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A Probabilistic Approach to the Descent Statistic

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We present a probabilistic approach to studying the descent statistic based upon a two-variable probability density. This density is log concave and, in fact, satisfies a higher order concavity condition. From these properties we derive quadratic inequalities for the descent statistic. Using Fourier series, we give exact expressions for the Euler numbers and the alternating r-signed permutations. We also obtain a probabilistic interpretation of the sin function.

1. INTRODUCTION

Given a permutation $\sigma = \sigma_1 \cdots \sigma_{n+1}$ in the symmetric group S_{n+1}, the *descent word* $u = u_1 \cdots u_n$ is the word in the variables a and b with $u_i = a$ if $\sigma_i > \sigma_{i+1}$ and b otherwise. This encodes the classical notion of the descent set [14]. For u an ab-word of length n, the *descent statistic* of u, denoted $[u]$, is the number of permutations in the symmetric group S_{n+1} having ab-word u.

A natural question to ask is for which ab-word u of length n is $[u]$ maximal. Historically this statistic has been studied in the language of the

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descent set. However, we will find it more convenient to use the descent word for stating and proving our results. Niven and de Bruijn [3, 11] independently showed that the descent statistic is maximized for alternating *ab*-words. Sagan, Yeh and Ziegler [13] showed $[uaav] \leqslant [uab\bar{v}]$, where $\bar{v}$ denotes the *ab*-word v with the a's and b's uniformly exchanged. This result also appears in the work of Viennot [17, 18]. Both of these inequalities follow from the non-negativity of the **cd**-index; see [12; 15, Corollary 2.9]. Gessel conjectured that among permutations with r runs, the descent statistic is maximized when the runs have roughly equal length. This was proven by Ehrenborg and Mahajan [5] by developing a large class of inequalities. Among them is the fact that the sequence in Corollary 3.6 is unimodal.

In this paper we derive quadratic inequalities for the descent statistic which can be thought of as a generalization of log concavity. The key to developing these inequalities is to recast the problem in terms of probabilities. Not only then does one have the power of probability theory at one's disposal, but also the great machinery of mathematical analysis.

For example, one efficient way to compute and study the descent statistic $[u]$ is by the Viennot triangle; see [4, 5, 10, 17, 18]. Using our probabilistic viewpoint, we develop a continuous analogue of the Viennot triangle. See identities (2.1) and (2.2).

The question of determining inequalities for the descent statistic is in part motivated by the study of flag vectors of polytopes. In fact, the descent set statistic is equivalent to the flag h-vector of the simplex. The classification of linear equalities among the flag vectors of polytopes is complete [2]. However, the question of determining inequalities is completely open for polytopes of dimension greater than three [1].

This paper is organized as follows. In Section 2 we introduce a two-variable probability density and state some of its properties related to the probabilistic interpretation of the descent statistic. In Section 3 we show concavity results of this density imply probabilistic quadratic inequalities. Using the theory of Fourier series, we then study the behavior of alternating permutations. One interesting result is a probabilistic interpretation of the sin function. In Section 5 we show these techniques also apply to the theory of r-signed permutations. Finally, in the last section, we translate our probabilistic results back to the descent statistic and conjecture a sharpening of one of the inequalities.

2. PRELIMINARIES ON DENSITY FUNCTIONS

Let a and b be noncommutative variables. For $u = u_1 \cdots u_n$ an *ab*-word of length n, we say a sequence of $n+1$ real numbers $x_1, ..., x_{n+1}$ has *descent word*

u if $x_i < x_{i+1}$ whenever $u_i = a$ and $x_i > x_{i+1}$ whenever $u_i = b$. Let S_n denote the symmetric group on n elements, that is, S_n consists of all $n!$ permutations. Since one can view a permutation as a sequence of numbers, one can associate to a given permutation a descent word in the obvious way. Thus for an *ab*-word u of length n, define the probability

$$\{u\} = P\{\text{a random permutation } \sigma \in S_{n+1} \text{ has } ab\text{-word } u\}.$$

Here random means that we pick a permutation with uniform distribution, that is, each permutation is equally likely to be chosen.

There are two natural involutions on *ab*-words. For $u = u_1 \cdots u_n$ an *ab*-word, let $u^* = u_n \cdots u_1$ denote the *ab*-word u read backwards and let $\bar{u}$ denote the operation of uniformly changing the a's in u to b's and the b's in u to a's. By symmetry of permutations we have that

$$\{u\} = \{u^*\} = \{\bar{u}\}.$$

Consider a continuous random variable with density function $f(x)$. Let $X_1, \ldots, X_{n+1}$ be a sequence of $n+1$ independent identically distributed random variables with density function $f(x)$. The probability that the sequence $X_1, \ldots, X_{n+1}$ has *ab*-word u is given by $\{u\}$. This follows from the fact that if we consider the relative order of the sequence, we obtain a random permutation.

In order to study the discrete statistic $\{u\}$, we can use continuous random variables. Define $f(u, x)$ to be the probability that the random sequence $X_1, \ldots, X_{n+1}$ has descent word u and the last entry X_{n+1} is equal to x. In other words, $f(u, x)$ is a two-variable density function where the first variable is discrete and the second is continuous.

We now state a number of identities concerning this two-variable probability density.

PROPOSITION 2.1. *Let u and v be two ab-words. Then*

$$f(1, x) = f(x),$$

$$f(ua, x) = \int_{-\infty}^{x} f(u, t) \cdot f(t)\, dt, \tag{2.1}$$

$$f(ub, x) = \int_{x}^{\infty} f(u, t) \cdot f(t)\, dt, \tag{2.2}$$

$$\{uv\} = \int_S \frac{f(u, t) \cdot f(\bar{v}^*, t)}{f(t)}\, dt, \tag{2.3}$$

where S is the support of the density function $f(x)$.

Proof. The first identity is direct. The second and third identities follow by conditioning on the last entry in the random sequence. The fourth identity follows from two facts. First, observe the two-variable density that $X_{n+1}, \ldots, X_{n+m+1}$ has *ab*-word v and the first entry is $x = X_{n+1}$ is $f(\bar{v}^*, x)$. Secondly, the conditional probability that $X_1, \ldots, X_{n+1}$ has *ab*-word u given the last entry X_{n+1} is x equals the expression $f(u, x)/f(x)$. ∎

The most natural probability distribution to work with is the uniform distribution. In Section 3 we use the uniform distribution on the unit interval $[0, 1]$. However, in Sections 4 and 5 it will be more convenient to work with the uniform distribution on the interval $[0, \pi/2]$.

COROLLARY 2.2. *Assume that the density $f(x)$ is uniform. If u is an ab-word of length n then $f(u, x)$ is a polynomial of degree n in the variable x.*

We end this section with an observation about symmetric density functions.

LEMMA 2.3. *If the density $f(x)$ is symmetric around c, that is, $f(x) = f(2 \cdot c - x)$ then*

$$f(\bar{u}, x) = f(u, 2 \cdot c - x).$$

3. QUADRATIC INEQUALITIES

In this section we restrict our attention to the uniform distribution on the interval $[0, 1]$, that is, $f(x) = 1$ for $x \in [0, 1]$ and $f(x) = 0$ otherwise. Hence we will only consider functions on the interval $[0, 1]$.

LEMMA 3.1. *Let g be an arbitrary non-negative C^2 function. Then $g^{1/n}$ is concave if and only if $g(x) \cdot g''(x) \leqslant \frac{n-1}{n} g'(x)^2$ for all x.*

Proof. This follows from noting that $g^{1/n}$ is concave if and only if $(g(x)^{1/n})'' \leqslant 0$ and from the fact

$$(g(x))^{1/n})'' = \frac{1}{n} g(x)^{\frac{1}{n}-1} \cdot \left(g(x) \cdot g''(x) - \frac{n-1}{n} g'(x)^2 \right). \quad ∎$$

THEOREM 3.2. *Let u be an ab-word of length n. Then $f(u, x)^{1/n}$ is a concave function of x.*

Proof. We proceed by induction on n. Without loss of generality, we may assume $w = ua$ with u an ab-word of length n. By equation (2.1) and the fact that $f(u, x) = g(x)^n$ for some concave function g, we have

$$f(w, x) = \int_0^x g(t)^n \, dt.$$

Thus

$$\begin{aligned}
&\frac{n}{n+1} f'(w, x)^2 - f''(w, x)\, f(w, x) \\
&\quad = \frac{n}{n+1} g(x)^{2n} - n g(x)^{n-1} g'(x) \cdot \int_0^x g(t)^n \, dt \\
&\quad = n g(x)^{n-1} \cdot \left(\frac{1}{n+1} g(x)^{n+1} - g'(x) \cdot \int_0^x g(t)^n \, dt \right) \\
&\quad = n g(x)^{n-1} \cdot \int_0^x \left(-g''(s) \cdot \int_0^s g(t)^n \, dt \right) ds,
\end{aligned}$$

where the last equality follows from integration by parts. Since g is concave, we know $-g''$ is non-negative, implying by Lemma 3.1 that $f(w, x)^{1/(n+1)}$ is a concave function in the variable x. ∎

Recall a positive function f is said to be *log-concave* if $\log f$ is concave.

PROPOSITION 3.3. *Let g be a positive C^2 function such that $g^{1/n}$ is concave. Then g is log-concave.*

Proof. We have

$$(\log g)'' = \frac{g \cdot g'' - g'^2}{g^2} \leqslant \frac{\frac{n-1}{n} g'^2 - g'^2}{g^2} \leqslant 0,$$

where the first inequality follows from Lemma 3.1. ∎

COROLLARY 3.4. *Let u be an ab-word. Then the probability density $f(u, x)$ as a function of x is log-concave.*

We are now ready to prove our main result.

THEOREM 3.5. *Let u and v be ab-words. Then the following probabilistic inequalities hold:*

$$\{uv\}\{uaav\} \leqslant \{uav\}^2, \tag{3.1}$$

$$\{uav\}\{ubv\} \leqslant \{uv\}\{uabv\}, \tag{3.2}$$

$$\{uaav\}\{ubbv\} \leqslant \{uabv\}^2. \tag{3.3}$$

Proof. Let $g(x) = f(ua, x)$ and $h(x) = f(\bar{v}^*b, x)$. By the Fundamental Theorem of Calculus, $f(u, x) = g'(x)$ and $f(\bar{v}^*, x) = -h'(x)$. By Corollary 3.4 the function $g(x)$ is log-concave. Thus the derivative of $\log g$ is a decreasing function, so we have

$$0 \leqslant g(x) \cdot g'(y) - g'(x) \cdot g(y) \qquad \text{for} \quad x \geqslant y.$$

The same inequality holds for the function h. Observe the product of these two inequalities is always non-negative regardless of whether $x \geqslant y$ or $x \leqslant y$. Integrating this product over the unit square $[0, 1]^2$, we obtain

$$0 \leqslant \int_0^1 \int_0^1 (g(x) \cdot g'(y) - g'(x) \cdot g(y)) \cdot (h(x) \cdot h'(y) - h'(x) \cdot h(y))\, dx\, dy$$

$$= 2 \int_0^1 g(x) \cdot h(x)\, dx \cdot \int_0^1 g'(x) \cdot h'(x)\, dx$$

$$-2 \int_0^1 g(x)\, h'(x)\, dx \cdot \int_0^1 g'(x)\, h(x)\, dx.$$

Dividing out the factors of 2 and using equation (2.3) to recast this expression in terms of probabilities, we see this is precisely the statement of inequality (3.1).

Letting $k(x) = f(\bar{v}^*a, x)$ and using a similar argument with the functions g and k, it is straightforward to see inequality (3.2) follows. Finally, to show inequality (3.3), square each side of (3.2). Multiply this resulting inequality with (3.1) and the dual version of (3.1), that is, $\{uv\}\{ubbv\} \leqslant \{ubv\}^2$. This gives

$$\{uv\}^2 \{uav\}^2 \{ubv\}^2 \{uaav\}\{ubbv\} \leqslant \{uv\}^2 \{uav\}^2 \{ubv\}^2 \{uabv\}^2.$$

Canceling out the non-negative term $\{uv\}^2 \{uav\}^2 \{ubv\}^2$ from both sides of this inequality yields inequality (3.3). ∎

Recall that a sequence of positive real numbers $x_0, x_1, \ldots$ is said to be log-concave if $x_{i-1} \cdot x_{i+1} \leqslant x_i^2$ for $i \geqslant 1$. By inequality (3.3) we immediately have the following result.

COROLLARY 3.6. *For $k = 0, \ldots, n$ the sequence $\{ua^k b^{n-k} v\}$ is log-concave.*

We now state a sharper version of (3.1) in Theorem 3.5 in the case $v = 1$. Notice this is a special case of Conjecture 6.2.

THEOREM 3.7. *Let u be an ab-word of length n. Then*

$$\{u\}\{uaa\} \leqslant \frac{n+2}{n+3} \cdot \{ua\}^2.$$

Proof. Define the function $g(x) = f(uaaa, x)$. Note that $g(1) = \{uaa\}$. By repeated use of the Fundamental Theorem of Calculus, we have $g'(x) = f(uaa, x)$ and $g''(x) = f(ua, x)$, implying $g'(1) = \{ua\}$ and $g''(1) = \{u\}$. By Theorem 3.2 the function $g(x)^{1/(n+3)}$ is a concave function of x. Hence by Lemma 3.1 the inequality

$$g(x) \cdot g''(x) \leqslant \frac{n+2}{n+3} \cdot g'(x)^2$$

holds for any x in the interval $[0, 1]$, in particular, when $x = 1$. ∎

4. DENSITY FOR ALTERNATING PERMUTATIONS

In this section we study the asymptotic behavior of alternating permutations. To do so, we will work with the uniform distribution on the interval $[0, \pi/2]$, that is, $f(x) = 2/\pi$ for $0 \leqslant x \leqslant \pi/2$ and $f(x) = 0$ otherwise. Let α_n be the alternating word

$$\alpha_n = \underbrace{\cdots baba}_{n}$$

of length n ending in the letter a.

THEOREM 4.1. *The function $f(\alpha_n, x)$ has the following Fourier series expansion:*

$$f(\alpha_n, x) = 2 \cdot \left(\frac{2}{\pi}\right)^{n+2} \cdot \sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \frac{(-1)^{n \cdot (k-1)/2}}{k^{n+1}} \cdot \sin(k \cdot x). \tag{4.1}$$

Moreover, for $n \geqslant 1$ the sum converges uniformly.

Proof. The cases $n = 0$ and $n = 1$ can be found in any standard text on Fourier series. Observe that the series $\sum_k 1/k^2$ converges. By the Weierstrass

M-test we obtain that for $n=1$ the series in Eq. (4.1) converges uniformly.

We prove the remaining cases by induction, where we have just completed the induction basis $n=1$. Substituting $\pi/2-t$ for x in Eq. (4.1), we obtain

$$f(\overline{\alpha_n}, t) = 2 \cdot \left(\frac{2}{\pi}\right)^{n+2} \cdot \sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \frac{(-1)^{(n+1)\cdot(k-1)/2}}{k^{n+1}} \cdot \cos(k \cdot t). \tag{4.2}$$

Integrating this uniformly convergent series from 0 to x, we obtain the desired result for $n+1$. ∎

The *Euler number* E_n is the number of permutations in the symmetric group S_n having *ab*-word α_{n-1}. As a corollary we obtain the following asymptotic expansion of the Euler numbers.

COROLLARY 4.2. *For $n \geqslant 0$, we have*

$$E_n = n! \cdot 2 \cdot \left(\frac{2}{\pi}\right)^{n+1} \cdot \sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \frac{(-1)^{(n+1)\cdot(k-1)/2}}{k^{n+1}}.$$

Proof. The cases $n=0$ and $n=1$ have to be considered separately and are straightforward to check. Recall that $E_n = n! \cdot \{\alpha_{n-1}\}$. Hence for $n \geqslant 2$ we have $E_n = n! \cdot \int_0^{\pi/2} f(\alpha_{n-1}, x) \quad dx$ and by Theorem 4.1 the result follows. ∎

This corollary is well known; see for instance [9, Eqs. (5) and (6) in Sect. 0.233, p. 9, and Eqs. (5) and (9) in Sect. 1.411, p. 42].

THEOREM 4.3. *For $n \geqslant 2$, we have the following estimate:*

$$\left| \frac{n!}{E_n} f(\alpha_{n-1}, x) - \sin(x) \right| < \frac{1}{3^{n-1}}.$$

In other words, choose an n-tuple $x_1, \ldots, x_n$ uniformly from the cube $[0, \pi/2]^n$. Given the condition that this n-tuple is alternating, that is, it satisfies $\cdots < x_{n-2} > x_{n-1} < x_n$, the last entry x_n has density approximated by $\sin(x)$ for $0 \leqslant x \leqslant \pi/2$.

Proof of Theorem 4.3. We first consider the case n is odd as then all the summands are positive. By Theorem 4.1 and its corollary, we have

$$\left|\frac{n!}{E_n} f(\alpha_{n-1}, x) - \sin(x)\right| = \left| \frac{\displaystyle\sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \frac{\sin(k \cdot x)}{k^n}}{\displaystyle\sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \frac{1}{k^{n+1}}} - \sin(x) \right|$$

$$= \frac{\left| \displaystyle\sum_{\substack{k \geqslant 3 \\ k \text{ odd}}} \frac{\sin(k \cdot x)}{k^n} - \sum_{\substack{k \geqslant 3 \\ k \text{ odd}}} \frac{1}{k^{n+1}} \cdot \sin(x) \right|}{\displaystyle\sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \frac{1}{k^{n+1}}}$$

$$\leqslant \sum_{\substack{k \geqslant 3 \\ k \text{ odd}}} \frac{1}{k^n} + \sum_{\substack{k \geqslant 3 \\ k \text{ odd}}} \frac{1}{k^{n+1}}.$$

By the integral test, we have

$$\sum_{\substack{k \geqslant 5 \\ k \text{ odd}}} \frac{1}{k^n} \leqslant \int_2^{\infty} \frac{1}{(2x-1)^n} \quad dx = \frac{1}{2(n-1)} \cdot \frac{1}{3^{n-1}}.$$

Hence, using the fact $n \geqslant 3$, our estimate becomes

$$\left|\frac{n!}{E_n} f(\alpha_{n-1}, x) - \sin(x)\right| \leqslant \frac{1}{3^n} + \frac{1}{2(n-1)} \cdot \frac{1}{3^{n-1}} + \frac{1}{3^{n+1}} + \frac{1}{2n} \cdot \frac{1}{3^n}$$

$$\leqslant \frac{3 + 9/4 + 1 + 1/2}{3^{n+1}}$$

$$= \frac{27/4}{3^{n+1}}.$$

By a similar argument, where we are now instead working with alternating sums, the estimate in the case n is even becomes

$$\left|\frac{n!}{E_n} f(\alpha_{n-1}, x) - \sin(x)\right| \leqslant \frac{\dfrac{1}{3^n} + \dfrac{1}{2(n-1)} \cdot \dfrac{1}{3^{n-1}} + \dfrac{1}{3^{n+1}}}{1 - \dfrac{1}{3^{n+1}}}$$

$$\leqslant \frac{17/2}{3^{n+1} - 1}$$

$$< \frac{9}{3^{n+1}}. \quad \blacksquare$$

5. ALTERNATING r-SIGNED PERMUTATIONS

Let r be a positive integer. An *r-signed permutation* is a list (i_1, σ_1), ..., (i_n, σ_n), where $\sigma_1 \cdots \sigma_n$ is a permutation and the signs i_j belong to the set $\{1, ..., r\}$; see [16]. Hence there are $r^n \cdot n!$ r-signed permutations.

Now let p be an integer such that $0 \leqslant p \leqslant r$. The descent word $u = u_1 \cdots u_n$ is defined by two cases. First, for $1 \leqslant j \leqslant n-1$ we say $u_j = a$ if (i_j, σ_j) precedes (i_{j+1}, σ_{j+1}) in the lexicographic order and $u_j = b$ otherwise. Finally, $u_n = a$ if $i_n \leqslant p$ and $u_n = b$ otherwise. This last comparison is called the augmentation. Since we are comparing the last sign with the integer p, the r-signed permutation is called p-augmented. The case $p = n-1$ corresponds to augmented r-signed permutations. See [6, 7] for a poset explanation for the augmentation. Note these references consider the augmentation at the beginning of the permutation, whereas we will find it more convenient to have the augmentation at the end.

Let $E_n^{r,p}$ denote the number of p-augmented r-signed permutations with the alternating descent word α_n. As a special case observe that $E_n^{1,1}$ is E_n. It was proven in [7] that the exponential generating function for $E_n^{r,p}$ is given by

$$\sum_{n \geqslant 0} E_n^{r,p} \cdot \frac{x^n}{n!} = \frac{\sin(p \cdot x) + \cos((r-p) \cdot x)}{\cos(r \cdot x)}.$$

Let $\{u\}^{r,p}$ be the probability that a uniformly random chosen p-augmented r-signed permutation has descent word u. We will now use the results in the previous section to establish an expression for $E_n^{r,p}$.

Let $f(x)$ be the density function for the uniform distribution on the interval $[0, c]$, that is, $f(x) = 1/c$ for $0 \leqslant x \leqslant c$. From the sequence of random variables $X_1, ..., X_n$ we can obtain a random r-signed permutation by the following method. Let the sign i_j be given by $\lfloor r/c \cdot X_j \rfloor + 1$. Let Y_j be the remainder in this division with c/r, that is, $Y_j = X_j - c/r \cdot \lfloor r/c \cdot X_j \rfloor$. Observe that Y_j is uniformly distributed on the interval $[0, c/r]$. Now determine the permutation $\sigma_1, ..., \sigma_n$ by the relative order of $Y_1, ..., Y_n$. Observe that each r-signed permutation $(i_1, \sigma_1), ..., (i_n, \sigma_n)$ occurs equally likely. Moreover, if $u = u_1 \cdots u_n$ is the descent word of this r-signed permutation, then $u_1 \cdots u_{n-1}$ is the descent word of the sequence $X_1, ..., X_n$. From this argument the r-signed analogue of identities (2.1) and (2.2) follow.

LEMMA 5.1. *Let $f(x)$ be the density function of the uniform distribution on the interval $[0, c]$. The following probabilistic identities hold for p-augmented r-signed permutations:*

$$\{ua\}^{r,p} = \int_0^{p/r \cdot c} f(u,x)\,dx,$$

$$\{ub\}^{r,p} = \int_{p/r \cdot c}^{c} f(u,x)\,dx.$$

THEOREM 5.2. *For $n \geqslant 2$, the number of alternating p-augmented r-signed permutations is given by the following expression:*

$$E_n^{r,p} = r^n \cdot n! \cdot 2 \cdot \left(\frac{2}{\pi}\right)^{n+1} \cdot \sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \frac{(-1)^{n \cdot (k-1)/2}}{k^{n+1}} \cdot \sin\left(\frac{k \cdot p \cdot \pi}{2 \cdot r}\right).$$

Proof. We have that $E_n^{r,p} = r^n \cdot n! \cdot \{\alpha_n\}^{r,p} = r^n \cdot n! \cdot \int_0^{p/r \cdot \pi/2} f(\overline{\alpha_{n-1}}, x)\,dx$. The result follows by integrating equation (4.2). ∎

From this theorem we obtain the asymptotic expression for $E_n^{r,p}$ which was derived in [7].

COROLLARY 5.3. *The asymptotic behavior of the number of alternating p-augmented r-signed permutations is given by the following expression:*

$$E_n^{r,p} \sim \frac{4}{\pi} \cdot \sin\left(\frac{p \cdot \pi}{2 \cdot r}\right) \cdot \left(\frac{2 \cdot r}{\pi}\right)^n \cdot n!.$$

6. CONCLUDING REMARKS

Let u be an *ab*-word of length n. Recall the descent statistic of u, denoted $[u]$, is the number of permutations in the symmetric group on $n+1$ elements, S_{n+1}, having *ab*-word u. Thus $[u] = (n+1)! \cdot \{u\}$.

We now translate the results on probability inequalities to inequalities on the descent statistic. Theorems 3.4 and 3.7 imply the following descent statistic inequalities.

THEOREM 6.1. *Let u and v be ab-words with the sum of their lengths equal to n. Then the following inequalities hold for the descent statistic:*

$$[uv][uaav] \leqslant \frac{n+3}{n+2} \cdot [uav]^2, \tag{6.1}$$

$$[uav][ubv] \leqslant \frac{n+2}{n+3} \cdot [uv][uabv], \tag{6.2}$$

$$[uaav][ubbv] \leqslant [uabv]^2, \tag{6.3}$$

$$[u][uaa] \leqslant [ua]^2. \tag{6.4}$$

Ehrenborg and Steingrímsson have conjectured similar inequalities for the excedance set statistic [8].

Inequality (6.4) can also be proven directly using the Viennot triangle. Notice that this inequality is a special case of a stronger version of inequality (6.1). We conjecture that the stronger version is true.

Conjecture 6.2. For *ab*-words u and v having the sum of their lengths equal to n, the inequality

$$[uv][uaav] \leqslant [uav]^2$$

holds. Equivalently,

$$\{uv\}\{uaav\} \leqslant \frac{n+2}{n+3} \cdot \{uav\}^2.$$

Conjecture 6.2 is implied from the following conjectured result.

Conjecture 6.3. Assume g and h are non-negative functions with $g^{1/(k+1)}$ and $h^{1/(m+1)}$ both concave and $k+m=n$. Then

$$\int_0^1 g'(x)\, h(x)\, dx \cdot \int_0^1 g(x)\, h'(x)\, dx$$
$$\leqslant \frac{n+3}{n+2} \cdot \int_0^1 g'(x)\, h'(x)\, dx \cdot \int_0^1 g(x)\, h(x)\, dx.$$

In the case that the functions g and h have the same monotone behavior, the conjecture is true. The interesting case is when these functions have different monotone behavior.

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REFERENCES

1. M. Bayer, The extended f-vectors of 4-polytopes, *J. Combin. Theory. Ser. A* **44** (1987), 141–151.
2. M. Bayer and L. Billera, Generalized Dehn–Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143–157.
3. N. G. de Bruijn, Permutations with given ups and downs, *Nieuw Arch. Wisk* **18** (1970), 61–65.

4. R. Ehrenborg, The asymptotics of almost alternating permutations, *Adv. in Appl. Math.*, to appear.
5. R. Ehrenborg and S. Mahajan, Maximizing the descent statistic, *Ann. Comb.* **2** (1998), 111–129.
6. R. Ehrenborg and M. Readdy, The **r**-cubical lattice and a generalization of the **cd**-index, *European J. Combin.* **17** (1996), 709–725.
7. R. Ehrenborg and M. Readdy, Sheffer posets and r-signed permutations, *Ann. Sci. Math. Québec* **19** (1995), 173–196.
8. R. Ehrenborg and E. Steingrímsson, The excedance set of a permutation, *Adv. in Appl. Math.* **24** (2000), 284–299.
9. I. S. Gradshteyn and I. M. Ryzhik, "Table of Integrals, Series and Products," 5th ed., Academic Press, San Diego, 1994.
10. J. Millar, N. J. A. Sloane, and N. E. Young, A new operation on sequences: The boustrophedon transform, *J. Combin. Theory Ser. A* **76** (1996), 44–54.
11. I. Niven, A combinatorial problem on finite sequences, *Nieuw Arch. Wisk.* **16** (1968), 116–123.
12. M. Readdy, Extremal problems in the face lattice of the n-dimensional octahedron, *Discrete Math.* **139** (1995), 361–380.
13. B. E. Sagan, Y.-N. Yeh, and G. Ziegler, Maximizing Möbius functions on subsets of Boolean algebras, *Discrete Math.* **126** (1994), 293–311.
14. R. P. Stanley, "Enumerative Combinatorics, Vol. I," Wadsworth and Brooks/Cole, Pacific Grove, 1986.
15. R. P. Stanley, A survey of Eulerian posets, *in* "Polytopes: Abstract, Convex and Computational, Scarborough, ON, 1993," NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 440, pp. 301–333, Kluwer Academic, Dordrecht, 1994.
16. E. Steingrímsson, Permutation statistics of indexed permutations, *European J. Combin.* **15** (1994), 187–205.
17. G. Viennot, Permutations ayant une forme donnée, *Discrete Math.* **26** (1979), 279–284.
18. G. Viennot, Équidistribution des permutations ayant une forme donnée selon les avances et coavances, *J. Combin. Theory Ser. A* **31** (1981), 43–55.