dyck paths, noncrossing partitions and 312-avoiding permutations. *Order*, 22(4):311–328 (2006), 2005.

[3] M. M. Bayer. The *cd*-index: a survey. In *Polytopes and discrete geometry*, volume 764 of *Contemp. Math.*, pages 1–19. Amer. Math. Soc., Providence, RI, 2021.

[4] M. M. Bayer and L. J. Billera. Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets. *Invent. Math.*, 79(1):143–157, 1985.

[5] M. M. Bayer and A. Klapper. A new index for polytopes. *Discrete Comput. Geom.*, 6(1):33–47, 1991.

[6] P. Biane. Some properties of crossings and partitions. *Discrete Math.*, 175(1-3):41–53, 1997.

[7] L. J. Billera and R. Ehrenborg. Monotonicity of the cd-index for polytopes. *Math. Z.*, 233(3):421–441, 2000.

[8] L. J. Billera, R. Ehrenborg, and M. Readdy. The *c*-2*d*-index of oriented matroids. *J. Combin. Theory Ser. A*, 80(1):79–105, 1997.

[9] G. Birkhoff. Abstract Linear Dependence and Lattices. *Amer. J. Math.*, 57(4):800–804, 1935.

[10] G. Birkhoff. *Lattice theory.* American Mathematical Society Colloquium Publications, Vol. XXV. American Mathematical Society, Providence, R.I., third edition, 1967.

[11] A. Björner. Posets, regular CW complexes and Bruhat order. *European J. Combin.*, 5(1):7–16, 1984.

[12] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, second edition, 1999.

[13] A. Björner and M. L. Wachs. Shellable nonpure complexes and posets. I. *Trans. Amer. Math. Soc.*, 348(4):1299–1327, 1996.

[14] A. Björner and M. L. Wachs. Shellable nonpure complexes and posets. II. *Trans. Amer. Math. Soc.*, 349(10):3945–3975, 1997.

[15] S. Chakravarty and Y. Kodama. Classification of the line-soliton solutions of KPII. *J. Phys. A*, 41(27):275209, 33, 2008.

[16] H. H. Crapo. Structure theory for geometric lattices. *Rend. Sem. Mat. Univ. Padova*, 38:14–22, 1967.

[17] H. H. Crapo and G.-C. Rota. *On the foundations of combinatorial theory: Combinatorial geometries.* The M.I.T. Press, Cambridge, Mass.-London, preliminary edition, 1970.

[18] P. Crawley and R. Dilworth. *Algebraic Theory of Lattices.* Prentice Hall, Inc., Englewood Cliffs, 1973.

[19] R. P. Dilworth. Lattices with unique irreducible decompositions. *Ann. of Math. (2)*, 41:771–777, 1940.

[20] P. H. Edelman. Meet-distributive lattices and the anti-exchange closure. *Algebra Universalis*, 10(3):290–299, 1980.

[21] J. Edmonds. Submodular functions, matroids, and certain polyhedra. In *Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)*, pages 69–87. Gordon and Breach, New York, 1970. reprinted in [22].

[22] J. Edmonds. Submodular functions, matroids, and certain polyhedra. In *Combinatorial optimization—Eureka, you shrink!*, volume 2570 of *Lecture Notes in Comput. Sci.*, pages 11–26. Springer, Berlin, 2003.

[23] R. Ehrenborg, M. Goresky, and M. Readdy. Euler flag enumeration of Whitney stratified spaces. *Adv. Math.*, 268:85–128, 2015.

[24] R. Ehrenborg and K. Karu. Decomposition theorem for the **cd**-index of Gorenstein posets. *J. Algebraic Combin.*, 26(2):225–251, 2007.

[25] R. Ehrenborg and M. Readdy. Coproducts and the *cd*-index. *J. Algebraic Combin.*, 8(3):273–299, 1998.

[26] R. Ehrenborg and M. Readdy. On the non-existence of an R-labeling. *Order*, 28(3):437–442, 2011.

[27] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[28] P. Hersh and R. Kenyon. Shellability of face posets of electrical networks and the CW poset property. *Adv. in Appl. Math.*, 127:Paper No. 102178, 37, 2021.

[29] D. A. Higgs. Strong maps of geometries. *J. Combinatorial Theory*, 5:185–191, 1968.

[30] J. F. P. Hudson. *Piecewise linear topology.* W. A. Benjamin, Inc., New York-Amsterdam, 1969. University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees.

[31] K. Karu. The *cd*-index of fans and posets. *Compos. Math.*, 142(3):701–718, 2006.

[32] D. E. Knuth. *The art of computer programming. Vol. 1.* Addison-Wesley, Reading, MA, 1997. Fundamental algorithms, Third edition [of MR0286317].

[33] G. Kreweras. Sur les partitions non croisées d'un cycle. *Discrete Math.*, 1(4):333–350, 1972.

[34] J. P. S. Kung. Critical problems. In *Matroid theory (Seattle, WA, 1995)*, volume 197 of *Contemp. Math.*, pages 1–127. Amer. Math. Soc., Providence, RI, 1996.

[35] T. Lam. The uncrossing partial order on matchings is Eulerian. *J. Combin. Theory Ser. A*, 135:105–111, 2015.

[36] T. Lam. Electroid varieties and a compactification of the space of electrical networks. *Adv. Math.*, 338:549–600, 2018.

[37] A. T. Lundell and S. Weingram. *The topology of CW complexes.* The University Series in Higher Mathematics. Van Nostrand Reinhold Co., New York, 1969.

[38] J. McCammond. Noncrossing partitions in surprising locations. *Amer. Math. Monthly*, 113(7):598–610, 2006.

[39] J. Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics.* Oxford University Press, Oxford, second edition, 2011.

[40] M. Purtill. André permutations, lexicographic shellability and the *cd*-index of a convex polytope. *Trans. Amer. Math. Soc.*, 338(1):77–104, 1993.

[41] N. Reading. The cd-index of Bruhat intervals. *Electron. J. Combin.*, 11(1):Research Paper 74, 25, 2004.

[42] N. Reading. Non-negative cd-coefficients of Gorenstein* posets. *Discrete Math.*, 274(1-3):323–329, 2004.

[43] R. P. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1(1):9–23, 1986.

[44] R. P. Stanley. Flag f-vectors and the *cd*-index. *Math. Z.*, 216(3):483–499, 1994.

[45] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[46] R. P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, second edition, 2012.

[47] R. P. Stanley. *Catalan numbers.* Cambridge University Press, New York, 2015.

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Publications

1. William Gustafson. Counterexample to a variant of a conjecture of Ziegler concerning a simple polytope and its dual. *Discrete & Computation Geometry*, 66:1470–1472, 2021.

2. William Gustafson. Polymatroids, closure operators and lattices. *Order*, 2022.

3. William Gustafson. Lattice minors and Eulerian posets. Submitted, arXiv:2205.01200

4. William Gustafson. The weak minor poset. In preparation.