Extremal problems for the Möbius function in the face lattice of the n-octahedron

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Abstract

We study extremal problems concerning the Möbius function μ of certain families of subsets from O_n, the lattice of faces of the n-dimensional octahedron. For lower order ideals $\mathcal{F}$ from O_n, $|\mu(\mathcal{F})|$ attains a unique maximum by taking $\mathcal{F}$ to be the lower two-thirds of the ranks of the poset. Stanley showed that the coefficients of the cd-index for face lattices of convex polytopes are non-negative. We verify an observation that this result implies the Möbius function is maximized over arbitrary rank-selections from these lattices by taking their odd or even ranks. Using recurrences by Purtill for the cd-index of B_n and O_n, we demonstrate that the alternating ranks are the only extremal configuration for these two face lattices.

Key Words: cd-index, convex polytopes, extremal configuration, lattice of faces, lower order ideals, maximum, Möbius function, octahedron, rank selections

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0 Notation

We follow [17] for most of the terminology and notation we will use in this paper. Let P be a partially ordered set (*poset*) that has unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$ (i.e., P is *bounded*). All the posets we study will be finite and graded, so we have an associated rank function. In particular, let O_n denote the lattice of faces of the n-dimensional octahedron. We can represent O_n as the poset of all signed subsets of $\{1, \ldots, n\}$ ordered by inclusion with the element $\hat{1}$ adjoined. In other words, a typical element x of $O_n \setminus \hat{1}$ is a subset of the set $\{\pm 1, \ldots, \pm n\}$, say

$$x = \{a_1, \ldots, a_k\},$$

where x cannot contain both the elements $+i$ and $-i$. The *rank* of any element $x \in O_n \setminus \hat{1}$ is given by $|x|$, where $|\cdot|$ denotes cardinality. The maximal element $\hat{1}$ has rank $n+1$ in O_n.

For a family $\mathcal{F} \subseteq P$ define its *completion* by

$$\hat{\mathcal{F}} = \mathcal{F} \cup \{\hat{0}, \hat{1}\}$$

and its *proper part* by

$$\overline{\mathcal{F}} = \mathcal{F} \setminus \{\hat{0}, \hat{1}\}.$$

With respect to the partial order $\leq$ of P, an *interval* $[x, y]$ in P is

$$[x, y] = \{z \in P : \ \ x \leq z \leq y\}, \text{ for } x \leq y.$$

Notice that $[x, x]$ consists of the element x. We do not allow the empty set to be an interval. Given a poset P, the *Möbius function* μ is defined recursively on intervals $[x, y]$ in P by

$$\begin{aligned} \mu(x, x) &= 1 \text{ for all } x \text{ in } P, \\ \mu(x, y) &= -\sum_{x \leq z < y} \mu(x, z). \end{aligned}$$

For brevity, let $\mu(P)$ denote the value of $\mu_P(\hat{0}, \hat{1})$. For x an element of P, $\mu(x)$ will denote $\mu_P(\hat{0}, x)$. Additionally, for any family $\mathcal{F}$ of elements of P let $\mu(\mathcal{F})$ equal $\mu_{\hat{\mathcal{F}}}(\hat{0}, \hat{1})$. (Here $\hat{\mathcal{F}}$ is $\mathcal{F}$ adjoined with $\hat{0}$ and $\hat{1}$, if necessary.)

Given non-negative integers i and j, let

$$[i] = \{1, 2, \ldots, i\}$$

and

$$[i, j] = \{i, i+1, \ldots, j\},$$

with the conventions that $[0] = \emptyset$ and $[i,j] = \emptyset$ for $j < i$. If $\text{rank}(\hat{1}) = n+1$, then for a family $\mathcal{F} \subseteq P$ and $S \subseteq [n]$, define the *rank-selected subposet*

$$\mathcal{F}(S) = \{x \in \mathcal{F} : \ \text{rank}(x) \in S\}.$$

In particular the i-th *rank level* of $\mathcal{F}$ is given by

$$\mathcal{F}(i) = \mathcal{F}(\{i\}) = \{x \in \mathcal{F} : \text{rank}(x) = i\},$$

and the *interval* $[i,j]$ *of ranks of* $\mathcal{F}$ (not to be confused with the closed interval $[x,y]$ of elements of P) is

$$\mathcal{F}[i,j] = \mathcal{F}([i,j]) = \{x \in \mathcal{F} : \text{rank}(x) \in [i,j]\}.$$

Also, we use the shorthand

$$\mathcal{F}[k] = \mathcal{F}([k]).$$

Finally, a *lower order ideal* $\mathcal{A}$ is a subset $\mathcal{A}$ of P such that if $x \in \mathcal{A}$ and $y \leq x$ then $y \in \mathcal{A}$.

1 Introduction

Stanley [17, Exercise 3.41a] posed the following extremal question: given a bounded poset P with a fixed number of elements, what is the maximum value of the Möbius function of P? Ziegler answered this question for both bounded posets and graded posets. He also determined the extremal configuration in each situation [21].

Recent work of Sagan, Yeh and Ziegler [15] approached extremal problems involving the Möbius function from a slightly different angle. They fixed the poset under consideration (in their case, the Boolean algebra B_n) and studied the maximum value attained by the Möbius function μ over certain subsets of B_n. More specifically, if $\mathcal{F}$ is a family of subsets contained in the Boolean algebra B_n, then the $\max_{\mathcal{F}} |\mu(\mathcal{F})|$ has been found for three categories of families:

i. lower order ideals
ii. intervals of ranks
iii. arbitrary rank-selections.

The maxima are obtained by taking roughly the lower half, middle third, and every other rank of B_n, respectively. The lower order ideal case was first solved by Eckhoff [11] and Scheid [1], and viewed in the context of the reduced Euler characteristic by Björner and Kalai [7]. Niven [12] and de Bruijn [10] had previously solved the arbitrary rank-selection case, while the interval of ranks case was a new result.

In this paper we will address analogous extremal problems for O_n, the lattice of faces of the n-dimensional octahedron. More specifically, by extending the techniques developed for B_n in [15] to O_n, we find the extremal configuration for lower order ideals is the lower two-thirds of the poset O_n. In section 3 we describe the solution to the arbitrary rank-selection case for n-polytopes. Finally, in section 4 we state our conjecture for the interval of ranks question and indicate future work.

I would like to thank my doctoral advisor Bruce Sagan for his guidance while this work was being completed at Michigan State University. While attending the 4^{th} Conference on Formal Power Series and Algebraic Combinatorics in Montréal (1992), Richard Stanley kindly brought to my attention Mark Purtill's work on the cd-index. This paper is dedicated to my brother and sister, Andy and Christine.

2 Lower Order Ideals

In this section we are concerned with maximizing $|\mu(\mathcal{F})|$ as $\mathcal{F}$ ranges over all lower order ideals in O_n. We first state the main result of this section:

Theorem 2.0.1 *If $\mathcal{F}$ is a lower order ideal in O_n, then*

$$|\mu(\mathcal{F})| \;\leq\; \left| \sum_{k=0}^{\lfloor \frac{2n}{3} \rfloor} \binom{n}{k} (-2)^k \right|,$$

with equality occurring if and only if

$$\overline{\mathcal{F}} = O_n[k] \;\; \textit{with}\; k = \lfloor \tfrac{2n}{3} \rfloor.$$

(Here $\lfloor \cdot \rfloor$ denotes the greatest integer function.)

Before proving Theorem 2.0.1, we will first specialize $\mathcal{F}$ to be a rank-selected lower order ideal, in other words, a lower order ideal of the form $O_n[k] \cup \{\hat{0}\}$. We show in Lemma 2.2.1 that $|\mu(O_n[k])|$ is maximized if we take k to be $\lfloor \frac{2n}{3} \rfloor$, i.e., the lower two-thirds of O_n. Once we generalize $\mathcal{F}$ to be any lower order ideal in O_n, we will see the ideal $O_n[\lfloor \frac{2n}{3} \rfloor] \cup \{\hat{0}\}$ is also the maximal configuration for Theorem 2.0.1.

2.1 Elementary Properties for O_n

We begin by determining the Möbius value μ of elements from O_n and rank-selected lower order ideals from O_n. For the most part, these results follow from known properties of the lattice O_n, the definition of the Möbius function, and the product theorem for the Möbius function.

The most important result we establish in this section is a recurrence for the $\mu(O_n[k])$'s (Corollary 2.1.5). We do this via a "reduced Euler characteristic" interpretation of the Möbius function. This recurrence, in its absolute value form (Corollary 2.1.7), allows us to quickly narrow down the possibilities for the maximum of the Möbius function for rank-selected lower order ideals. The summation formula for the $\mu(O_n[k])$'s allows us to sharpen and complete the argument for Theorem 2.0.1.

We first determine the Möbius value of elements from O_n.

Proposition 2.1.1 *Let x be an element of $O_n \setminus \hat{1}$ with rank k. Then $\mu(x) = (-1)^k$.*

Proof. Recall

$$O_n \setminus \hat{1} \cong \underbrace{V \times \cdots \times V}_{n},$$

where V is the poset with Hasse diagram

$$V = \begin{array}{cc} {\scriptstyle +1} & {\scriptstyle -1} \\ \multicolumn{2}{c}{{\scriptstyle 0}} \end{array}$$

Applying the product theorem for the Möbius function [17, Prop. 3.8.2] gives the result. □

We are able to express $\mu(O_n[k])$ in two ways: in terms of a summation formula (Corollary 2.1.2) or a recurrence (Corollary 2.1.5).

Corollary 2.1.2 *The following summation formula holds for $\mu(O_n[k])$:*

$$\mu(O_n[k]) = -\left(\sum_{j=0}^{k} \binom{n}{j} (-2)^j\right), \quad 0 \le k \le n.$$

Proof. Use Proposition 2.1.1 and the fact there are $\binom{n}{j} 2^j$ elements of rank j in O_n. □

Corollary 2.1.3 $\mu(O_n) = (-1)^{n+1}$, $n \ge 1$.

Proof. It is enough to observe $\mu(O_n) = \mu(O_n[n])$ (recall $\hat{1}$ has rank $n+1$ in O_n). The result then immediately follows once we apply the binomial theorem to the summation expression for $\mu(O_n[n])$ given in Corollary 2.1.2. □

Combining Proposition 2.1.1 and Corollary 2.1.3 yields the following proposition:

Proposition 2.1.4 *Let x be an element of O_n with rank k. Then $\mu(x) = (-1)^k$.*

Corollary 2.1.5 *The $\mu(O_n[k])$'s satisfy the recurrence*

$$\mu(O_n[k]) = -2\mu(O_{n-1}[k-1]) \; + \; \mu(O_{n-1}[k]),$$

where $n \geq 2,\ 0 < k < n$, *with boundary conditions*

$$\mu(O_n[0]) = -1 \ \ for\ n \geq 0$$

and

$$\mu(O_n[n]) = (-1)^{n+1} \ \ for\ n \geq 1.$$

We have two different proofs of this recurrence. The first applies the interpretation of the Möbius function as counting certain chains in a poset. The second, which we omit, is a direct application of the summation formula for $\mu(O_n[k])$ and induction on n.

Before beginning our proof, we briefly review the aforementioned interpretation of the the Möbius function. For P a bounded poset, let $c_i = c_i(P)$ denote the number of chains of length i in $P \setminus \{\hat{0}, \hat{1}\}$. Here a chain of length i is a totally ordered subset of $P \setminus \{\hat{0}, \hat{1}\}$ containing $i+1$ elements.

A well-known result [17, Prop. 3.8.5] which relates chains with the Möbius function is

$$\mu(P) = -1 + c_0 - c_1 + c_2 - c_3 + \ldots \tag{1}$$

We are now ready to use this formulation of the Möbius function for our proof.

Proof. Let

$$\begin{aligned} a_j &= \#\text{ of chains of length } j \text{ in } O_{n-1}[k-1] \\ b_j &= \#\text{ of chains of length } j \text{ in } O_{n-1}[k] \\ c_j &= \#\text{ of chains of length } j \text{ in } O_n[k]. \end{aligned}$$

We will refer to chains of "type c_j" to mean those chains of length j in $O_n[k]$. Similar terminology will be used for a_j and b_j. Using equation (1), we can translate the desired recurrence for $\mu(O_n[k]$ in terms of chain notation, then regroup terms to give

$$-1+c_0-c_1+c_2-\ldots = -1+(2(1)+b_0)-(2a_0+b_1)+(2a_1+b_2)-(2a_2+b_3)+\ldots \tag{2}$$

The chains of type $c_j:\ x_1 < x_2 < \ldots < x_{j+1}$ consist of those chains of length j in $O_n[k]$ belonging to one of the following four classes:

(i)	neither $\{+n\}$ nor $\{-n\} \in x_i$ for any i
(ii)	$x_1 = \{+n\}$ ("the chain starts with $\{+n\}$")
(iii)	$x_1 = \{-n\}$ ("the chain starts with $\{-n\}$")
(iv)	all other remaining chains not accounted for in (i), (ii) or (iii).

If we let d_j denote the cardinality of the last category (iv) of chains, then we claim

$$\begin{aligned} c_j &= b_j + a_{j-1} + a_{j-1} + d_j \\ &= b_j + 2a_{j-1} + d_j, \end{aligned} \tag{3}$$

with the convention that $a_{-1} = 1$ counts the number of empty chains in $O_{n-1}[k-1]$. Clearly the chains of type b_j and category (i) have the same cardinality via the identity map. The collection of chains starting with $\{+n\}$ in $O_n[k]$ has the same cardinality as the type a_{j-1} chains by simply removing the $+n$ which occurs in every element of the category (ii) chains. Similarly, the cardinality of category (iii) chains equals a_{j-1}.

Substituting equation (3) for each of the c_j's appearing in the left side of equation (2), we see that all the terms on the right side of equation (2) cancel. The remaining terms consist of an alternating sum of the d_j's, so the corollary will follow once we have shown this sum equals zero, i.e.

$$d_0 - d_1 + d_2 - d_3 + \ldots + (-1)^{k-1} d_{k-1} = 0. \tag{4}$$

To complete this argument, we construct a bijection λ between certain chains of type d_j and type d_{j+1}, according to where $\{\pm n\}$ first occurs in each type of chain. Here $\pm n$ stands for either $+n$ or $-n$ (but not both), depending upon which occurs in the chain. An important observation to make is that if $x_1 < \ldots < x_{j+1}$ is a chain of length j in $O_n[k]$, then rank $x_i \geq i$.

We divide the chains of type d_j into two classes: $d_{j,0}$ and $d_{j,1}$. Let $d_{j,0}$ denote the cardinality of those type d_j chains $x_1 < \ldots < x_{j+1}$ in $O_n[k]$ with the property that if x_h is the least element in the chain containing $\pm n$ then $x_h = x_{h-1} \cup \{\pm n\}$. Let $d_{j,1}$ be the complementary set of chains. For $h = 1$ we let $x_0 = \emptyset$. Note that $d_j = d_{j,0} + d_{j,1}$ for all j.

From the definitions of $d_{j,0}$ and $d_{j,1}$, it is easy to see that $d_{0,0} = 0$ and $d_{k-1,1} = 0$. We complete the proof by showing

$$d_{j,1} = d_{j+1,0} \quad (0 \leq j \leq k-2) \tag{5}$$

holds. Define the map λ : *type* $d_{j,1}$ *chains* $\longrightarrow$ *type* $d_{j+1,0}$ *chains* by

$$x_1 < \ldots < x_{h-1} < x_h < \ldots < x_{j+1} \longmapsto x_1 < \ldots < x_{h-1} < x_h \backslash \{\pm n\} < x_h < \ldots < x_{j+1}$$

where h is the first position where $\pm n$ occurs in a type $d_{j,1}$ chain. Define the inverse map λ^{-1} : *type* $d_{j+1,0}$ *chains* $\longrightarrow$ *type* $d_{j,1}$ *chains* by

$$y_1 < \ldots < y_{h-1} < y_h < y_{h+1} < \ldots < y_{j+2} \longmapsto y_1 < \ldots < y_{h-1} < y_{h+1} < \ldots < y_{j+2}$$

where $h+1$ is the first position where $\pm n$ occurs in a type $d_{j+1,0}$ chain. It is easy to check $\lambda^{-1} \circ \lambda$ is the identity on type $d_{j,1}$ chains and $\lambda \circ \lambda^{-1}$ is the identity on type $d_{j+1,0}$ chains, so λ is a bijection. Hence, the sum in equation (4) telescopes. $\Box$

An easy result that we will need to complete the proof of Theorem 2.0.1 is the following corollary:

Corollary 2.1.6 *When k is odd $\mu(O_n[k])$ is positive and when k is even $\mu(O_n[k])$ is negative ($n \geq 1$, $0 \leq k \leq n$).*

Proof. The proof easily follows by induction on n. Alternatively, it is true because for P a poset of rank n and $S \subseteq [n]$, $\mu(P(S)) = (-1)^{|S|+1}|\mu(P(S))|$ [17, Chapter 3, equation (34)]. $\Box$

Putting Corollaries 2.1.5 and 2.1.6 together, we see that the $|\mu(O_n[k])|$'s also satisfy a recurrence.

Corollary 2.1.7 *The $|\mu(O_n[k])|$ satisfy the recurrence*

$$|\mu(O_n[k])| = 2|\mu(O_{n-1}[k-1])| + |\mu(O_{n-1}[k])| \tag{6}$$

with boundary conditions

$$|\mu(O_n[0])| = 1 \;\; for\ n \geq 0$$

and

$$|\mu(O_n[n])| = 1 \;\; for\ n \geq 1.$$

2.2 Rank-Selected Lower Order Ideals

Before we determine the extremal configuration for $|\mu(\mathcal{F})|$, where $\mathcal{F}$ is an arbitrary lower order ideal, it is natural to first study "simpler" ideals. In particular, we study ideals $\mathcal{F}$ which are rank-selected lower order ideals, in other words, lower order ideals of the form $\mathcal{F} = O_n[k] \cup \hat{0}$.

In order to state a special case of the main theorem of this chapter, we make a few more definitions. We say a sequence $a_0, a_1, \ldots, a_n$ of real numbers is *unimodal* if for some k, $0 \leq k \leq n$, we have

$$a_0 \leq \ldots \leq a_k \geq \ldots \geq a_n.$$

Table 1: $|\mu(O_n[k])|$ for $n = 2, 3, 4$

	k=1	2	3	4
n =2	3	1	0	0
3	5	7	1	0
4	7	17	15	1

Similarly, a sequence is *strictly unimodal* if we replace the inequalities by strict inequalities in the definition of unimodal. Finally, we say a sequence is *almost strictly unimodal* if the sequence is strictly unimodal or is of the form

$$a_0 < \ldots < a_k = a_{k+1} > \ldots > a_n.$$

We are now ready to state a rank-selection lemma:

Lemma 2.2.1 *For fixed* $n \geq 2$, $|\mu(O_n[k])|$ *is strictly unimodal with unique maximum occurring when* $k = \lfloor \frac{2n}{3} \rfloor$.

Proof. The proof for rank-selected lower order ideals proceeds by induction on n. By the recurrence for $|\mu(O_n[k])|$, we quickly conclude the sequence $\{|\mu(O_{n+1}[k])|\}_{k=0}^{n+1}$ is almost unimodal and narrow down its maximum to one of two possibilities. We will complete the argument by considering the equivalence class of n modulo 3 and apply the summation formulas and recurrence relation for $\mu(O_n[k])$ determined in the previous section.

For $n = 2, 3$, and 4, the lemma is easily checked to be true. (Refer to Table 1.) Fix n and let k^* be the index k for which $|\mu(O_n[k])|$ is a maximum. By the induction hypothesis the sequences $\{\mu(O_n[k])\}_{k=1}^{k*}$ and $\{\mu(O_n[k])\}_{k=k*}^{n}$ are strictly increasing and decreasing, respectively. Applying the recurrence in Corollary 2.1.7 to these monotone sequences, we conclude the sequences

$$\{\ |\mu(O_{n+1}[k])|\ \}_{k=1}^{k^*} \tag{7}$$

and

$$\{\ |\mu(O_{n+1}[k])|\ \}_{k=k^*+1}^{n+1} \tag{8}$$

are strictly increasing and strictly decreasing, respectively. Thus, equations (7) and (8) imply we have pinned down the index k corresponding to the maximum $|\mu(O_{n+1}[k])|$ as one of two possibilities: k^* or $k^* + 1$.

For ease in notation, let

$$\begin{aligned} a_k &= |\mu(O_{n-1}[k])| \\ b_k &= |\mu(O_n[k])| \\ c_k &= |\mu(O_{n+1}[k])|. \end{aligned}$$

First suppose $n \equiv 1(3)$. Since $n+1 \equiv 2(3)$, we wish to show

$$c_{k^*+1} > c_{k^*}.$$

Applying the recurrence (6), it suffices to show

$$2b_{k^*} + b_{k^*+1} > 2b_{k^*-1} + b_{k^*},$$

in other words,

$$b_{k^*} > 2b_{k^*-1} - b_{k^*+1}.$$

The summation formulas given in Corollary 2.1.2 and the fact $n \equiv 1(3)$ enable us to rewrite this as

$$\begin{aligned} b_{k^*} &> b_{k^*-1} + (b_{k^*-1} - b_{k^*+1}) \\ &= b_{k^*-1} + \sum_{j=k^*}^{k^*+1} \binom{n}{j} (-2)^j, \end{aligned}$$

since k^* is even. By the induction hypothesis we have $b_{k^*} > b_{k^*-1}$. It is easy to see that the quantity $\sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j$ is negative. Thus this case is finished.

Consider next $n \equiv 2(3)$. Since $n+1 \equiv 0(3)$, we wish to show

$$c_{k^*+1} > c_{k^*}. \tag{9}$$

Using the same method as in the case for $n \equiv 1(3)$ (and noting k^* is now odd), we rewrite equation (9) as

$$b_{k^*} > b_{k^*-1} - \sum_{j=k^*}^{k^*+1} \binom{n}{j} (-2)^j.$$

By the induction hypothesis $b_{k^*} > b_{k^*-1}$, so it remains to show $\sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j$ is non-negative. But in fact, this sum is equal to zero. In other words $b_{k^*-1} = b_{k^*+1}$ when $n \equiv 2(3)$. We will need this result for the next case.

For $n \equiv 0(3)$ we wish to show

$$c_{k^*+1} < c_{k^*},$$

or

$$b_{k^*} < 2b_{k^*-1} - b_{k^*+1}. \tag{10}$$

Rewriting equation (10) in terms of elements from the $(n-1)$st row, we have

$$a_{k^*+1} < 4a_{k^*-2} - 3a_{k^*}. \tag{11}$$

Observe $n - 1 \equiv 2(3)$, so by our previous observation we can conclude $a_{k^*-2} = a_{k^*}$. (Recall here that the index k^* is the index of the maximum in the nth, not the $(n-1)$st, row. The index corresponding to the maximum in the $(n-1)$st row is $k^* - 1$.) Hence equation (11) becomes

$$a_{k^*+1} < a_{k^*},$$

which holds by the induction hypothesis on the $(n-1)$st row. $\square$

2.3 Arbitrary Lower Order Ideals

In this section we give the proof of Theorem 2.0.1. To do so, we develop the notions of the shadow $\Delta(S)$ and the dual shadow $\nabla(S)$ of elements S of a fixed rank from O_n. Using bipartite graph arguments, we obtain estimates for $\Delta(S)$ and $\nabla(S)$ reminiscent of Bollobás' work on shadows for hypergraphs [9].

For the proof of Theorem 2.0.1 we suppose we have a lower order ideal $\mathcal{F} \subseteq O_n$ which maximizes $|\mu(\mathcal{F})|$. By the shadow lemmas we find bounds for the "shape" of $\mathcal{F}$, allowing us to find rank-selected lower order ideals contained in $\mathcal{F}$ and containing $\mathcal{F}$. A lemma due to Baclawski [2, Lemma 4.6] and independently to Stečkin [20] enable us to "peel off" elements from these ideals so that we can compare the Möbius values of these configurations with $\mathcal{F}$ and complete the proof.

Suppose we are given elements S all of the same rank in a poset. Since we are working with lower order ideals, we would naturally like to be able to estimate the number of elements in the poset covered by S. More formally, we define the *shadow* of a subset S of rank r in O_n by

$$\Delta(S) = \{B \in O_n(r-1) : \quad B \subseteq A \text{ for some } A \in S\}.$$

We then have an inequality involving $|\Delta(S)|$ and $|S|$.

Lemma 2.3.1 *(Shadow Lemma for O_n) If $S \subseteq O_n(r)$, where $r \geq \frac{2n+2}{3}$, then $|\Delta(S)| \geq |S|$ with equality only when $n \equiv 2(3)$ and $S = O_n(\frac{2n+2}{3})$.*

Proof. For the inequality we utilize an edge-counting argument. Consider the bipartite graph G formed in the Hasse diagram of O_n by S and $\Delta(S)$. Each vertex $A \in S$ has degree r, so the graph G has exactly $r|S|$ edges. Also, every vertex $B \in \Delta(S)$ has degree at most $2(n-r+1)$, so the number of edges in G is at most $2(n-r+1)|\Delta(S)|$. Thus when $r > \frac{2n+2}{3}$, the first part of the lemma follows.

If $n \equiv 2(3)$ and $r = \frac{2n+2}{3}$, then the above argument works as long as some vertex in $\Delta(S)$ does not have degree $\frac{2n+2}{3}$. If every vertex $B \in \Delta(S)$ has this degree, then in O_n the vertices of $\Delta(S)$ are only adjacent to vertices of S (and vice-versa). Hence if $S \subset O_n(r)$ (strict containment), this would contradict shellability of the chain complex of O_n ([5], [6]). $\square$

In an analogous manner, we define the *dual shadow* of S, where S is a subset of rank r in O_n, by

$$\nabla(S) = \{B \in O_n(r+1) : \quad B \supseteq A \text{ for some } A \in S\}.$$

We also have a dual shadow lemma for O_n. Since the proof is virtually identical to that of Lemma 2.3.1, we shall simply state the result.

Lemma 2.3.2 *(Dual Shadow Lemma for O_n) If $S \subseteq O_n(r)$ where $r \leq \frac{2n-1}{3}$, then $|\nabla(S)| \geq |S|$ with equality only when $n \equiv 2(3)$ and $S = O_n(\frac{2n-1}{3})$.* $\square$

Now we are ready to give a proof of Theorem 2.0.1. Let $\mathcal{F} \subseteq O_n$ be a lower order ideal with maximum $|\mu(\mathcal{F})|$ and let k be the maximum rank of an element in $\mathcal{F}$. We will first derive some expressions that will enable us to compare $\mu(\mathcal{F})$ with $\mu(\mathcal{F}[k-1])$ and $\mu(\mathcal{F}[k-2])$, yielding an upper bound for k. The Shadow Lemma for O_n and the following proposition, due to Baclawski and independently to Stečkin, enable us to do this. Here max P denotes the set of maximal elements of the poset P. Recall that for P a bounded poset, $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$.

Proposition 2.3.3 *[2, Lemma 4.6] [20] Let P be a bounded poset. If $T \subseteq \max \bar{P}$ then*

$$\mu(P) = \mu(P \setminus T) - \sum_{x \in T} \mu(\hat{0}, x). \tag{12}$$

This result follows by counting the chains in P not containing elements of T and the chains in P containing elements of T. Proposition 2.3.3 is useful in that it enables us to see how the Möbius function of a poset changes if we "peel off" some (or all) of its top elements.

Applying equation (12) to $P = \mathcal{F}$, $T = \mathcal{F}(k)$, and recalling $\mu(\hat{0}, x) = (-1)^k$ for $x \in O_n$ of rank k, gives

$$\mu(\mathcal{F}[k-1]) = \mu(\mathcal{F}) + (-1)^k |\mathcal{F}(k)|. \tag{13}$$

Similarly applying equation (12) to $P = \mathcal{F}[k-1]$, $T = \mathcal{F}(k-1)$, substituting for $\mu(\mathcal{F}[k-1])$ in equation (13), and solving for $\mu(\mathcal{F}[k-2])$ gives

$$\mu(\mathcal{F}[k-2]) = \mu(\mathcal{F}) - (-1)^k (|\mathcal{F}(k-1)| - |\mathcal{F}(k)|). \tag{14}$$

Since $\mathcal{F}$ is a lower order ideal, we have $\Delta\mathcal{F}(k) \subseteq \mathcal{F}(k-1)$, so

$$|\mathcal{F}(k-1)| - |\Delta\mathcal{F}(k)| \geq 0.$$

Suppose $k > \lceil \frac{2n}{3} \rceil$. By the Shadow Lemma 2.3.1 we know $|\Delta\mathcal{F}(k)| > |\mathcal{F}(k)|$. Hence $|\mathcal{F}(k-1)| - |\mathcal{F}(k)| > 0$. After considering all the possibilities for the sign of $\mu(\mathcal{F})$ and the parity of k, we see one of equations (13), (14) implies $|\mu(\mathcal{F})|$ is not a maximum, contradicting the fact that $\mathcal{F} \subseteq O_n$ is an ideal with maximum $|\mu(\mathcal{F})|$. Hence $k \leq \lceil \frac{2n}{3} \rceil$.

We will now work with the Dual Shadow Lemma to extract further information about the structure of $\mathcal{F}$. Let

$$\mathcal{F}^c = \{B : \quad B \in O_n \setminus \hat{\mathcal{F}}\}$$

and define

$$\ell = \min\{\text{rank}(B) : \quad B \in \mathcal{F}^c\}.$$

Form $\mathcal{G} = \mathcal{F} \cup \mathcal{F}^c(\ell)$ and $\mathcal{H} = \mathcal{G} \cup \mathcal{F}^c(\ell+1)$. As before, we apply equation (12) first to $P = \mathcal{G}$, $T = \mathcal{F}^c(\ell)$ and then to $P = \mathcal{H}$, $T = \mathcal{F}^c(\ell+1)$ to obtain equations resembling equations (13) and (14):

$$\mu(\mathcal{G}) = \mu(\mathcal{F}) - (-1)^\ell |\mathcal{F}^c(\ell)| \tag{15}$$

$$\mu(\mathcal{H}) = \mu(\mathcal{F}) + (-1)^\ell (|\mathcal{F}^c(\ell+1)| - |\mathcal{F}^c(\ell)|). \tag{16}$$

Now $\nabla\mathcal{F}^c(\ell) \subseteq \mathcal{F}^c(\ell+1)$, implying $|\mathcal{F}^c(\ell+1)| - |\nabla\mathcal{F}^c(\ell)| \geq 0$. Suppose $\ell < \frac{2n-1}{3}$. We apply the Dual Shadow Lemma 2.3.2 to conclude

$$|\mathcal{F}^c(\ell+1)| - |\mathcal{F}^c(\ell)| > 0.$$

Once we consider all the possibilities for the sign of $\mu(\mathcal{F})$ and the parity of ℓ, we see that one of equations (15), (16) implies $|\mu(\mathcal{F})|$ is not a maximum. Hence we must have $\ell \geq \frac{2n-1}{3}$.

To finish this argument, we reason in the following manner: for each equivalence class of n modulo 3, the bounds for k and ℓ will enable us to find rank-selected lower order ideal configurations containing $\mathcal{F}$ and contained in $\mathcal{F}$, respectively. As before, we will apply equation (12) to obtain expressions for the Möbius function of these two lower order ideal configurations, apply a parity and sign argument, and then the rank-selection Lemma 2.2.1 to derive the required result.

By definition of k and ℓ, we have $k \geq \ell - 1$. We first consider the case $n \equiv 0(3)$. We have $k \geq \ell - 1 \geq \frac{2n}{3} - 1$, and from before $k \leq \frac{2n}{3}$, implying $k = \frac{2n}{3}$ or $\frac{2n}{3} - 1$. For convenience, let $r = \frac{2n}{3}$. Then we have

$$O_n([r-1]) \subseteq \overline{\mathcal{F}} \subseteq O_n([r]),$$

where the first containment follows from the definition of ℓ and its bounds, while the second from the definition of k and its bounds. Note that r is even in this case, so by equation (13) we have

$$\mu(O_n[r-1]) = \mu(\mathcal{F}) + |\mathcal{F}(r)|. \tag{17}$$

By the same token, equation (15) becomes

$$\mu(O_n[r]) = \mu(\mathcal{F}) - |O_n(r) \setminus \mathcal{F}(r)|. \tag{18}$$

If $\mu(\mathcal{F}) > 0$, then equation (17) and the maximality of $|\mu(\mathcal{F})|$ imply $\mathcal{F}(r) = \emptyset$. Hence $\overline{\mathcal{F}} = O_n[r-1] = O_n[\lfloor \frac{2n}{3} \rfloor - 1]$, contradicting the rank-selection Lemma 2.2.1. Otherwise $\mu(\mathcal{F}) < 0$, so equation (18) plus the maximality of $|\mu(\mathcal{F})|$ imply $O_n(r) = \mathcal{F}(r)$. Thus $\overline{\mathcal{F}} = O_n[r] = O_n[\lfloor \frac{2n}{3} \rfloor]$, as desired.

Suppose $n \equiv 1(3)$. Again, by the inequalities derived for k and ℓ and their definitions, we obtain $k = \frac{2n+1}{3}$ or $\frac{2n+1}{3} - 1$. Letting $r = \frac{2n+1}{3}$, we have

$$O_n([r-1]) \subseteq \overline{\mathcal{F}} \subseteq O_n([r]).$$

Notice r is odd in this case, so again equations (13) and (15) reduce to

$$\mu(O_n[r-1]) = \mu(\mathcal{F}) - |\mathcal{F}(r)| \tag{19}$$

and

$$\mu(O_n[r]) = \mu(\mathcal{F}) + |O_n(r) \setminus \mathcal{F}(r)|. \tag{20}$$

If $\mu(\mathcal{F}) > 0$, then equation (20) and the maximality of $|\mu(\mathcal{F})|$ imply $O_n(r) = \mathcal{F}(r)$, so $\overline{\mathcal{F}} = O_n[\lfloor \frac{2n}{3} \rfloor + 1]$, contradicting Lemma 2.2.1. If $\mu(\mathcal{F}) < 0$, then equation (19) and the maximality of $|\mu(\mathcal{F})|$ imply $\mathcal{F}(r) = \emptyset$. Thus $k = r - 1$ and $\overline{\mathcal{F}} = O_n[\lfloor \frac{2n}{3} \rfloor]$, as desired.

Finally, suppose $n \equiv 2(3)$. In this case $k = \frac{2n-1}{3} - 1$, $\frac{2n-1}{3}$, or $\frac{2n-1}{3} + 1$. Letting $r = \frac{2n-1}{3}$ we have

$$O_n([r-1]) \subseteq \overline{\mathcal{F}} \subseteq O_n([r+1]).$$

Suppose $k = r+1$. We will see that this will lead to a contradiction. The choices for k will then reduce to $r-1$ or r, and the remainder of the argument will proceed as in the $n \equiv 0(3)$ and $n \equiv 1(3)$ cases. Since r is odd, equations (13) and (14) become

$$\mu(\mathcal{F}[r]) = \mu(\mathcal{F}) + |\mathcal{F}(r+1)| \tag{21}$$

and

$$\mu(O_n[r-1]) = \mu(\mathcal{F}) + |\mathcal{F}(r+1)| - |\mathcal{F}(r)|. \tag{22}$$

If $\mu(\mathcal{F}) > 0$ then equation (21) and the maximality of $|\mu(\mathcal{F})|$ imply $\mathcal{F}(r+1) = \emptyset$, contrary to our assumption on k. Thus $\mu(\mathcal{F}) < 0$. However, recall that the Shadow Lemma applied to $S = \mathcal{F}(r+1)$ has

$$|\mathcal{F}(r+1)| \leq |\Delta\mathcal{F}(r+1)| \leq |\mathcal{F}(r)|,$$

implying the difference $|\mathcal{F}(r+1)| - |\mathcal{F}(r)| \leq 0$. If the difference is negative, this contradicts the maximality of $|\mu(\mathcal{F})|$. If the difference is zero, then (by the Shadow Lemma again) $\mathcal{F}(r+1) = O_n(\frac{2n+2}{3})$. So $\overline{\mathcal{F}} = O_n[\frac{2n+2}{3}]$, contradicting Lemma 2.2.1.

The present situation is k is $r-1$ or r, so we have

$$O_n([r-1]) \subseteq \overline{\mathcal{F}} \subseteq O_n([r]).$$

Since r is odd, the same reasoning as in the $n \equiv 1(3)$ case shows that equations (19) and (20) continue to hold. Now if $\mu(\mathcal{F}) < 0$, equation (19) and the maximality of $|\mu(\mathcal{F})|$ imply $\mathcal{F}(r) = \emptyset$. Thus $\overline{\mathcal{F}} = O_n[r-1] = O_n[\lfloor \frac{2n}{3} \rfloor - 1]$, contradicting the rank-selection Lemma 2.2.1. Therefore, $\mu(\mathcal{F}) > 0$, so the maximality of $|\mu(\mathcal{F})|$ implies $O_n(r) \setminus \mathcal{F}(r) = \emptyset$, i.e., $\overline{\mathcal{F}} = O_n[r] = O_n[\lfloor \frac{2n}{3} \rfloor]$.

We thus conclude the extremal configuration occurring in Lemma 2.2.1 coincides with the extremal configuration for Theorem 1. □

3 Arbitrary Rank-selections

Now we address the arbitrary rank-selection case for the n-octahedron. We present an approach based upon a non-commutative polynomial Φ called the *cd*-index. Recently Purtill [13] showed that for certain lattices L, $\Phi(L)$ has non-negative coefficients. Stanley [18] generalized Purtill's results to face lattices of convex polytopes, which permitted him to observe the following theorem.

Theorem 3.0.4 *(Stanley) Let L_P be the lattice of faces of a convex polytope P, where the rank of L_P is $n+1$. For arbitrary rank-selections $S \subseteq [n]$ of L_P, $|\mu(L_P(S))|$ attains a maximum when we take S to be*

$$S = \{1, 3, 5, \ldots\} \cap [n] \text{ or } S = \{2, 4, 6, \ldots\} \cap [n].$$

We reconstruct the proof behind Stanley's observation for face lattices of convex polytopes of rank $n+1$ (i.e., *n-polytopes*). Furthermore, we show a *uniqueness result*, in other words, that the alternating ranks are the only rank-selection which maximize the Möbius function over arbitrary rank-selections from O_n.

Theorem 3.0.5 *For arbitrary rank-selections $S \subseteq [n]$ of O_n, $|\mu(O_n(S))|$ attains a unique maximum when we take S to be*

$$S = \{1, 3, 5, \ldots\} \cap [n] \text{ or } S = \{2, 4, 6, \ldots\} \cap [n].$$

In this case $|\mu(O_n(S)| = E_n^{\pm}$, the nth signed Euler number.

3.1 The *ab*-index and the *cd*-index

This section serves as a brief introduction to the *ab*-index and the *cd*-index. We follow [18] for all notation and terminology related to the *cd*-index. For those interested in studying the *cd*-index's origins, we refer to the paper of Bayer and Klapper [4].

Let P be a finite, graded poset of rank $n+1$ that is bounded. For $S \subseteq [n]$ let $\alpha(S) = \alpha_P(S)$ denote the number of maximal chains of $P(S) \cup \{\hat{0}, \hat{1}\}$. The *beta invariant* $\beta(S) = \beta_P(S)$ is defined by

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha(T).$$

By the Principle of Inclusion and Exclusion,

$$\alpha(S) = \sum_{T \subseteq S} \beta(T).$$

Two vectors associated with these statistics are the *flag f-vector* and *flag h-vector* of P, where $\alpha, \beta : 2^n \to \mathbb{Z}$ and $S \mapsto \alpha(S)$, and $S \mapsto \beta(S)$, respectively.

We now encode the flag h-vector (equivalently, the flag f-vector) of the poset P. For $S \subseteq [n]$, first define a monomial u_S in the non-commutative variables a and b by

$$u_S = u_1 \cdots u_n,$$

where

$$u_i = \begin{cases} a, & \text{if } i \notin S \\ b, & \text{if } i \in S. \end{cases}$$

(For our purposes, it will be helpful to think of a as "ascent" and b as "descent".) As an example, if $n = 5$ and $S = \{1, 4, 5\}$, then $u_S = baabb$. Form a non-commutative polynomial, called the *ab-index*, by

$$\Psi_P(a, b) = \sum_{S \subseteq [n]} \beta_P(S) u_S.$$

We define the degree of both a and b be 1 so that $\Psi_P(a, b)$ is homogeneous of degree n.

When P is an Eulerian poset (refer to [17, Chapter 3] for terminology), Bayer and Billera [3] showed the flag h-vector β_P satisfies certain linear relations called the *generalized Dehn-Sommerville equations*. In the literature these equations are also referred to as the *Bayer-Billera relations*. Fine observed that having β_P satisfy the Bayer-Billera relations is equivalent to having the ab-index contained in the algebra generated by the two elements $c = a + b$ and $d = ab + ba$.

Proposition 3.1.1 *[4, Theorem 4] (Fine) Let P be a finite graded poset that is also bounded. Then the flag h-vector β_P satisfies the Bayer-Billera relations if and only if $\Psi_P(a, b)$ can be written as a polynomial $\Phi_P(c, d)$ in the non-commutative variables $c = a + b$ and $d = ab + ba$.*

We call the polynomial $\Phi_P(c, d)$ the *cd-index* of P.

As noted in [18] the cd-index has some very nice algebraic properties. If $\Phi_P(c, d)$ exists (which it does for Eulerian posets such as B_n and O_n), then it is unique. (As non-commutative polynomials over any field K, $c = a + b$ and $d = ab + ba$ are algebraically independent.) If we define the degree of c to be 1 and the degree of d to be 2, then $\Phi_P(c, d)$ is homogeneous of degree n with integer coefficients. By the very nature of the variables $c = a + b$ and $d = ab + ba$, if $\Phi_P(c, d)$ exists then its associated ab-index $\Psi_P(a, b)$ is symmetric in a and b.

Purtill proved the following result about the cd-index. For shorthand we write $\Phi(P)$ for $\Phi_P(c, d)$.

Theorem 3.1.2 *[13, Theorem 6.1] For B_n and O_n, its cd-index $\Phi(B_n)$, respectively $\Phi(O_n)$, has non-negative coefficients.*

He also established recurrences for $\Phi(B_n)$ and $\Phi(O_n)$.

Proposition 3.1.3 *[13, Corollary 5.8] The following recurrence holds for the cd-index of the Boolean algebra:*

$$\Phi(B_{n+2}) = c\Phi(B_{n+1}) + \sum_{j=1}^{n} \binom{n}{j} \Phi(B_j) d \Phi(B_{n+1-j}), \quad n \geq 0,$$

with $\Phi(B_1) = 1$.

Proposition 3.1.4 *[13, Corollary 5.12] The following recurrence holds for the cd-index of the $n+1$-octahedron:*

$$\Phi(O_{n+1}) = c\Phi(O_n) + \sum_{j=1}^{n} 2^j \binom{n}{j} \Phi(B_j) d\Phi(O_{n-j}), \quad n \geq 0,$$

with $\Phi(O_0) = 1$.

Some values for $\Phi(B_n)$ and $\Phi(O_n)$ are given below:

$$\begin{aligned}
&\Phi(B_1) = 1\\
&\Phi(B_2) = c\\
&\Phi(B_3) = c^2 + d\\
&\Phi(B_4) = c^3 + 2cd + 2dc\\
&\Phi(B_5) = c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2
\end{aligned}$$

$$\begin{aligned}
&\Phi(O_0) = 1\\
&\Phi(O_1) = c\\
&\Phi(O_2) = c^2 + 2d\\
&\Phi(O_3) = c^3 + 6cd + 4dc\\
&\Phi(O_4) = c^4 + 14c^2d + 16cdc + 6dc^2 + 20d^2\\
&\Phi(O_5) = c^5 + 30cdc^2 + 48c^2dc + 30c^3d + 100cd^2 + 64d^2c + 80dcd + 8dc^3
\end{aligned}$$

3.2 Arbitrary Rank-selections: L_P and O_n

In this section we will prove the arbitrary rank-selection result for face lattices of convex polytopes L_P. We begin by making some simple observations about monomials in the non-commutative variables c and d. If $w = w_1 \cdots w_n$ is an ab-word of degree n, then we say w has a *double ascent at position i* if $w_i = w_{i+1} = a$ and has a *double descent at position i* if $w_i = w_{i+1} = b$.

Lemma 3.2.1 *Given any monomial w of degree n in the non-commutative variables $c = a + b$ and $d = ab + ba$, its ab-expansion satisfies*

(i) the alternating initial ascent word of length n occurs exactly once
(ii) the alternating initial descent word of length n occurs exactly once
(iii) any given ab-word of length n occurs at most once.
(iv) If $w = w'dw''$ with deg $w' = i - 1$ then its ab-expansion contains no ab-word with a double ascent or double descent at position i.

Here the *alternating initial ascent word* of length n is $\underbrace{ababab\ldots}_{n}$, while the *alternating initial descent word* of length n is $\underbrace{bababa\ldots}_{n}$. We say an ab-word of length n is *alternating* if it is the alternating initial descent or ascent word of length n.

Proof. Properties (i) through (iii) follow by induction on the degree n of a monomial in c and d. Given a monomial w in the variables c and d of degree $n \geq 1$, remove its first letter. Apply the induction hypothesis for words of degree $n-1$ or $n-2$, depending upon whether w began with the letter c or d.

Property (iv) follows from direct observation.□

The uniqueness result in Theorem 3.0.5 requires two key observations about $\Phi(O_n)$.

Lemma 3.2.2 *The following two properties hold for* $\Phi(O_n)$*:*

(i) *all possible monomials of degree* n *in the variables* c *and* d *occur at least once in* $\Phi(O_n)$

(ii) *for* $n \geq 0$ *if* m *is some monomial of length* n *in the variables* a *and* b *that is not alternating, then there exists a monomial* w *in* $\Phi(O_n)$ *whose* ab*-expansion makes no contribution to* m*.*

Proof. We first show (i) holds by induction on n. Property (i) is vacuously true for $n = 0$ since $\Phi(O_0) = 1$.

Now assume $n \geq 1$. By the recurrence of Proposition 3.1.4

$$\Phi(O_n) = c\Phi(O_{n-1}) + \sum_{j=1}^{n-1} 2^j \binom{n-1}{j} \Phi(B_j) d \Phi(O_{n-j-1}),\ n \geq 1, \tag{23}$$

with $\Phi(O_0) = 1$. Suppose w is a monomial of degree n in the variables c and d. If w begins with the variable c, then by the induction hypothesis the term $c\Phi(O_{n-1})$ appearing in equation (23) contains the monomial $w = cw'$. We do not have to worry about any possible cancellation since the coefficients of $\Phi(O_n)$ are clearly non-negative from equation (23). If w begins with the variable d, then a similar argument using the term corresponding to $j = 1$ in equation (23) completes (i).

Property (ii) is vacuously true when $n = 0$ since there are no non-alternating words of length 1 in the variables a and b. For $n \geq 1$, let m be a monomial of length n in the variables a and b that is not alternating. Then m has a double ascent or double descent. Without loss of generality we may assume m has a double ascent aa at the ith position. (The same argument works when m has a double descent bb at the ith position.) By property (i) we know every monomial of degree n in the variables c and d occurs at least once in $\Phi(O_n)$, including monomials of the form $w = w'dw''$ with $deg\ w' = i - 1$. By Lemma 3.2.1 the ab-expansion of any such w makes no contribution to m. □

An analogous version of Lemma 3.2.2 holds for B_n. Its proof is omitted since it is virtually identical to that of Lemma 3.2.2.

Lemma 3.2.3 *The following two properties hold for $\Phi(B_n)$:*

(i) *all possible monomials of degree $n-1$ in the variables c and d occur at least once in $\Phi(B_n)$*

(ii) *for $n \geq 1$ if m is some monomial of length $n-1$ in the variables a and b that is not alternating, then there exists a monomial w in $\Phi(B_n)$ whose ab-expansion makes no contribution to m.* $\square$

From Lemmas 3.2.1 and 3.2.2, we can now very easily give the proofs of the arbitrary rank-selection theorems for L_P and O_n, Theorems 3.0.4 and 3.0.5. We first consider the arbitrary rank-selection question for L_P, and then prove the uniqueness result for O_n.

Proof of Theorems 3.0.4 and 3.0.5. It is a well-known fact that L_P is Eulerian. Thus, $\Phi(L_P)$ exists by Bayer and Billera's work in [3] and Fine's equivalence in Proposition 3.1.1. By Stanley [18, Theorem 3.1.2], the coefficients of $\Phi(L_P)$ are non-negative.

Consider the ab-expansion of $\Phi(L_P)$ into $\Psi(L_P)$. By Lemma 3.2.1 each monomial in $\Phi(L_P)$ contributes exactly one term to the alternating initial ascent word of length n, exactly one term to the alternating initial descent word of length n, and at most one term to all other words of length n in the variables a and b. Thus, the coefficient of the alternating permutations of length n with initial ascent (equivalently, with initial descent) equals the number of monomials appearing in $\Phi(L_P)$. In turn, this number is just the sum of all the coefficients of $\Phi(L_P)$, i.e., $\Phi_{L_P}(1,1)$. (Here we are formally substituting $c=1$ and $d=1$ in $\Phi_{L_P}(c,d)$.) Since L_P is CL-shellable (Theorem [8, Theorem 4.5], the coefficient $\beta(S)$ of any ab-word u_S in $\Psi(L_P)$ equals $|\mu(L_P(S))|$.

We have shown the alternating ab-words of length n receive the greatest contribution from the coefficients of $\Phi(L_P)$. In fact, they receive the maximum possible contribution, since the expansion of each monomial in $\Phi(L_P)$ contributes at most one to each ab-word. Thus, two of the extremal configuration for rank-selections $S \subseteq [n]$ from L_P are

$$S = \{1,3,5,\ldots\} \cap [n] \text{ or } S = \{2,4,6,\ldots\} \cap [n].$$

This is Theorem 3.0.4.

We next show that for O_n the odd and even alternating rank-selections are the only extremal configurations. Suppose m is not an alternating word in the variables a and b. By property (ii) of Lemma 3.2.2 there exists a monomial w in $\Phi(O_n)$ whose ab-expansion does not make any contribution to m. Thus the coefficient of m in $\Psi(O_n)$ is less than that of the alternating initial ascent (or descent) ab-word of length n. $\square$

As a comment, we could use Lemma 3.2.3, the B_n counterpart of Lemma 3.2.2, in the argument just given to reprove the comparable result for the Boolean algebra [15, Theorem 1.2, part 2]. Also, in the proof of Theorem 3.0.5 we did not have to appeal to Stanley's non-negativity of the coefficients of $\Phi(L_p)$ and the CL-shellability of L_p. Instead, we could have used Purtill's non-negativity of the coefficients of $\Phi(O_n)$ and the fact O_n has an R-labeling.

In a personal communication with Richard Stanley, we brought to his attention that we cannot conclude uniqueness of the extremal configuration for L_P other than the cases when L_P is B_n or O_n. Consequently, Stanley has made the following conjecture:

Conjecture 3.2.4 *[19, Conjecture 2.7] Among all n-polytopes (or more generally Gorenstein* lattices of rank $n+1$), the simplex (i.e., the Boolean algebra) has the least cd-index coefficient-wise.*

Stanley's conjecture immediately implies the uniqueness result for Theorem 3.0.4: it implies that every cd-word occurs at least once in $\Phi(L_P)$, since every cd-word occurs at least once in $\Phi(B_n)$ with positive coefficients. This would give an L_P counterpart of part (i) of Lemma 3.2.2. Moreover, since part (ii) of Lemma 3.2.2 follows from part (i) of the same lemma and from Lemma 3.2.1, Lemma 3.2.2 would hold for any L_P, not just O_n and B_n.

4 Comments and Questions

A natural question to ask is the following: for which posets P will the extremal configuration maximizing the Möbius function over lower order ideals in P correspond to a rank-selected lower order ideal. We are currently working on classifying these posets.

It is interesting to note that for both the Boolean algebra and the n-octahedron, the lower order extremal configuration is of the form $P[1,a] \cup \{\hat{0}\}$. Here a corresponds to the rank in the respective poset P having the greatest number of elements. We would like to determine which posets have this property.

The question of maximizing the Möbius function for intervals of rank-selections from O_n (i.e., rank-selected subposets of the form $O_n[i,j] \cup \{\hat{0},\hat{1}\}$) is also being considered. For completeness, we state our conjecture here:

Conjecture 4.0.5 *For intervals of rank-selections $[i,j] \subseteq [n]$ of O_n, where $n > 0$ is fixed and $n \neq 2$, $|\mu(O_n(S))|$ achieves a unique maximum when*

$$S = [\lfloor \tfrac{2n+5}{5} \rfloor, \lfloor \tfrac{4n+1}{5} \rfloor].$$

For $n = 2$, the maxima occur when

$$S = [1,1] \text{ or } [2,2].$$

Extremal questions for q-analogues of the Boolean algebra and the n-octahedron are also being studied. These posets are the lattice of subspaces of an n-dimensional vector space over a finite field, and the poset of isotropic subspaces. For the lattice of subspaces the entire poset is the extremal configuration in the lower order ideal case [15]. All of the ranks of the subspace lattice is also the extremal configuration for its interval of ranks and arbitrary rank-selection cases [14]. We expect similar results for the poset of isotropic subspaces.

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