# College Algebra 

by Avinash Sathaye, Professor of Mathematics ${ }^{1}$ Department of Mathematics, University of Kentucky



This book may be freely downloaded for personal use from the author's web site www.msc.uky.edu/sohum/ma109_fa08/fa08_edition/ma109fa08.pdf.

Any commercial use must be preauthorized by the author.
Send an email to sathaye@uky.edu for inquiries.
September 18, 2008

[^0]
## Introduction.

This book represents a significant departure from the current crop of commercial college algebra textbooks. In our view, the core material for the (non-remedial) courses defined by these tomes is but a shadow of that traditionally covered material in a reasonable high school program. Moreover, much of the material is substantially repeated from earlier study and it proceeds at a slow pace with extensive practice and a large number of routine exercises. As taught, such courses tend to be ill-advised attempts to prepare the student for extensive calculations using calculators, with supposed "real life" examples offered for motivation and practice. Given the limited time and large number of individual topics to study, the average student emerges, perhaps, with the ability to answer isolated questions and the well-founded view that the rewards of the study of algebra (and of mathematics in general) lie solely in the experience of applying opaque formulas and mysterious algorithms in the production of quantitative answers.

As rational, intelligent individuals with many demands on their time, students in such an environment are more than justified when they say to the teacher: "don't tell me too many ways of doing something; don't tell me how the formula is derived; just show me how to do the problems which will appear on the test!. Individuals, who experience only this type of mathematics leave with a static collection of tools and perhaps the ability to apply each to one or two elementary or artificial situations. In our view, a fundamental objective of the students mathematical development should be an understanding of how mathematical tools are made and the experience of working as an apprentice to a teacher, learning to build his or her own basic tools from "the ground up. Students imbued with this philosophy are prepared to profit as much from their incorrect answers, as from their correct ones. They are able to view a small number of expected outcomes of exercises as a validation of their understanding of the underlying concepts. They are further prepared to profit from those "real world applications through an understanding of them as elementary mathematical models and an appreciation of the fact that only through a fundamental understanding of the underlying mathematics can one understand the limits of such models. They understand how to participate in and even assume responsibility for their subsequent mathematical education.

This text is intended to be part of a College Algebra course which exposes students to this philosophy. Such a course will almost certainly be a compromise, particularly if it must be taught in a lecture/recitation format to large numbers of students.

The emphasis in this course is on mastering the Algebraic technique. Algebra is a discipline which studies the results of manipulating expressions (according to a set of rules which may vary with the context) to put them in convenient form, for enhanced understanding. In this view, Algebra consists of looking for ways of finding information about various quantities, even though it is difficult or even impossible to explicitly solve for them. Algebra consists of finding multiple expressions for the same quantities, since the comparison of different expressions often leads to new discoveries.

Here is our sincere request and strong advice to the reader:

- We urge the readers to approach this book with an open mind. If you do, then you will find new perspective on known topics.
- We urge the reader to carefully study and memorize the definitions. A majority of mistakes are caused by forgetting what a certain term means.
- We urge the reader to be bold. Don't be afraid of a long involved calculation. Exercises designed to reach an answer in just a step or two, often hide the true meaning of what is going on.
As far as possible, try to do the derivations yourself. If you get stuck, look up or ask. The derivations are not to be memorized, they should be done as a fresh exercise in Algebra every time you really need them; regular exercise is good for you!
- We urge the reader to be inquisitive. Don't take anything for granted, until you understand it. Don't ever be satisfied by a single way of doing things; look for alternative shortcuts.
- We also urge the reader to be creatively lazy. Look for simpler (yet correct, of course) ways of doing the same calculations. If there is a string of numerical calculations, don't just do them. Try to build a formula of your own; perhaps something that you could then feed into a computer some day.

A warning about graphing. Graphs are a big help in understanding the problem and they help you set up the right questions. They are also notorious for misleading people into wrong configurations or suggesting possible wrong answers. Never trust an answer until it is verified by theory or straight calculations.

Calculators are useful for getting answers but in this course most questions are designed for precise algebraic answers. You can and should use the calculators freely to do tedious numerical calculations or to verify your work or intuition. But you should not feel compelled to convert every answer into a decimal number, however precise. In this course $\frac{6}{8}$ and $\frac{1+\sqrt{(2)}}{1-\sqrt{(7)}}$ are perfectly acceptable answers unless the instructions specify a specific form. A computer system which is capable of infinite precision calculations can be used for study and is recommended. But make sure that you understand the calculations well.

A suggestion about proofs. We do value the creation and understanding of a proof, but often it is crucial that you get good at calculations before you know what they mean and why they are valid. Throughout the book you will find sections billed as "optional" or "can be omitted in a first reading". We strongly urge that you master the calculations first and then return to these for further understanding.

In many places, you will find challenges and comments for attentive or alert readers. They can appear obscure if you are new to the material, but will become clear
as you get the "feel" of it. Some of these are subtle points which may occur to you long after the course is finished! In other words, don't be discouraged if you don't get these right away.

The reader will quickly note that although there are numerous worked examples, there are very few exercises in this text. In particular there are no collections of problems from which the instructor might assign routine homework. Those exist, but they are in electronic form and provided through the Web Homework System (WHS) at (http://www.mathclass.org). The system was created by my University of Kentucky (UK) colleague Dr. Ken Kubota. The problems themselves were prepared by myself and UK colleague Dr. Paul Eakin using the Maple problem solving system with the MCtools macro package developed by our UK colleague Dr. Carl Eberhart.

The initial version of this text was used in pilot sections of College Algebra taught at UK in spring 2005 by Paul Eakin, our colleague and department chair, Dr. Rick Carey, and by then pre-service teachers: Amy Heilman and Sarah Stinson. ${ }^{2}$

Following that pilot, the text was extensively revised and used for about 1500 students in a large-lecture format at UK in fall 2005 and about 450 students in spring 2006. The results for fall were: $31 \% \mathrm{As}, 20 \% \mathrm{Bs}, 14 \% \mathrm{Cs}, 6 \% \mathrm{Ds}$, and $9 \%$ Fs (UK calls them Es), and $20 \%$ Ws.

In the spring the outcomes were: $16.5 \% \mathrm{As}, 20.1 \% \mathrm{Bs}, 15 \% \mathrm{Cs}, 13.5 \% \mathrm{Ds}$, and $16.2 \%$ Fs, and $18.8 \%$ Ws. In the spring, six Eastern Kentucky high school students took the course by distance learning: four made As and two dropped because of conflicts with sports practice.

In summer 2006, I worked with a team of 22 mathematicians: eighteen high school teachers, one mathematics education doctoral student, and three UK math faculty went through a week-long ( 30 hour) seminar which went (line by line, page by page) through the entire text and all of the online homework problems, discussing in detail such course characteristics as the underlying philosophy, the mathematical content, topic instructional strategies, alignment of homework and text material, alignment with high school curricula, etc. The results of that tremendous amount of effort were incorporated into the third major iteration of the text and course.

The third edition was used in Fall 2006 and Spring 2007 for both college and secondary students in a program called "Access to Algebra which is sponsored by the National Science Foundation (NSF) and the University of Kentucky (UK). In that program secondary students, mentored by their school math teachers, take the UK College Algebra course at no cost. The students take the same course in lockstep with a matched cohort of college students, doing the same homework and taking the same uniformly graded (and hand graded) examinations on the same schedule.

[^1]The program was coordinated by Lee Alan Roher. She was assisted by Jason Pridemore, Beth Kirby, and April Pilcher. The outcomes were

|  | A | B | C | D | E(F) | W |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Fall 2006: 45 college students | $33 \%$ | $27 \%$ | $18 \%$ | $0 \%$ | $16 \%$ | $7 \%$ |
| Fall 2006: 41 secondary students | $58 \%$ | $7 \%$ | $10 \%$ | $7 \%$ | $16 \%$ | $17 \%$ |
| Spring 2007: 60 college students | $50 \%$ | $33 \%$ | $4 \%$ | $0 \%$ | $4 \%$ | $8 \%$ |
| Spring 2007: 24 secondary students | $58 \%$ | $18 \%$ | $20 \%$ | $3 \%$ | $14 \%$ | $15 \%$ |
| Fall 2007: 33 college students | $39 \%$ | $29 \%$ | $12 \%$ | $10 \%$ | $2 \%$ | $7 \%$ |
| Fall 2007: 45 secondary students | $43 \%$ | $19 \%$ | $10 \%$ | $6 \%$ | $5 \%$ | $17 \%$ |
| Spring 2008: 41 secondary students | $34 \%$ | $17 \%$ | $16 \%$ | $9 \%$ | $5 \%$ | $18 \%$ |

The outcomes in the general, conventional college program which uses a commercial text were:

|  | A | B | C | D | E(F) | W |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Fall 2006: 1449 college students | $17 \%$ | $23 \%$ | $21 \%$ | $12 \%$ | $12 \%$ | $17 \%$ |
| Spring 2007: 663 college students | $17 \%$ | $21 \%$ | $18 \%$ | $14 \%$ | $17 \%$ | $14 \%$ |
| Fall 2007: 1608 college students | $21 \%$ | $26 \%$ | $23 \%$ | $11 \%$ | $9 \%$ | $10 \%$ |

The member of the summer 2005 development team members were:
Andrea OBryan of East Jessamine High School, Jessamine County, Ky; Charlotte Moore, and Sharon Vaughn of Allen Central High School, Floyd County, Ky; Karen Heavin, Marcia Smith, and Mark Miracle of West Jessamine High, Jessamine County, Ky; Cheryl Crowe and Susan Popp of Woodford High School, Woodford County, Ky; Clifton Green of Owsley County High School, Owsley County, Ky; Gina Kinser of Powell County High School, Powell County, Ky; Jennifer Howard of Magoffin County High School, Magoffin County, Ky; Joanne Romeo of Washburn High School, Grainger County, Tn, Lee Alan Roher, Paul Eakin, Ken Kubota, and Carl Eberhart of the University of Kentucky; Lisa Sorrell and Teresa Plank of Rowan County Senior High School, Rowan County, Ky; Patty Marshall of Johnson Central High School, Johnson County, Ky; Roxanne Johnson of Wolfe County High School, Wolfe County, Ky; and Sarah Stinson of Paul Laurence Dunbar High School, Lexington, Ky.

The college algebra program continued during the academic years 2007-2008 and 2008-2009.

Teachers who have joined the Access to Algebra Team:
Scott Adams of Rockcastle County High School, Rockcastle County, Ky. Teresa Combs of Knott County High School, Knott County, Ky. Brent West of Corbin County Independent High School, Corbin County, Ky.

## Acknowledgment

I want to express my sincere appreciation to these colleagues and to the National Science Foundation and U.S. Departments of Education which have been the principal sponsors of the project, including the associated delivery technology and development seminars.

I would also like to express a special appreciation to Jason Pridemore for carefully going through the current edition and making suggestions for improvement based on his experience in teaching from the past editions.

## What is Algebra?

Algebra is the part of mathematics dealing with manipulation of expressions and solutions of equations. Since these operations are needed in all branches of mathematics, algebraic skill is a fundamental need for doing mathematics and therefore for working in any discipline which substantially requires mathematics. It is the foundation and the core of higher mathematics. A strong foundation in Algebra will help individuals become better mathematicians, analyzers, and thinkers.

The name algebra itself is a shortened form of the title of an old Arabic book on what we now call algebra, entitled al jabr wa al muk $\bar{a} b a l \bar{a}$ which roughly means manipulation (of expressions) and comparison (of equations).

The fundamental operations of algebra are addition, subtraction, multiplication and division (except by zero). An extended operation derived from the idea of repeated multiplication is the exponentiation (raising to a power).

We start Algebra with the introduction of variables. Variables are simply symbols used to represent unknown quantities and are the building blocks of Algebra.

Next we combine these variables into mathematical expressions using algebraic operations and numbers. In other words, we learn to handle them just like numbers. The real power of Algebra comes in when we learn tools and techniques to solve equations involving various kinds of expressions. Often, the expected solutions are just numbers, but we need to develop a finer idea of what we can accept as a solution. This kind of equation solving is the art of Algebra! ${ }^{3}$

A student in a College Algebra course is expected to be familiar with basic algebraic operations and sufficiently skilled in performing them with ease and speed. The

[^2]first homework and diagnostic test will help evaluate the level of these skills. Regardless of your level of preparation, there will likely become frustrated at some point. That is both natural and expected. If things begin to become frustrating, remember that learning Algebra is much like learning a new language. It is important to first have all the letters, sounds, words, and punctuation down before you can make a correct sentence, or in our case, write and solve an equation. Of course, there are people who can pick up a language without ever learning grammar, by being good at imitating others and picking up on what is important. You may be one of these few lucky ones, but be sure to verify your feeling with the correct rules. Unlike a spoken language, mathematics has a very precise structure!

## Contents

1 Polynomials, the building blocks of algebra ..... 1
1.1 Underlying field of numbers. ..... 1
1.2 Indeterminates, variables, parameters ..... 2
1.3 Basics of Polynomials ..... 4
1.3.1 Rational functions. ..... 7
1.4 Working with polynomials ..... 8
1.5 Examples of polynomial operations. ..... 11
Example 1. Polynomial operations. ..... 11
Example 2. Collecting coefficients. ..... 12
Example 3. Using algebra for arithmetic. ..... 13
Example 4. The Binomial Theorem. ..... 13
Example 5. Substituting in a polynomial. ..... 17
Example 6. Completing the square. ..... 18
2 Solving linear equations. ..... 21
2.1 What is a solution? ..... 22
2.2 One linear equation in one variable. ..... 24
2.3 Several linear equations in one variable ..... 25
2.4 Two or more equations in two variables. ..... 26
2.5 Several equations in several variables. ..... 27
2.6 Solving linear equations efficiently. ..... 29
Elimination Method. Manipulation of equations. ..... 29
Determinant Method. Cramer's Rule. ..... 30
Example 1. Exceptions to Cramer's Rule. ..... 32
Example 2. Cramer's Rule with many variables. ..... 33
3 The division algorithm and applications ..... 35
3.1 Division algorithm in integers. ..... 35
Example 1. GCD calculation in Integers ..... 38
3.2 Aryabhaṭa algorithm: Efficient Euclidean algorithm. ..... 39
Example 2. Kuttaka or Chinese Remainder Theorem. ..... 42
Example 3. More Kuttaka problems. ..... 43
3.3 Division algorithm in polynomials. ..... 44
3.4 Repeated Division. ..... 48
3.5 The GCD and LCM of two polynomials. ..... 50
Example 4. Efficient division by a linear polynomial ..... 51
Example 5. Division by a quadratic polynomial. ..... 53
4 Introduction to analytic geometry. ..... 57
4.1 Coordinate systems. ..... 57
4.2 Geometry: Distance formulas ..... 58
4.3 Change of coordinates on a line. ..... 61
4.4 Change of coordinates in the plane. ..... 62
4.5 General change of coordinates. ..... 63
Examples. Changes of coordinates. ..... 64
5 Equations of lines in the plane. ..... 67
5.1 Parametric equations of lines. ..... 67
Examples. Parametric equations of lines. ..... 69
5.2 Meaning of the parameter $\mathbf{t}$ : ..... 71
Examples. Special points on parametric lines ..... 72
5.3 Comparison with the usual equation of a line. ..... 74
5.4 Examples of equations of lines ..... 79
Example 1. Points equidistant from two given points. ..... 80
Example 2. Right angle triangles. ..... 81
6 Special study of Linear and Quadratic Polynomials. ..... 85
6.1 Linear Polynomials. ..... 85
6.2 Factored Quadratic Polynomial. ..... 86
Interval notation. Intervals on real line. ..... 87
6.3 The General Quadratic Polynomial. ..... 90
6.4 Examples of quadratic polynomials. ..... 92
7 Functions ..... 95
7.1 Plane algebraic curves ..... 95
7.2 What is a function? ..... 97
7.3 Modeling a function. ..... 99
8 The Circle ..... 103
8.1 Circle Basics. ..... 103
8.2 Parametric form of a circle. ..... 104
8.3 Application to Pythagorean Triples. ..... 106
Pythagorean Triples. Generation of. ..... 107
8.4 Examples of equations of a circle. ..... 110
Example 1. Intersection of two circles ..... 112
Example 2. Line joining through the intersection of two circles. ..... 113
Example 3. Circle through three given points. ..... 113
Example 4. Exceptions to a circle through three points. ..... 114
Example 5. Smallest circle with a given center meeting a given line. ..... 114
Example 6. Circle with a given center and tangent to a given line. ..... 115
Example 7. The distance between a point and a line. ..... 116
Example 8. Half plane defined by a line. ..... 118
8.5 Trigonometric parameterization of a circle. ..... 121
Definition. Trigonometric Functions. ..... 124
8.6 Basic Formulas for the Trigonometric Functions ..... 126
8.7 Connection with the usual Trigonometric Functions ..... 128
8.8 Important formulas described ..... 129
8.9 Using trigonometry. ..... 139
8.10 Proof of the Addition Formula. ..... 142
8.11 Identities Galore. ..... 143
9 Looking closely at a function ..... 149
9.1 Introductory examples ..... 149
Parabola. Analysis near its points. ..... 149
Circle. Analysis near its points ..... 152
9.2 Analyzing a general curve $y=f(x)$ near a point $(a, f(a))$. ..... 154
9.3 The slope of the tangent, calculation of the derivative. ..... 156
9.4 Derivatives of more complicated functions. ..... 159
9.5 General power and chain rules. ..... 160
9.6 Using the derivatives for approximation. ..... 164
Linear Approximation. Examples. ..... 164
10 Root finding ..... 167
10.1 Newton's Method ..... 167
10.2 Limitations of the Newton's Method ..... 169
11 Summation of series. ..... 171
11.1 Application of polynomials to summation formulas. ..... 172
11.2 Examples of summation of series. ..... 173

## Chapter 1

## Polynomials, the building blocks of algebra

### 1.1 Underlying field of numbers.

We begin work in Algebra with an agreement on a field of numbers to work in. In our field we will want numbers which allow the four basic operations: addition, subtraction, multiplication, and division (except by 0). We typically use the real numbers, denoted by $\Re$, the rational numbers, denoted by $\mathbb{Q}$, or (less often) the complex numbers, denoted by $\mathbb{C}$. Of these $\Re$ is probably the most familiar.
$\Re$, the set of real numbers may be thought of as the set of all possible decimal numbers (with expansions of unlimited length). Within $\Re$ are the integers $\mathbb{Z}$, which do not form a field since they don't have division in the sense that the result of dividing one integer by another (e.g. dividing 1 by 2) may not be an integer. Put another way, the integers aren't quite a field since not all "fractions" are integers. However the set of all fractions is.

The set (field) of rational numbers, denoted by $\mathbb{Q}$, consists of all numbers of the form $\pm a / b$ where $a, b$ are non negative integers with a non zero $b$.

Rational numbers are real numbers. You might have already seen that a rational number may be expressed as a finite decimal (for example $\frac{3}{4}=0.75$ or a repeating decimal $\left.\frac{4}{3}=1.333 \overline{3}\right) .{ }^{1}$ There are real numbers like $\sqrt{2}$ which have decimal expansions which are not finite and which do not have a repeating pattern. ${ }^{2}$

In a slightly more advanced setting you may find the set of complex numbers $\mathbb{C}$. The complex numbers are obtained from $\Re$ by throwing in one extra number, $i$ which satisfies $i^{2}=-1$.

By the known algebraic operations, it is easy to see that every number of the form $u+i v$ where $u, v$ are real numbers must be in $\mathbb{C}$. Conversely, it is not hard to

[^3]convince yourself that all the four algebraic operations on numbers of the form $u+i v$ result in numbers of the same form.

Indeed, it is not hard to verify the following formulas, if you don't worry about what " $i$ " means.

Formula to multiply two complex numbers and get back a number of the same form.

$$
(a+i b)(c+i d)=(a c-b d)+i(a d+b c)
$$

Formula to take the reciprocal of a complex number and get back a number of the same form.

$$
\frac{1}{(a+i b)}=\frac{a}{\left(a^{2}+b^{2}\right)}-\frac{b}{\left(a^{2}+b^{2}\right)} i
$$

It is easy to see that now all algebraic operations can be easily performed.
To verify your understanding, assign yourself the task of verifying the following set of formulas to verify: ${ }^{3}$

$$
\frac{1}{1+i}=\frac{1-i}{2},(-1 \pm \sqrt{3} i)^{3}=8,(1+i)^{4}=-4
$$

Occasionally, we may denote our set of underlying numbers by the letter $F$ (to remind us of its technical mathematical term - a field, or perhaps to emphasize that what we are doing at that time would work in any field). Usually, we may just call them numbers and most often will be talking about the field of real numbers i.e. $F=\Re$.

In modern algebra, you may find examples of finite fields. These are fields which have only finitely many numbers, the smallest one has just two numbers! They are abstract, but fun to work with and very useful in many branches of mathematics as well as applications.

### 1.2 Indeterminates, variables, parameters

When we see something like $a x^{2}+b x+c$ we usually think of this as an expression in a variable $x$ with "constants" $a, b, c$. At this stage, however, $a, b, c$ as well as $x$ are unspecified. Depending on the context it may be that $x, b, c$ are the constants and

[^4]By simple expansion, the right hand side (usually shortened to RHS) is

$$
(1)(1-i)+(i)(1-i)=1-i+i-i^{2}=1-i^{2} .
$$

Using the basic fact $i^{2}=-1$, we see that this becomes $1+1=2$ or the left hand side (LHS). Done!
$a$ is the variable. In algebraic expressions, the idea of who is constant and who is a variable is a matter of declaration or convention.

## Variables

A variable is a symbol (typically represented by a Greek or Standard English letter), used to represent an unknown quantity. However, not all symbols are variables. We need additional information or a simple declaration to tell us which is a variable and which is a constant.

Take this familiar expression for example:

$$
m x+c .
$$

On the face of it, all the letters " $m, c, x$ " are potential variables. That is, in the absence of any additional information all of them are variables.

However the expression may be encountered in connection with the equation of a line. Some of you may have seen the expression $m x+b$ instead; just read $b$ for $c$ while reading this discussion!. In that case the $m$ in $m x+c$ would be a constant and not a variable as it would be the slope of a certain line. It would be a fixed number when we know the line. Similarly, in the context of equations of lines, the letter $c$ in $m x+c$ would be a constant, the $y$-intercept (i.e. the $y$-coordinate of the intersection of the line and the $y$-axis). When we know the line, we know its specific value. Finally, when talking about equations of lines, the $x$ in $m x+c$ is the variable in the expression and represents a changing value of the $x$-coordinates as we move among different points of the line. The expression $m x+c$ then gives the $y$-coordinates of the corresponding points on the line.

Thus, when dealing with a specific line, only one of the three letters is a true "variable".

## Parameters

So, what do we call these letters like $m, c$ ? Letters which represent quantities like the slope and intercept of a general line but whose exact values are not specified. Such quantities are often called parameters. Here the idea is that a parameter is a variable which is not intended to be pinned down to a specific single value, but is expected to move thru a preassigned set of values generating interesting objects (like lines in our example). Parameters carry with them the idea of families of objects and one moves through the family as the parameter(s) change.

In the above example of $m x+c$ we may think of $m, c$ as parameters. When the quantity $m$ changes, it describes the equations of all lines with different slopes. But once a line is pinned down, the quantity $m$ is fixed. Similarly, the quantity $c$ describes families of lines with different $y$-intercepts as it changes. Once a line is pinned down, so is the quantity $c$. For a fixed line, only $x$ stays a variable.

As another example, let $1+2 t$ be the distance of a particle from a starting position at time $t$. Then $t$ might be thought of as a parameter describing the motion of the particle. It is pinned down as soon as we locate the moving point. At that instance, $t$ has a specific value.

Now, if we ask a question like:
"At what time is the distance equal to 5?",
then the same $t$ becomes a variable. To answer the question, we write an equation $1+2 t=5$ and by algebraic manipulation, determine that $t=2$.

## Indeterminates

An indeterminate usually is a symbol which is not intended to be substituted with values; thus it is like a variable in appearance, but we don't care to assign or change values for it.

Thus the statement

$$
X^{2}-a^{2}=(X-a)(X+a)
$$

is a formal statement about expansion and $X, a$ can be thought of as indeterminates.
When we use it to factor $y^{2}-9$ as $(y-3)(y+3)$ we are turning $X, a$ into variables and substituting values $y, 3$ to them! ${ }^{4}$

### 1.3 Basics of Polynomials

We now show how polynomials are constructed and handled. This material is very important for future success in Algebra.

## Monomials

Suppose that we are given a variable $x$, a constant $c$ and a non negative integer $n$. Then the expression $c x^{n}$ is said to be a monomial.

- The constant $c$ is said to be its coefficient .(Sometimes we call it the coefficient of $x^{n}$ ).
- If the coefficient $c$ is non zero, then the degree of the monomial is defined to be $n$.

If $c=0$, then we get the zero monomial and its degree is undefined.

- It is worth noting that $n$ is allowed to be zero, so 2 is also a monomial with coefficient 2 and degree 0 in any variable (or variables) of your choice!

Sometimes, we need monomials with more general exponents, meaning we allow $n$ to be a negative integer as well. The rest of the definition is the same.

Sometimes, we may even allow $n$ to be a positive or negative fraction or even a real number. Formally, this is easy, but it takes some effort and great care to give a meaning to the resulting quantity.

We will always make it clear if we are using such general exponents.
For example, consider the monomial $4 x^{3}$. Its variable is $x$, its coefficient is 4 and its degree is 3 .

[^5]Consider another example $\frac{2}{3} x^{5}$. Its variable is $x$, its coefficient is $\frac{2}{3}$ and its degree is 5 .

Consider a familiar expression for the area of a circle of radius $r$, namely $\pi r^{2}$. This is a monomial of degree 2 in $r$ with coefficient $\pi$. Thus, even though it is a Greek letter, the symbol $\pi$ is a well defined fixed real number. ${ }^{5}$

This basic definition can be made fancier as needed.
Usually, $n$ is an integer, but in higher mathematics, $n$ can be a more general object. As already stated, we can accept $-\frac{5}{2 x^{3}}=(-5 / 2) x^{-3}$ as an acceptable monomial with general exponents. We declare that its variable is $x$ and it has degree -3 with coefficient $-5 / 2$.

Notice that we collected all parts (including the minus sign) of the expression except the power of $x$ to build the coefficient!

## Monomials in several variables.

It is permissible to use several variables and write a monomial of the form $c x^{p} y^{q}$. This is a monomial in $x, y$ with coefficient $c$. Its degree is said to be $p+q$. Its exponents are said to be $(p, q)$ respectively (with respect to variables $x, y$ ). As before, more general exponents may be allowed, if necessary. ${ }^{6}$

## Example.

Consider the following expression which becomes a monomial after simplification. expression in $\mathrm{x}, \mathrm{y}$.

$$
\frac{21 x^{3} y^{4}}{-3 x y}
$$

First, we must simplify the expression thus:

$$
\frac{21 x^{3} y^{4}}{-3 x y}=\frac{21}{-3} \cdot \frac{x^{3}}{x} \cdot \frac{y^{4}}{y}=-7 x^{2} y^{3}
$$

Now it is a monomial in $x, y$.
What are the exponents, degree etc.? Its coefficient is -7 , its degree is $2+3=5$ and the exponents are $(2,3)$.

What happens if we change our idea of who the variables are?
Consider the same simplified monomial $-7 x^{2} y^{3}$.

[^6]Let us think of it as a monomial in $y$. Then we rewrite it as:

$$
\left(-7 x^{2}\right) y^{3} .
$$

Thus as a monomial in $y$ alone, its coefficient is $-7 x^{2}$ and its degree is 3 .
What happens if we take the same monomial $-7 x^{2} y^{3}$ and make $x$ as our variable?

Then we rewrite it as:

$$
\left(-7 y^{3}\right) x^{2}
$$

Be sure to note the rearrangement!
Thus as a monomial in $x$ alone, its coefficient is $-7 y^{3}$ and its degree is 2.
Here is yet another example. Consider the monomial

$$
\frac{2 x}{y}=2 x^{1} y^{(-1)} .
$$

Think of this as a monomial in $x, y$ with more general exponents allowed.
As a monomial in $x, y$ its degree is $1-1=0$ and its coefficient is 2 . Its exponents are $(1,-1)$ respectively.

## Binomials, Trinomials, $\cdots$ and Polynomials

We now use the monomials as building blocks to build other algebraic structures.
When we add monomials together, we create binomials, trinomials, and more generally polynomials.

It is easy to understand these terms by noting that the prefix "mono" means one. The prefix "bi" means two, so it is a sum of two monomials. The prefix "tri" means three, so it is a sum of three monomials. ${ }^{7}$

In general "poly" means many so we define:

## Definition: Polynomial

A polynomial is defined as a sum of finitely many monomials whose exponents are all non negative. Thus $f(x)=x^{5}+2 x^{3}-x+5$ is a polynomial.

The four monomials $x^{5}, 2 x^{3},-x, 5$ are said to be the terms of the polynomial $\mathrm{f}(\mathrm{x})$.

Usually, a polynomial is defined to be simplified if all terms with the same exponent are combined into a single term.

For example, our polynomial is the simplified form of $x^{5}+x^{3}+x^{3}+3 x-4 x+5$ and many others.

An important convention to note: Before you apply any definitions like the degree and the coefficients etc., you should make sure that you have collected all like terms together and identified the terms which are non zero after this collection.

For example, $f(x)$ above can also be written as

$$
f(x)=x^{5}+2 x^{3}+x^{2}-x+5-x^{2}
$$

[^7]but we don't count $x^{2}$ or $-x^{2}$ among its terms!
A polynomial with no terms has to admitted for algebraic reasons and it is denoted by just 0 and called the zero polynomial .

About the notation for a polynomial: When we wish to identify the main variable of a polynomial, we include it in the notation. If several variables are involved, several variables may be mentioned. Thus we may write: $p(x)=x^{3}+a x+5, h(x, y)=$ $x^{2}+y^{2}-r^{2}$ and so on. With this notation, all variables other than the ones mentioned in the left hand side are treated as constants for that polynomial.

The coefficient of a specific power (or exponent) of the variable in a polynomial is the coefficient of the corresponding monomial, provided we have collected all the monomials having the same exponent.

Thus the coefficient of $x^{3}$ in $f(x)=x^{5}+2 x^{3}-x+5$ is 2 . Sometimes, we find it convenient to say that the $x^{3}$ term of $f(x)$ is $2 x^{3}$.

The coefficient of the missing monomial $x^{4}$ in $f(x)$ is declared to be 0 . The coefficient of $x^{100}$ in $f(x)$ is also 0 by the same reasoning.

Indeed, we can think of infinitely many monomials which have zero coefficients in a given polynomial. They are not to be counted among the terms of the polynomial.

What is the coefficient of $x^{3}$ in $x^{4}+x^{3}-1-2 x^{3}$. Remember that we must collect like terms first to rewrite

$$
x^{4}+\left(x^{3}-2 x^{3}\right)-1=x^{4}-x^{3}-1
$$

and then we see that the coefficient is -1 .

### 1.3.1 Rational functions.

Just as we create rational numbers from ratios of two integers, we create rational functions from ratios of two polynomials.

Definition: Rational function A rational function is a ratio of two polynomials $\frac{p(x)}{q(x)}$ where $q(x)$ is assumed to be a non zero polynomial.

Examples.

$$
\frac{2 x+1}{x-5}, \frac{1}{x^{3}+x}, \frac{x^{3}+x}{1}=x^{3}+x
$$

Suppose we have rational functions $h_{1}(x)=\frac{p_{1}(x)}{q_{1}(x)}$ and $h_{2}(x)=\frac{p_{2}(x)}{q_{2}(x)}$.
We define algebraic operations of rational functions, just as in rational numbers.

$$
h_{1}(x) \pm h_{2}(x)=\frac{p_{1}(x) q_{2}(x) \pm p_{2}(x) q_{1}(x)}{q_{1}(x) q_{2}(x)}
$$

and

$$
h_{1}(x) h_{2}(x)=\frac{p_{1}(x) p_{2}(x)}{q_{1}(x) q_{2}(x)} .
$$

It is easy to deduce that $h_{1}(x)=h_{2}(x)$ or $h_{1}(x)-h_{2}(x)=0$ if and only if the the numerator of $h_{1}(x)-h_{2}(x)$ is zero, i.e.

$$
p_{1}(x) q_{2}(x)-p_{2}(x) q_{1}(x)=0
$$

For a rational number, we know that multiplying the numerator and the denominator by the same non zero integer does not change its value. (For example $\frac{4}{5}=\frac{8}{10}$.)

Similarly, we have:

$$
\frac{p(x)}{q(x)}=\frac{d(x) p(x)}{d(x) q(x)}
$$

for any non zero polynomial $d(x)$. (For example $\frac{x}{x+1}=\frac{x^{2}-x}{x^{2}-1}$.)

### 1.4 Working with polynomials

Given several polynomials, we can perform the usual operations of addition, subtraction and multiplication on them. We can also do division, but if we expect the answer to be a polynomial again, then we have a problem. We will discuss these matters later. ${ }^{8}$

As a simple example, let

$$
f(x)=3 x^{5}+x, g(x)=2 x^{5}-2 x^{2}, h(x)=3, \quad \text { and } w(x)=2 .
$$

Then
$f(x)+g(x)=3 x^{5}+x+2 x^{5}-2 x^{2}$ and after collecting terms $f(x)+g(x)=5 x^{5}-2 x^{2}+x$.
Notice that we have also arranged the monomials in decreasing degrees, this is a recommended practice for polynomials.

What is $w(x) f(x)-h(x) g(x)$ ?
We see that

$$
(2)\left(3 x^{5}+x\right)-(3)\left(2 x^{5}-2 x^{2}\right)=6 x^{5}+2 x-6 x^{5}+6 x^{2}=6 x^{2}+2 x .
$$

Definition: Degree of a polynomial with respect to a variable $x$ is defined to be the highest degree of any monomial present in the simplified form of the polynomial (i.e. any term $c x^{m}$ with $c \neq 0$ in the simplified form of the polynomial). We shall write $\operatorname{deg}_{x}(p)$ for the degree of $p$ with respect to $x$.

[^8]The degree of a zero polynomial is defined differently by different people. Some declare it not to have a degree, others take it as -1 and yet others take it as $-\infty$. We shall declare it undefined and hence we always have to be careful to determine if our polynomial reduces to 0 .

We shall use:

## Definition: Leading coefficient of a polynomial

For a polynomial $p(x)$ with degree $n$ in $x$, by its leading coefficient, we mean the coefficient of $x^{n}$ in the polynomial.

Thus given a polynomial $x^{3}+2 x^{5}-x-1$ we first rewrite it as $2 x^{5}+x^{3}-x-1$ and then we can say that its degree is 5 , and its leading coefficient is 2 .

The following are some of the evident facts about polynomials and their degrees.
Assume that

$$
u=a x^{n}+\cdots, v=b x^{m}+\cdots
$$

are non zero polynomials in $x$ of degrees $\mathbf{n}, \mathbf{m}$ respectively.
We shall need the following important observations.

1. If we say that " $u$ has degree $n$ ", then we mean that the coefficient $a$ of $x^{n}$ is non zero and that none of the other (monomial) terms has degree as high as $n$.
Similarly, if we say that " $v$ has degree $m$ ", then we mean that $b \neq 0$ and no other terms in $v$ are of degree as high as $m$.
2. If $c$ is a non zero number, then

$$
c u=c a x^{n}+\cdots
$$

has the same degree $n$ and its leading coefficient is $c a$.
For example, if $c$ is a non zero number, then the degree of $c\left(2 x^{5}+x-2\right)$ is always 5 and the leading coefficient is $(c)(2)=2 c$.
For $c=0$ the degree becomes undefined!
3. Suppose that the degrees $n, m$ of $u, v$ are unequal.

Given constants $c, d$, what is the degree of $c u+d v$ ?
If $c, d$ are non zero, then the degree of $c u+d v$ is the maximum of $n, m$. If one of $c, d$ is zero, then we have to look closely.
For example, if $u=2 x^{5}+x-2$ and $v=-x^{3}+3 x-1$, then the degree of $c u+d v$ is determined thus:

$$
c u+d v=c\left(2 x^{5}+x-2\right)+d\left(-x^{3}+3 x-1\right)=(2 c) x^{5}+(-d) x^{3}+(c+3 d) x+(-2 c-d)
$$

Thus, if $c \neq 0$, then degree will be 5 . The leading coefficient is $2 c$.

If $c=0$ but $d \neq 0$, then

$$
c u+d v=(-d) x^{3}+(3 d) x+(-d)
$$

and clearly has degree 3 . The leading coefficient is $-d$.
If $c, d$ are both zero then $c u+d v=0$ and the degree becomes undefined.
4. Now suppose that the degrees of $u, v$ are the same, i.e. $n=m$.

Let $c, d$ be constants and consider $h=c u+d v$.
Then the degree of $h$ needs a careful analysis.
Since $m=n$, we see that

$$
h=(c a+d b) x^{n}+\cdots
$$

and hence we have:

- either $h=0$ and hence $\operatorname{deg}_{x}(h)$ is undefined, or
- $0 \leq \operatorname{deg}_{x}(h) \leq n=m$.

For example, let $u=-x^{3}+3 x^{2}+x-1$ and $v=x^{3}-3 x^{2}+2 x-2$. Calculate $c u+d v$ with $c, d$ constants.

$$
\begin{aligned}
h=c u+d v & =c\left(-x^{3}+3 x^{2}+x-1\right)+d\left(x^{3}-3 x^{2}+2 x-2\right) \\
& =(-c+d) x^{3}+(3 c-3 d) x^{2}+(c+2 d) x+(-c-2 d)
\end{aligned}
$$

We determine the degree of the expression thus:

- If $-c+d \neq 0$, i.e. $c \neq d$ then the degree is 3 .
- If $c=d$, then the $x^{2}$ term also vanishes and we get

$$
c u+d v=(c+2 d) x+(-c-2 d)=(3 d) x+(-3 d) .
$$

Thus if $d=0$ then the degree is undefined and if $d \neq 0$ then the degree is 1.

You are encouraged to figure out the formula for the leading coefficients in each case.

You should also experiment with other polynomials.
5. The product rule for degrees. The degree of the product of two non zero polynomials is always the sum of their degrees. This means:

$$
\operatorname{deg}_{x}(u v)=\operatorname{deg}_{x}(u)+\operatorname{deg}_{x}(v)=m+n
$$

Indeed, we can see that

$$
\begin{aligned}
u v & =\left(a x^{n}+\text { terms of degree less than } n\right)\left(b x^{m}+\text { terms of degree less than } m\right) \\
& =(a b) x^{(m+n)}+\text { terms of degree less than } m+n
\end{aligned}
$$

The leading coefficient of the product is $a b$, i.e. the product of their leading coefficients. Thus we have an interesting principle.
The degree and the leading coefficient of a product of two non zero polynomials can be calculated by ignoring all but the leading terms in each!

By repeated application of this principle, we can see the
The power rule for degrees. If $u=a x^{n}+\cdots$ is a polynomial of degree $n$ in $x$, then for any positive integer $d$, the polynomial $u^{d}$ has degree $d n$ and leading coefficient $a^{d}$.
You should think and convince yourself that it is natural to define $u^{0}=1$ for all non zero polynomials $u$. Thus, the same formula can be assumed to hold for $d=0$.

### 1.5 Examples of polynomial operations.

- Example 1: Consider $p(x)=x^{3}+x+1$ and $q(x)=-x^{3}+3 x^{2}+5 x+5$. What are their degrees?
Also, calculate the following expressions:

$$
p(x)+q(x), \quad p(x)+2 q(x) \text { and } p(x)^{2}
$$

and their degrees.
Determine the leading coefficients for each of the answers.
Answers: The degrees of $p(x), q(x)$ are both 3 and their leading coefficients are respectively $1,-1$.
The remaining answers are:

$$
\begin{array}{ll}
p(x)+q(x) & =x^{3}(1-1)+x^{2}(0+3)+x(1+5)+(1+5) \\
& =3 x^{2}+6 x+6 \\
& =x^{3}(1-2)+x^{2}(0+6)+x(1+10)+(1+10) \\
& =-x^{3}+6 x^{2}+11 x+11 \\
\hline p(x)+2 q(x) & \\
\hline\left(x^{3}+x+1\right)\left(x^{3}+x+1\right) & \left.=x^{6}+x^{4}+x^{3}\right)+\left(x^{4}+x^{2}+x\right)+\left(x^{3}+x+1\right) \\
& =x^{6}+2 x^{4}+2 x^{3}+x^{2}+2 x+1
\end{array}
$$

The respective degrees are, $2,3,6$. The leading coefficients are, respectively, $3,-1,1$.

Here is an alternate technique for finding the last answer. It will be useful for future work.

First, note that the degree of the answer is $3+3=6$ from the product rule above. Thus we need to calculate coefficients of all powers $x^{i}$ for $i=0$ to $i=6$.
Each $x^{i}$ term comes from multiplying an $x^{j}$ term and an $x^{i-j}$ term from $p(x)$. Naturally, both $j$ and $i-j$ need to be between 0 and 3 .

Thus the only way to get $x^{0}$ in the answer is to take $j=0$ and $i-j=0-0=0$. Thus the $x^{0}$ term of the answer is the square of the $x^{0}$ term of $p(x)$, hence is $1^{2}=1$.

Similarly, the $x^{6}$ term can only come from $x^{3}$ term multiplied by the $x^{3}$ term and is $x^{6}$.

Now for the $x^{5}$ term, the choices are $j=3, i-j=2$ or $j=2, i-j=3$ and in $p(x)$ the coefficient of $x^{2}$ is 0 , so both these terms are 0 . Hence the $x^{5}$ term is missing in the answer.
Now for $x^{4}$ terms, we have three choices for $(j, i-j)$, namely $(3,1),(2,2),(1,3)$. We get the corresponding coefficients $(1)(1)+(0)(0)+(1)(1)=2$.

The reader should verify the rest.

- Example 2: Use the above technique to answer the following:

You are given that:

$$
\text { Let } \begin{aligned}
& \left(x^{20}+2 x^{19}-x^{17}+\cdots+3 x+5\right) \\
\times & \left(3 x^{20}+6 x^{19}-4 x^{18}+\cdots+5 x+6\right) \\
= & a x^{40}+b x^{39}+c x^{38}+\cdots+d x+e
\end{aligned}
$$

Determine $a, b, c, d, e$.
Note that the middle terms are not even given and not needed for the required answers. The only term contributing to $x^{40}$ is $\left(x^{20}\right) \cdot\left(3 x^{20}\right)$, so $a=3$.
The only terms producing $x^{39}$ are

$$
\left(x^{20}\right) \cdot\left(6 x^{19}\right) \text { and }\left(2 x^{19}\right) \cdot\left(3 x^{20}\right)
$$

and hence, the coefficient for $x^{39}$ is $(1)(6)+(2)(3)=12$, so $b=12$.
Similarly, check that $c=(1)(-4)+(2)(6)+(0)(3)=8$. Calculate $d=(3)(6)+$ $(5)(5)=43$ and $e=(5)(6)=30$.

- Example 3: The technique of polynomial multiplication can often be used to help with integer calculations. Remember that a number like 5265 is really a polynomial $5 d^{3}+2 d^{2}+6 d+5$ where $d=10$.

Now, we can use our technique of polynomial operations to add or multiply numbers. The only thing to watch out for is that since $d$ is a number and not really a variable, there are "carries" to worry about.

Here we derive the well known formula for squaring a number ending in 5. Before describing the formula, let us give an example.

Say, you want to square the number 25 . Split the number as $2 \mid 5$. From the left part 2 construct the number $(2)(3)=6$. This is obtained by multiplying the left part with itself increased by 1 . Simply write 25 next to the current calculation, so the answer is 625 .

The square of 15 by the same technique shall be obtained thus:
Split it as $1 \mid 5 .(1)(1+1)=2$. So the answer is 225 .
The general rule is this:
Let a number $n$ be written as $p \mid 5$ where $p$ is the part of the number after the units digit 5 .

Then $n^{2}=(p)(p+1) 25$, i.e. $(p)(p+1)$ becomes the part of the number from the 100s digit onwards.
To illustrate, consider $45^{2}$. Here $p=4$, so $(p)(p+1)=(4)(5)=20$ and so the answer is 2025 . Also $105^{2}=(10)(11) 25=11025$. Similarly $5265^{2}=$ (526)(527)25, i.e. 27720225.

What is the proof? Since $n$ splits as $p \mid 5$, our $n=p d+5$. Using that $d=10$ we make the following calculations.

$$
\begin{aligned}
n^{2} & =(p d+5)(p d+5) \\
& =p^{2} d^{2}+(2)(5)(p d)+25 \\
& =p^{2}(100)+p(100)+25 \\
& =\left(p^{2}+p\right) 100+25 .
\end{aligned}
$$

Thus the part after the 100 s digits is $p^{2}+p=p(p+1)$.
You can use such techniques for developing fast calculation methods.

- Example 4: Let us begin by $f(x)=x^{n}$ for $n=1,2, \cdots$ and let us try to calculate the expansions of $f(x+t)$. For $n=1$ we simply get $f(x+t)=x+t$. For $n=2$ we get $f(x+t)=(x+t)^{2}=x^{2}+2 x t+t^{2}$.

For $n=3$ we get $f(x+t)=(x+t)^{3}$. Let us calculate this as follows:

$$
\begin{aligned}
(x+t)(x+t)^{2}= & (x+t)\left(x^{2}+2 x t+t^{2}\right) \\
= & x\left(x^{2}+2 x t+t^{2}\right) \\
& +t\left(x^{2}+2 x t+t^{2}\right) \\
= & x^{3}+2 x^{2} t+x t^{2} \\
& +x^{2} t+2 x t^{2}+t^{3} \\
= & x^{3}+3 x^{2} t+3 x t^{2}+t^{3}
\end{aligned}
$$

For $n=4$ we invite the reader to do a similar calculation and deduce ${ }^{9}$

$$
\begin{aligned}
(x+t)^{4} & =(x+t)\left(x^{3}+3 x^{2} t+3 x t^{2}+t^{3}\right) \\
& =x^{4}+\left(3 x^{3} t+x^{3} t\right)+\left(3 x^{2} t^{2}+3 x^{2} t^{2}\right)+\left(x t^{3}+3 x t^{3}\right)+t^{4} \\
& =x^{4}+4 x^{3} t+6 x^{2} t^{2}+4 x t^{3}+t^{4}
\end{aligned}
$$

Let us step back and list our conclusions.

| $n$ |  |  | Coefficient list |  |  |  |  |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 1 | 1 |  |  |  |  |  |
| 2 | 1 | 2 | 1 |  |  |  |  |
| 3 | 1 | 3 | 3 | 1 |  |  |  |
| 4 | 1 | 4 | 6 | 4 | 1 |  |  |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 |  |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 |
|  |  |  |  |  |  |  |  |

We note a pattern from first few rows, as well as recalling how we collected the coefficients for small values of $n$. The pattern is that a coefficient is the sum of the one above and the one in its northwest corner (above-left), where any coefficient which is missing is treated as zero.
We have then boldly gone where we had not gone before and filled in the rows for $n=5,6,7$. What do they tell us?
The row for $n=5$ says that

$$
(x+t)^{5}=(1) x^{5}+(5) x^{4} t+(10) x^{3} t^{2}+(10) x^{2} t^{3}+(5) x t^{4}+t^{5}
$$

[^9]Notice that we start with the highest power of $x$, namely $x^{5}$ and using the coefficients write successive terms by reducing a power of $x$ while increasing a power of $t$, until we end up in $t^{5}$.
Clearly, we can continue this pattern as long as needed. This arrangement of coefficients is often referred to as the Pascal ( 17th Century) triangle, but it was clearly known in China (13th century) and India (10th century or earlier). A better name would be the Halāyudha arrangement to be named after the commentator who gave it in his tenth century commentary on a very old work by Pingala (from the second century BC).

The general formula called Binomial Theorem is this:
$(x+t)^{n}={ }_{n} C_{0} x^{n}+{ }_{n} C_{1} x^{n-1} t+{ }_{n} C_{2} x^{n-2} t^{2}+\cdots+{ }_{n} C_{n-2} x^{2} t^{n-2}+{ }_{n} C_{n-1} x t^{n-1}+{ }_{n} C_{n} t^{n}$.

The binomial coefficients ${ }_{n} C_{r}$ (read as "en cee ar") will now be described.
Recall that for any positive integer $m$, we write $m$ ! for (1)(2) $\cdots(m)$. We also make a special convention that $0!=1$.
The formula for the binomial coefficients is: ${ }^{10}$

$$
{ }_{n} C_{r}=\frac{n!}{r!(n-r)!} \text { or equivalently } \frac{n(n-1) \cdots(n-r+1)}{r!} .
$$

(Hint: To get the second form, write out the factors in the first formula and cancel terms.)

We shall not worry about a formal proof, but the reader can use the idea of Induction to prove the formula. ${ }^{11}$
Note. The notation, ${ }_{n} C_{r}$ is convenient, but many variants of the same are common. Many people use $\binom{\mathbf{n}}{\mathbf{r}}$.

[^10]If the complicated format of this notation is not easy to implement, a simpler $\mathbf{C}(\mathbf{n}, \mathbf{r})$ is also used.
Why are we giving two different looking formulas, when one would do? The first formula is easy to write down, but the second is easier to evaluate, since it has already done many of the cancellations.
There is a more subtle reason as well; the second formula can be easily evaluated when $n$ is not even an integer and indeed there is a famous Binomial Theorem due to Newton which will let you compute $(1+x)^{n}$ for any power $n$, provided you take $x$ small (i.e. between -1 and 1 ) and provided you get clever enough to add infinitely many terms. We shall discuss this later in the Appendix.
First let us calculate a few binomial coefficients:

$$
{ }_{3} C_{2}=\frac{3!}{2!1!}=\frac{1 \cdot 2 \cdot 3}{1 \cdot 2 \cdot 1}=3
$$

Similarly,

$$
{ }_{5} C_{3}=\frac{5!}{3!2!}=\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2}=\frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3}=\frac{2 \cdot 5}{1}=10 .
$$

How do we use these? The meaning of these coefficients is that when we expand $(x+t)^{n}$ and collect the terms $x^{r} t^{n-r}$ then its coefficient is ${ }_{n} C_{r}$.
Consider:
$(x+t)^{10}={ }_{10} C_{0} x^{10}+{ }_{10} C_{1} x^{9} t+{ }_{10} C_{2} x^{8} t^{2}+$ more terms with higher powers of $t$.
Note that ${ }_{10} C_{0}=\frac{10!}{0!10!}=1$. Indeed, ${ }_{n} C_{0}=1={ }_{n} C_{n}$ for all $n$.
The value of ${ }_{10} C_{1}=\frac{10}{1!}=10$.
Note that we are using the second form of the formula with $n=10, r=1$, so the numerator $n(n-1) \cdots(n-r+1)$ reduces to just 10 . Indeed ${ }_{n} C_{1}=n$ is easy to see for all $n$.

$$
x^{10}+10 x^{9} t+\frac{10 \cdot 9}{1 \cdot 2} x^{8} t^{2}=x^{10}+10 x^{9} t+45 x^{8} t^{2}
$$

You should verify the set of eleven coefficients:

$$
\left(\begin{array}{rrrrrrrrrrr}
{ }_{10} C_{0} & { }_{10} C_{1} & { }_{10} C_{2} & { }_{10} C_{3} & { }_{10} C_{4} & { }_{10} C_{5} & { }_{10} C_{6} & { }_{10} C_{7} & { }_{10} C_{8} & { }_{10} C_{9} & { }_{10} C_{10} \\
1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1
\end{array}\right) .
$$

Note the neat symmetry of the coefficients around the middle. It is fun to deduce it from the formula alone.

Exercise based on the above formula. Expand $(5-x)^{4}$ and $(2 x+3)^{5}$, using the binomial theorem.

## Answer:

For the first expansion, we try to use:

$$
(x+t)^{4}=x^{4}+4 x^{3} t+6 x^{2} t^{2}+4 x t^{3}+t^{4} .
$$

We can clearly match the expression $x+t$ with $5-x$ by replacing $x \rightarrow 5$ and $t \rightarrow-x$.

Thus, after a careful substitution, we get:

$$
(5-x)^{4}=(5)^{4}+4(5)^{3}(-x)+6(5)^{2}(-x)^{2}+4(5)(-x)^{3}+(-x)^{4} .
$$

After simplification, we conclude:

$$
(5-x)^{4}=625-500 x+150 x^{2}-20 x^{3}+x^{4} .
$$

Try to deduce the same answer by using a different substitution: $x \rightarrow-x, t \rightarrow$ 5.

For the second exercise, we make the substitution $x \rightarrow 2 x, t \rightarrow 3$ in the formula for $(x+t)^{5}$.
Thus, we get:

$$
(2 x+3)^{5}=(2 x)^{5}+5(2 x)^{4}(3)+10(2 x)^{3}(3)^{2}+10(2 x)^{2}(3)^{3}+5(2 x)(3)^{4}+(3)^{5} .
$$

This simplifies to:

$$
(2 x+3)^{5}=32 x^{5}+240 x^{4}+720 x^{3}+1080 x^{2}+810 x+243 .
$$

- Example 5: Substituting in a polynomial. Given a polynomial $p(x)$ it makes sense to substitute any valid algebraic expression for $x$ and recalculate the resulting expression. It is important to replace every occurrence of $x$ by the substituted expression enclosed in parenthesis and then carefully simplify.
For example, for

$$
\begin{aligned}
p(x)= & 3 x^{3}+x^{2}-2 x+5 \text { we have } \\
p(-2 x)= & 3(-2 x)^{3}+(-2 x)^{2}-2(-2 x)+5 \\
& \text { and this simplifies to } \\
= & 3(-8) x^{3}+(4) x^{2}-2(-2) x+5 \\
= & -24 x^{3}+4 x^{2}+4 x+5 .
\end{aligned}
$$

Given below are several samples of such operations. Verify these:

$$
\begin{array}{lcl}
\text { Polynomial } & \text { Substitution for } x & \text { Answer } \\
(x+5)(x+1) & x+2 & (x+7)(x+3) \\
x^{2}+6 x+5 & x-3 & x^{2}-4 \\
x^{2}+6 x+5 & 2 x-3 & 4 x^{2}-4 \\
x^{2}+6 x+5 & -3 x & 9 x^{2}-18 x+5
\end{array}
$$

Polynomial Substitution for $x$ Answer

$$
\begin{array}{lll}
x^{2}+6 x+5 & \frac{1}{x} & \frac{1}{x^{2}}+\frac{6}{x}+5=\frac{5 x^{2}+6 x+1}{x^{2}} \\
x^{2}+6 x+5 & \frac{-2}{x} & \frac{4-12 x+5 x^{2}}{x^{2}} \\
x^{2}+6 x+5 & \frac{1}{x+1} & \frac{12+16 x+5 x^{2}}{(x+1)^{2}} \\
(x+5)(x+1) & -5 & 0
\end{array}
$$

As an example of substituting in a rational function, note that substitution of $(x-5)$ for $x$ in $\frac{x+2}{(x+5)(x+1)}$ can be conveniently written as $\frac{x-3}{(x)(x-4)}$.
Note that, if convenient, we leave the answer in factored form. Indeed, the factored form is often better to work with, unless one needs coefficients.

The substitutions by rational expressions are prone to errors and the reader is advised to work these out carefully!

## - Example 6: Completing the square.

Consider a polynomial $q(x)=a x^{2}+b x+c$ where $0 \neq a, b, c$ are constants.
Find a substitution $x=u+s$ such that the substituted polynomial $q(u+s)$ has no $u$-term (i.e. it has no monomial in $u$ of degree 1).
In a more colorful language, we say that the $u$-term in $q(u+s)$ is killed!
Find the resulting polynomial $q(u+s)$.
Answer. Check out the work:

$$
q(u+s)=a(u+s)^{2}+b(u+s)+c=a u^{2}+(2 a s+b) u+\left(a s^{2}+b s+c\right)
$$

We want to arrange $2 a s+b=0$ and this is true if $2 a s=-b$, i.e. $s=-b /(2 a)$.
We substitute this value of $s$ in the final form above. ${ }^{12}$

[^11]\[

$$
\begin{aligned}
q\left(u-\frac{b}{2 a}\right) & =a u^{2}+\left(a\left(-\frac{b}{2 a}\right)^{2}+b\left(-\frac{b}{2 a}\right)+c\right) \\
& =a u^{2}+\frac{b^{2}}{4 a}-\frac{b^{2}}{2 a}+c \\
& =a u^{2}-\frac{b^{2}}{4 a}+c=a u^{2}-\left(\frac{b^{2}-4 a c}{4 a}\right) .
\end{aligned}
$$
\]

Sometimes, it is better not to bring in a new letter $u$. So, note that $x=u+s=$ $u-\frac{b}{2 a}$ means $u=x+\frac{b}{2 a}$.
Thus our final formula can be rewritten as:

$$
q(x)=a\left(x+\frac{b}{2 a}\right)^{2}+\frac{b^{2}-4 a c}{4 a}
$$

Thus, we have rewritten our expression $q(x)$ in the form
"(constant)(square) + constant".

Hence, this process is also called completing of the square.

## Exercise based on the above.

Let $q(x)=2 x^{2}+5 x+7$.
Find a substitution $x=u+s$ which makes $q(u+s)$ have no $u$-term.
Answer: Here $a=2, b=5, c=7$, so $s=-\frac{b}{2 a}=-\frac{5}{4}$.
Thus

$$
q\left(u-\frac{5}{4}\right)=a u^{2}-\left(\frac{b^{2}-4 a c}{4 a}\right)=2 u^{2}-\frac{5^{2}-4 \cdot 2 \cdot 7}{4 \cdot 2}=2 u^{2}+\frac{31}{8} .
$$

Note that we have $x=u-\frac{5}{4}$ and hence $u=x+\frac{5}{4}$.
Thus, our final answer can also be written as:

$$
q(x)=2\left(x+\frac{5}{4}\right)^{2}+\frac{31}{8}
$$

A more general idea: Indeed, the same idea can be used to kill the term whose degree is one less than the degree of the polynomial.
For example, For a cubic $p(x)=2 x^{3}+5 x^{2}+x-2$, the reader should verify that $p(u-5 / 6)$ will have no $u^{2}$ term.
Challenge! We leave the following two questions as a challenge to prove:

- Given a cubic $p(x)=a x^{3}+b x^{2}+$ lower terms with $a \neq 0$, find the value of $s$ such that $p(u+s)=a u^{3}+(0) u^{2}+$ lower terms. Answer: $s=-\frac{b}{3 a}$.
- More generally, do the same for a polynomial $p(x)=a x^{n}+b x^{(n-1)}+$ lower terms. This means find the $s$ such that $p(u+s)=a u^{n}+(0) u^{(n-1)}+$ lower terms. Answer: $s=-\frac{b}{n a}$.


## Chapter 2

## Solving linear equations.

As explained in the beginning, solving equations is a very important topic in algebra. In this section, we learn the techniques to solve the simplest types of equations called linear equations.

Definition: Linear Equation An equation is said to be linear in its listed variables $x_{1}, \cdots, x_{n}$, if each term of the equation is either free of the listed variables or is a monomial of degree 1 involving exactly one of the $x_{1}, \cdots, x_{n}$. ${ }^{1}$

Examples Each of these equations are linear in indicated variables:

1. $y=4 x+5$ : Variables $x, y$.
2. $2 x-4 y=10:$ Variables $x, y$.
3. $2 x-3 y+z=4+y-w$ : Variables $x, y, z$.

Now we explain the importance of listing the variables.

1. $x y+z=5 w$ is not linear in $x, y, z, w$. This is because the term $x y$ has degree 2 in $x, y$.
2. The same expression $x y+z=5 w$ is linear in $x, z, w$. It is also linear in $y, z, w$. We can also declare it as simply linear in $y$. What we are trying to emphasize we don't have to list every visible symbol as a variable.
Indeed, it is linear in any choice of variables from among $x, y, z, w$, as long as we don't include both $x$ and $y$ !

[^12]3. Sometimes, equations which appear as non linear may reduce to linear ones; but technically they are considered non linear. Thus, $(x-1)(x-5)=(x-2)(x-7)$ is clearly non linear as it stands, but simplifies to $x^{2}-6 x+5=x^{2}-7 x+14$ or $-6 x+5=-7 x+14$ after cancelling the $x^{2}$ term.
4. Sometimes, we can make equations linear by creating new names. Thus:
$$
\frac{5}{x}+\frac{3}{y}=1
$$
is certainly not linear, but we can set $u=\frac{1}{x}$ and $v=\frac{1}{y}$ and say that the resulting equation $5 u+3 v=1$ is linear in $u$, $v$. Such tricks let us solve some non linear equations as if they are linear!

### 2.1 What is a solution?

We will be both simplifying and solving equations throughout the course. So what is the difference in merely simplifying and actually solving an equation?

Definition: A solution to an equation. Given an equation in a variable, say $x$, by a solution to the equation we mean a value of $x$ which makes the equation true.

For example, the equation:

$$
3 x+4=-2 x+14
$$

has a solution given by $x=2$, since substituting $x=2$ makes the equation:

$$
6+4=-4+14 \text { which simplifies to } 10=10
$$

and this is a true equation.
On the other hand, $x=3$ is not a solution to this equation since it leads to:

$$
9+4=-6+14 \text { which simplifies to } 13=8
$$

which is obviously false.
How did we find the said solution? In this case we simplified the equation in the following steps:

| LHS | RHS | Explanation |
| :--- | :--- | :--- |
| $3 x+4$ | $=$ | $-2 x+14$ |
| Original equation |  |  |
| $3 x+2 x+4$ | $=14$ | Collect $x$ terms on LHS |
| $3 x+2 x$ | $=$ | $-4+14$ |
| $5 x$ | $=10$ | Move constants to RHS |
| $x$ | $=$ | Simplify |
|  |  | Divide by 5 on both sides |

Usually, the above process is described by the phrase "By simplifying the equation we get $x=2 "$. The reader is expected to carry out the steps and get the final answer.

What does it mean to simplify? It is a process of changing the original equation into a sequence of other equations which have the same set of solutions but whose solutions are easier to deduce.

As another example of a solution, consider the equation $3 x+2 y=7$ and check that $x=1, y=2$ gives a solution.

As yet another example, consider $x^{2}+y z=16$ with solution $x=2, y=3, z=4$.
Note that these last two examples can have many different solutions besides the displayed ones.

Definition: Consistent equation. A single equation in variables $x_{1}, \cdots, x_{r}$ is said to be consistent if it has at least one solution $x_{1}=a_{1}, \cdots, x_{r}=a_{r}$.

It is said to be inconsistent if there is no solution.
For example the equation $x^{2}+y^{2}+5=0$ cannot have any (real) solution, since the left hand side (LHS) is always positive and the right hand side (RHS) is zero. Thus it is inconsistent when we are working with real numbers.

Definition: System of equations By a system of equations in variables $x_{1}, \cdots, x_{r}$ we mean a collection of equations $E_{1}, E_{2}, \cdots, E_{s}$ where each $E_{i}$ is an equation in $x_{1}, \cdots, x_{r}$.

Thus

$$
E_{1}: 3 x+4 y=-1, E_{2}: 2 x-y-z=1, E_{3}: x+y+z=2, E_{4}: y+2 z=3
$$

is a system of four equations in three variables $x, y, z$.
Definition: Solution of a system of equations By a solution of a system of equations $E_{1}, E_{2}, \cdots, E_{s}$ in variables $x_{1}, \cdots, x_{r}$ we mean values for the variables $x_{1}=a_{1}, \cdots, x_{r}=a_{r}$ which make all the equations true.

For instance, you can check that $x=1, y=-1, z=2$ is a solution of the above system.

Definition: Consistent system of equations. A system of equations is said to be consistent if it has at least one solution and is said to be inconsistent otherwise.

For example the system of equations

$$
x+y=3, x+y=4
$$

is easily seen to be inconsistent.
In general, we want to learn how to find and describe all the solutions to a system of linear equations. In this course, we learn about equations in up to three variables. The general analysis is left for an advanced course in linear algebra.

We discuss methods of solution for systems of linear equations in the following sections.

### 2.2 One linear equation in one variable.

A linear equation in one variable, say $x$, can be solved by simple manipulation. We rewrite the equation in the form $a x=b$ where all $x$ terms appear on the left and all terms free of $x$ appear on the right. This is often referred to as isolating the variable.

Then the solution is simply $x=b / a$, provided $a \neq 0$. We come up with this solution by dividing both the right hand side (RHS) of the equation and the left hand side (LHS) by a. Thus $a x / a=x$ and $b / a$ is simply $b / a$.

If $a=0$ and $b \neq 0$, then the equation is inconsistent and has no solution.
If both $a, b$ are zero, then we have an identity - an equation which is valid for all values of the variable! Here, any value substituted for $x$ makes the equation true, therefore there are infinitely many solutions. (We are, of course assuming that we are working with an infinite set of numbers like the usual real numbers.)

This is the principle of $0,1, \infty$ number of solutions!
It states that any system of linear equations in any number of variables always falls under one of three cases: ${ }^{2}$

1. No solution (inconsistent system) or
2. a unique solution or
3. infinitely many solutions.

Example: The equation

$$
-5 x+3+2 x=7 x-8+9 x
$$

can be rearranged to:

$$
x(-5+2-7-9)=-8-3
$$

or even more simplified

$$
-19 x=-11
$$

and hence $x=11 / 19$ is a unique solution
The reader should note the efficient collection of terms. The reader should also note that the solution is really another equation! Thus, to solve an equation really means to change it to a convenient but equivalent form.

Example: The equation

$$
(3 x+4)+(5 x-8)=(5 x-4)+(3 x+8)
$$

[^13]would lead to
$$
x(3+5-5-3)=-4+8-4+8
$$
or
$$
0=8
$$
an inconsistent equation; hence it has no solution.
Example: The equation:
$$
(3 x+4)+(5 x-8)=(5 x+4)+(3 x-8)
$$
reduces to
$$
8 x-4=8 x-4
$$
and is an example of an identity. It gives all values of $x$ as valid solutions!

### 2.3 Several linear equations in one variable.

If we are given several equations in one variable, in principle, we can solve each of them and if our answers are consistent, then we can declare the common solution as the final answer. If not, we shall declare them as inconsistent.

Example: Given equations:

$$
2 x=4,6 x=12 .
$$

Solve each equation and determine if they are consistent or inconsistent. $2 x=4$ gives the solution $x=2$ and $6 x=12$ also gives the solution $x=2$. So, we have $x=2$ as a unique solution.

Example:Given equations:

$$
2 x=4,6 x=8
$$

Working as above, we get: $2 x=4$ gives the solution $x=2$, while $6 x=8$ gives the solution $x=4 / 3$ Since the equations give us different solutions, they are inconsistent.

One systematic way of checking this is to solve the first one as $x=2$ and substitute it in the second to see if we get a valid equation. In this case, we get $6(2)=8$ or $12=8$ an inconsistent equation!

Thus, we did not actually solve the second equation, just verified its consistency!
This is a case of the Substitution method of solving linear equations. The method consists of solving one equation for some variable and substituting (plugging in ) the answer into other equations for verification or further processing. You will see more instances of this method below.

### 2.4 Two or more equations in two variables.

Suppose we wish to solve

$$
x+2 y=10 \text { and } 2 x-y=5 \text { together. }
$$

We first think of them as equations in $x$ alone. Solution of the first equation in $x$ is then $x=10-2 y$. If we plug this into the second, we get

$$
2(10-2 y)-y=5
$$

or, by simplification

$$
20-4 y-y=5
$$

Collecting like terms we get:

$$
-5 y=-15 \text { or } y=3
$$

Thus, unless $y=3$ we have an inconsistent system on our hands.
Further, when $y=3$, our value for $x$ becomes $x=10-2 y=10-2(3)=4$.
Thus the unique solution to the system of the two equations is $x=4, y=3$. Notice the substitution method being used repeatedly.

Note: We could have solved the above two linear equations $x+2 y=10$ and $2 x-y=5$ by a slightly different strategy; namely interchanging the roles played by $x$ and $y$.

We could have solved the first equation for $y$ as

$$
y=\frac{10-x}{2}=5-\frac{x}{2} .
$$

Substitute this answer in the second to get

$$
2 x-\left(5-\frac{x}{2}\right)=5
$$

and deduce that $\frac{5 x}{2}=10$ or $x=4$. Then finally plug this back into $y=5-\frac{x}{2}$ to give $y=3$. Thus we get the same solution again.

Thus, the choice of the first variable is up to you, but sometimes it may be easier to solve for one particular variable. With practice, you will be able to pick the most convenient variable to solve first.

## Example:

Solve $x+2 y=10,2 x-y=5$ and $3 x+4 y=10$ together.
We already know the unique solution of the first two; namely $x=4, y=3$. So, all we need to do is plug it in the third to get $3(4)+4(3)=10$ or $24=10$. Since this is not true, we get inconsistent equations and declare that that there is no solution.

## Example:

Solve $x+2 y=10,2 x-y=5$ and $3 x-4 y=0$ together;
As above, the first two equations give $x=4, y=3$. Plugging this into the third, we get:

$$
3(4)-4(3)=0
$$

which reduces to $0=0$ a consistent equation; hence the unique answer is $x=4, y=3$.

### 2.5 Several equations in several variables.

Here is an example of solving three equations in $x, y, z$.
Solve:

$$
\begin{array}{lr}
\text { Eq1: } & x+2 y+z=4 \\
\text { Eq2: } & 2 x-y=3 \\
\text { Eq3: } & y+z=7
\end{array}
$$

We shall follow the above idea of solving one variable at a time.
We solve Eq1 for $x$ to get
Solution1.

$$
x=4-2 y-z
$$

Substitution into Eq2 gives a new equations in $y, z$ :

$$
2(4-2 y-z)-y=3 \text { or } y(-4-1)-2 z=3-8 \text { or } 5 y+2 z=5 .
$$

The third equation Eq3 has no $x$ anyway, so it stays $y+z=7$.
Now we solve the new equations:

$$
E q 2^{*}: 5 y+2 z=5, E q 3^{*}: y+z=7
$$

Our technique says to solve one of them for one of the variables. Note that it is easier to solve the equation $E q 3^{*}$ for $y$ or $z$, so we solve it for $y$ :

## Solution2. <br> $$
y=7-z
$$

Finally, using this, $E q 2^{*}$ becomes $5(7-z)+2 z=5$ or $z(-5+2)=5-35$.
Thus $z=-30 /(-3)=10$. So we have:
Solution3. $\quad z=10$
Notice our three solutions. The third pins $z$ down to 10 .
Then the second says $y=7-z=7-10=-3$.

Finally, the first one gives $x=4-2 y-z=4-2(-3)-(10)=0$ giving the: ${ }^{3}$

$$
\text { Final solution. } \quad x=0, y=-3, z=10
$$

This gives the main idea of our solution process. We summarize it formally below, as an aid to understanding and memorization.

1. Pick some equation and a variable, say $x$, appearing in it.
2. Solve the chosen equation for $x$ and save this as the first solution.

This solution was $x=4-2 y-z$ in our example. Note that the value involves the remaining variables $y, z$.

Substitute the first solution in all the other equations.
In our example, this is the system

$$
E q 2^{*}: 5 y+2 z=5 \text { and } E q 3^{*}: y+z=7
$$

These equations are now free of $x$ and represent a system with fewer variables. This is sometimes described as "eliminate $x$ from the system". The new equations with the value of $x$ plugged in them are said to form the system obtained by eliminating $x$.
3. Continue to solve the smaller system by choosing a new variable etc. as described above.

In our example, this gave the solution $y=-3, z=10$.
4. When the smaller system gets solved, plug its solution into the original value of $x$ to finish the solution process.

This is often called the back substitution.
In our example, this gave us $x=4-2 y-z=0$.

[^14]
### 2.6 Solving linear equations efficiently.

- Elimination Method. The process of solving for a variable and substituting in the other can sometimes be done more efficiently by manipulating the whole equations. Here is an example.

Solve E1: $2 x+5 y=3$, E2: $4 x+9 y=5$.

We note that solving for $x$ and plugging it into the other equation is designed to get rid of $x$ from the other equation. So, if we can get rid of $x$ from one of the equation with less work, we should be happy.
One permissible operation (which still leads to the same final answers) is to change one equation by adding a suitable multiple of one (or more) other equation(s) to it.
There are two more natural operations which are very familiar. One is to add equal quantities to both sides of the equation and the other is to multiply the whole equation by a non zero number. ${ }^{4}$
These three types of operations can indeed lead to a complete analysis of solutions for any system of linear equations.
By inspection, we see that E2-2E1 will get rid of $x$. So, we replace E2 by E2-2E1. We calculate this efficiently by collecting the coefficients as we go:

$$
\begin{array}{l|lll}
E 1 & 2 x & +5 y & =3 \\
E 2 & 4 x & +9 y & =5 \\
E 2-2 E 1 & 4 x-2(2 x) & +9 y-2(5 y) & =5-2(3) \\
\hline E 2-2 E 1 & 0 & -y & =-1
\end{array}
$$

Thus we have $-y=-1$ or $y=1$.
The value of $x$ is then easily determined by plugging this into either E1 or E2. Using E1, we get:

$$
2 x+5(1)=3 \text { so } 2 x=-2 \text { or } x=-1 .
$$

The reader should compare this with the old method of eliminating $x$ and notice its efficiency.

[^15]- Determinant Method. There is an even more efficient calculation for two equations in two variables using what is known as the Cramer's Rule.

First we need a definition of a $2 \times 2$ determinant.
Definition: Determinant of a 2 by 2 matrix An array of four numbers: $M=\left(\begin{array}{cc}x & y \\ z & w\end{array}\right)$ arranged in two rows and two columns is said to be a $\mathbf{2}$ by $\mathbf{2}$ matrix.
Its determinant is denoted as $\operatorname{det}(M)=\left|\begin{array}{rr}x & y \\ z & w\end{array}\right|$ and is defined to be equal to the quantity $x w-y z$.

Thus, for example

$$
\operatorname{det}\left(\left(\begin{array}{rr}
2 & 5 \\
4 & 13
\end{array}\right)\right)=\left|\begin{array}{rr}
2 & 5 \\
4 & 13
\end{array}\right|=(2)(13)-(5)(4)=6 .
$$

A way to recall this value is to remember it as "the product of the entries on the main diagonal minus the product of the entries on the opposite diagonal".
Now we are ready to state the Cramer's Rule to solve two linear equations in two variables.

Cramer's Rule The solution to

$$
\begin{aligned}
a x+b y & =c \text { and } p x+q y=r \text { is given by } \\
x & =\frac{\left|\begin{array}{ll}
c & b \\
r & q
\end{array}\right|}{\left|\begin{array}{ll}
a & b \\
p & q
\end{array}\right|}, y=\frac{\left|\begin{array}{ll}
a & c \\
p & r
\end{array}\right|}{\left|\begin{array}{ll}
a & b \\
p & q
\end{array}\right|} .
\end{aligned}
$$

This looks very complicated, but there is an easy trick to remember it!

The denominator is the determinant formed by the coefficients of the two variables in the two equations. For convenience, let us call it

$$
\Delta=\left|\begin{array}{ll}
a & b \\
p & q
\end{array}\right| .
$$

To get the value of $x$ we start with $\Delta$ and replace the coefficients of $x$ by the right hand sides. We get the determinant which we call

$$
\Delta_{x}=\left|\begin{array}{cc}
c & b \\
r & q
\end{array}\right|
$$

Then the value of $x$ is $\frac{\Delta_{x}}{\Delta}$.
To get the $y$-value, we do the same with the $y$ coefficients. Namely, we get

$$
\Delta_{y}=\left|\begin{array}{ll}
a & c \\
p & r
\end{array}\right|
$$

after replacing the coefficients of $y$ by the right hand side.
Then the value of $y$ is $\frac{\Delta_{y}}{\Delta}$.
To illustrate, let us solve the above equations again:

$$
\begin{aligned}
& 2 x+5 y=3 \\
& 4 x+9 y=5
\end{aligned}
$$

Here the denominator is

$$
\Delta=\left|\begin{array}{ll}
2 & 5 \\
4 & 9
\end{array}\right|=(2)(9)-(5)(4)=-2 .
$$

The numerator for $x$ is

$$
\Delta_{x}=\left|\begin{array}{ll}
3 & 5 \\
5 & 9
\end{array}\right|=(3)(9)-(5)(5)=2 .
$$

The numerator for $y$ is

$$
\Delta_{y}=\left|\begin{array}{ll}
2 & 3 \\
4 & 5
\end{array}\right|=(2)(5)-(3)(4)=-2 .
$$

So

$$
x=\frac{\Delta_{x}}{\Delta}=\frac{2}{-2}=-1 \text { and } y=\frac{\Delta_{y}}{\Delta}=\frac{-2}{-2}=1 .
$$

Exceptions to Cramer's Rule. Our answers $x=\frac{\Delta_{x}}{\Delta}$ and $y=\frac{\Delta_{y}}{\Delta}$ are clearly the final answers as long as $\Delta \neq 0$.
Thus we have a special case if and only if $\Delta=a q-b p=0 .{ }^{5}$
If one of the numerators ( $\Delta_{x}$ or $\Delta_{y}$ ) is non zero, then there is no solution, i.e. the system is inconsistent! If both ( $\Delta_{x}$ and $\Delta_{y}$ ) are also zero, then we have infinitely many solutions; indeed, one of the equations is simply a multiple of the other!
The reader can easily verify these facts.
We illustrate them by some examples.

[^16]
## - Example 1.

Solve the system of equations:

$$
2 x+y=5,4 x+2 y=k
$$

for various values of the parameter $k$.
If we apply the Cramer's Rule, then we get

$$
x=\frac{\Delta_{x}}{\Delta}=\frac{\left|\begin{array}{ll}
5 & 1 \\
k & 2
\end{array}\right|}{\left|\begin{array}{ll}
2 & 1 \\
4 & 2
\end{array}\right|}=\frac{10-k}{0}
$$

Similarly, we get ${ }^{6}$

$$
y=\frac{\Delta_{y}}{\Delta}=\frac{\left|\begin{array}{ll}
2 & 5 \\
4 & k
\end{array}\right|}{\left|\begin{array}{ll}
2 & 1 \\
4 & 2
\end{array}\right|}=\frac{2 k-20}{0}
$$

We see that when $k \neq 10$ then this leads to illegal values for $x$ and $y$, while if $k=10$ then we are left with $x=\frac{0}{0}$ and $y=\frac{0}{0}$.
We should not rush into an answer right away, but try to analyze the situation further.

A little thought shows that for $k=10$ the second equation $4 x+2 y=20$ is simply twice the first equation $2 x+y=5$ and so it can be forgotten as long as we use the first equation!
Thus the solution set consists of all values of $x, y$ which satisfy $2 x+y=5$. It is easy to describe these by saying:
Take any value, say $t$ for $y$ and then take $x=\frac{5-t}{2}$. Some of the concrete answers are $x=\frac{5}{2}, y=0$ using $t=0$, or $x=2, y=1$ using $t=1$, or $x=3, y=-1$ using $t=-1$ and so on. Indeed it is customary to declare: ${ }^{7}$

$$
x=\frac{5-t}{2}, y=t \text { where } t \text { is arbitrary. }
$$

[^17]Thus, we have infinitely many solutions.

## Many linear equations in many variables.

There is a similar theory of solving many equations in many variables using higher order determinants. The interested reader may look up books on determinants or linear algebra.

- Example 2. Here, we shall only illustrate how the Cramer's Rule mechanism works beautifully for more variables, but we shall refrain from including the details of calculations. The reader is encouraged to search for the details from higher books.
Solve the three equations in three variables:

$$
x+y+z=6, x-2 y+z=0 \text { and } 2 x-y-z=-3 .
$$

As before, we make a determinant of the coefficients

$$
\Delta=\left|\begin{array}{rrr}
1 & 1 & 1 \\
1 & -2 & 1 \\
2 & -1 & -1
\end{array}\right|
$$

As before, we get $\Delta_{x}$ by replacing the $x$ coefficients by the right hand sides:

$$
\Delta_{x}=\left|\begin{array}{rrr}
6 & 1 & 1 \\
0 & -2 & 1 \\
-3 & -1 & -1
\end{array}\right| .
$$

Once you get the right definition for a 3 by 3 determinant, it is easy to calculate $\Delta=9=\Delta_{x}$, so $x=\frac{9}{9}=1$.
Also, we get

$$
\Delta_{y}=\left|\begin{array}{rrr}
1 & 6 & 1 \\
1 & 0 & 1 \\
2 & -3 & -1
\end{array}\right|=18
$$

Finally, we have

$$
\Delta_{z}=\left|\begin{array}{rrr}
1 & 1 & 6 \\
1 & -2 & 0 \\
2 & -1 & -3
\end{array}\right|=27
$$

This gives us $y=\frac{18}{9}=2$ and $z=\frac{27}{9}=3$. The reader can at least verify the answers directly by substitution. The elimination method will also work. Thus, we know many methods already, but the Cramer's Rule has the advantage of writing down the final answer in form and letting us determine the value of any desired variable, without getting the whole solution. It also lets us analyze the behavior of the solution when one or more coefficients is a parameter; as we observed in the discussion of the exceptions to the Cramer's Rule above.

## Chapter 3

## The division algorithm and applications

### 3.1 Division algorithm in integers.

Given two integers $u, v$ where $v$ is positive, we already know how to divide $u$ by $v$ and get a quotient (division) and a remainder.

This was taught as the long division of integers in schools.
For instance, when we divide 23 by 5 we get the quotient 4 and remainder 3. In equation form, we write this as

$$
23=(4) 5+3 \text { or } 23-(4) 5=3
$$

For the readers who like to use their calculators, here is a simple recipe for finding the quotient and the remainder.

- Have the calculator evaluate $u / v$. the quotient $q$ is simply the integer part of the answer. Thus $23 / 5=4.6$, so $q=4$.
- Then calculate $u-q v$. This is the remainder $r$. Thus, we get $23-(4) 5=3$, the remainder.

While useful, the above recipe can become unwieldy for large or complicated integers, so we urge the readers to not get dependent on it.

Here is an example: Let $u=1+3+3^{2}+3^{3}+\cdots+3^{1000}$ and $v=3$. Divide $u$ by $v$ and find the division $q$ and the remainder $r$. The person with a calculator might be working a long time, just to determine what $u$ is and may get overflow error messages from the calculator.

A person who looks at the form can easily see that we want

$$
u=1+3\left(1+3+\cdots+3^{999}\right)=r+3 q .
$$

Note that $0<1<3$, so we can happily declare: $r=1$ and $q=\left(1+3+\cdots+3^{999}\right)$.

You may not have seen much of the variations which we discuss next.
Our chosen $u$ can be negative, say -23 and then we get:

$$
-23=(-5) 5+2
$$

so now the quotient is -5 and remainder is 2 !
The calculator user has to note that now $u / v=-4.6=-5+0.4$. So the division is now taken as -5 , since we want our remainder to be bigger than or equal to zero.

Then $u-q v=-23-(-5) 5=-23+25=2$ so $r=2$.
For better understanding, we record the definitions.
Definition: Division Algorithm in Integers. Given two integers $u, v$ with $v \neq 0$, there are unique integers $q, r$ satisfying the following conditions:

1. $u=q v+r$ and
2. $0 \leq r<|v|$ where, as usual, $|v|$ denotes the absolute value of $v$.

The integer $q$ is called the quotient and $r$ is called the remainder. These are often said in a more convenient fashion as follows.

We may say that $\mathbf{u}$ is equal to $\mathbf{r}$ modulo $\mathbf{v}$ or that the remainder of $u$ modulo $v$ is $r$.

Here are some examples to illustrate how the definition works for negative integers.

- Let $u=-23, v=5$. Then we get:

$$
-23=(-5) 5+2
$$

so now the quotient is -5 and remainder is 2 !
The calculator user has to note that now $u / v=-4.6=-5+0.4$. So the division is now taken as -5 , since we want our remainder to be bigger than or equal to zero.
Then $u-q v=-23-(-5) 5=-23+25=2$ so $r=2$.

- Now take $u=23, v=-5$ and note that $|v|=|-5|=5$. Thus the remainder has to be one of $0,1,2,3,4$.

We get:

$$
23=(-4)(-5)+3 \text { so } q=-4 \text { and } r=3
$$

The calculator work will give:
The ratio

$$
u / v=\frac{23}{-5}=-4.6=-4-0.6
$$

If we take $q=-4$, then we get $u-q v=23-(-4)(-5)=23-20=3$ so $r=3$. Note that if we try taking $q=-5$, we would get a negative remainder, hence we need $q=-4$.

- Let $u=-23, v=-5$. Now again $|v|=5$. We get:

$$
-23=(5)(-5)+2 \text { so } q=-5 \text { and } r=2 .
$$

The calculator work will give:
The ratio ${ }^{1}$

$$
u / v=\frac{-23}{-5}=4.6=5-0.4
$$

If we take $q=5$, then we get $u-q v=-23-(5)(-5)=-23+25=2$ so $r=2$.

Definition: Divisibility of integers. We shall say that $v$ divides $u$ if $u=k v$ for some integer $k$. We write this in notation as $v \mid u$. Sometimes, this is also worded as " $u$ is divisible by $v$ ".

Thus, if $v \neq 0$ then we see that $v \mid u$ exactly means that, when $u$ is divided by $v$, the remainder $r$ becomes 0 .

It is clear that the only integer divisible by 0 is 0 .
Exercise. We leave it as an exercise to the reader to explain the following definition and provide suitable examples.

Let us now apply the division algorithm to find the GCD of two integers. For completeness, we record the definition:

Definition: GCD of integers. Given two integers $u, v$ we say that an integer $d$ is their GCD if the following conditions hold:

1. $d$ divides both $u, v$.
2. If $x$ divides both $u$ and $v$ then $x$ divides $d$.
3. Either $d=0$ or $d>0$.

As before, for convenience, we define:
Definition: LCM of integers. Given two integers $u, v$ we say that an integer $L$ is their LCM if the following conditions hold:

1. Each of $u, v$ divides $L$.
2. If each of $u, v$ divide $x$, then $L$ divides $x$.
3. Either $L=0$ or $L>0$.
[^18]As before, we have a simple fact:
Calculation of LCM.
Actually, it is easy to find LCM of two non zero integers $u, v$ by a simple calculation. The simple formula is:

$$
\operatorname{LCM}(u, v)=c \frac{u v}{\operatorname{GCD}(u, v)}
$$

Here $c$ is equal to 1 or -1 and is chosen such that the LCM comes out positive.
Remarks. First note that the definitions of GCD and LCM can be easily extended to include several integers rather than just two $u, v$. The reader is encouraged to do this. Next note that the GCD becomes zero exactly when $u=v=0$.

Important Fact. If we had to check these definitions every time, then the GCD would be a complicated concept. However, we show how to effectively find the GCD using a table based on the division algorithm as above.

Example 1: Find the GCD of 7553 and 623.
Answer:
We apply the same algorithm as we did in polynomials.
We write the two given integers on top, divide the bottom into the top and write the negative of the quotient on the left and finally write the remainder below the top two integers.

$$
\left[\begin{array}{l|r|r}
\text { Step no. } & \text { Minus Quotients } & \text { Integers } \\
0 & \text { Begin } & 7553 \\
1 & -12 & 623 \\
2 & \text { End of step 1 } & 77
\end{array}\right]
$$

You could, of course, do this with a calculator:

$$
7553 / 623=12.12 \text { so } q=12 \text { and } r=7553-(12)(623)=77
$$

Don't forget to enter the negative of $q$, i.e. -12 in the column.
Then we treat the next two numbers the same way; i.e. we divide 77 into 623 and get the quotient 8 and remainder 7 .

Finally, when we divide 7 into 77 , we have a quotient 11 and remainder 0 , ending the process.

$$
\left[\begin{array}{l|r|r}
\text { Step no. } & \text { Minus Quotients } & \text { Integers } \\
0 & \text { Begin } & 7553 \\
1 & -12 & 623 \\
2 & -8 & 77 \\
3 & -11 & 7 \\
4 & \text { End of process. } & 0
\end{array}\right]
$$

As before, we argue that 7 must be their GCD. ${ }^{2}$

[^19]
## 3.2 Āryabhaṭa algorithm: Efficient Euclidean algorithm.

Āryabhaṭa algorithm gives us a way to solve the following problem. ${ }^{3}$
Problem. Suppose that you are given two non zero integers $u, v$ and let $d$ be their GCD. Any integer which is a combination $a u-b v$ of the two integers is obviously divisible by $d$.

The Kuttaka problem asks us to write any multiple of $d$ as a combination of $u, v$.

In symbols, we need to write an integer multiple $s d$ as $s d=x u+y v$ for some integers $x, y$.

The problem known as Chinese Remainder Problem asks us to find an integer $n$ which has assigned remainders $p, q$ when divided by $u, v$ respectively. This gives a pair of equations:

$$
n=a u+p \text { and } n=b v+q \text { for some integers } a, b
$$

The Chinese Remainder Problem can always be solved by reducing to a Kuttaka problem as follows:

- Subtract the second from the first, then we get:

$$
0=a u+p-b v-q \text { or } q-p=a u-b v .
$$

- Solve the Kuttaka problem of writing $q-p$ as a combination of $u, v$.

Of course, it must satisfy the GCD condition that $q-p$ must be divisible by $d=\operatorname{GCD}(u, v)$.
If the condition fails, then the original Chinese Remainder Problem is also unsolvable.

- Once we have $q-p=a u-b v$ our $n$ is equal to $a u+p$ which is also equal to $b v+q$.
must divide the next one below and continuing must divide the bottom non zero integer 7. Also, the bottom 7 clearly divides everything above and is positive by construction! So it is the answer.
${ }^{3}$ Indeed, this algorithm is usually known as the Euclidean algorithm. Our version is a more convenient arrangement of the algorithm given by an Indian Astronomer/Mathematician in the fifth century A.D., some 800 years after Euclid. It was, however, traditionally given in a much more complicated form by commentators. The original text is only one and a half verses long! We also note that the usual Euclidean Algorithm has two components: one is how to find the GCD of two given numbers and the second is how to express the GCD as a combination of the two numbers. Euclid did not state the second part at all. Indeed his geometric terminology did not even provide for a natural statement of this property since the combination always includes a negative number and since for Euclid all numbers were lengths of lines, a negative number was not a valid object.

Thus, the A A ryabhaṭa algorithm represents the first complete formulation of what is now called the Eucliden algorithm!

We begin by illustrating the technique with our solved example with $u=7553, v=$ $623, d=7$ above.

We shall start with a similar table, but drop the step numbers and put a pair of blank columns for answers next to the numbers. We fill in the numbers as shown in the answers' columns.

$$
\left[\begin{array}{r|r|r|r}
\text { Minus Quotients } & \text { Answer 1 } & \text { Answer 2 } & \text { Integers } \\
\text { Begin } & 1 & 0 & 7553 \\
-12 & 0 & 1 & 623
\end{array}\right]
$$

Now, here is how we fill in the next entries in the Answers' and Integers columns.
The entry in a slot is obtained by multiplying the entry above by the "minus quotient" on the left and then adding the entry above to it.

Thus the entry in the Answer 1 column shall be $(-12)(0)+1=1$. Similarly, the entry in the Answer 2 column shall be $(-12)(1)+0=-12$. The entry in the Integers column shall be $(-12)(623)+7553=77$ the remainder that we had already figured out.

Thus we get:

$$
\left[\begin{array}{r|r|r|r}
\text { Minus Quotients } & \text { Answer 1 } & \text { Answer 2 } & \text { Integers } \\
\text { Begin } & 1 & 0 & 7553 \\
-12 & 0 & 1 & 623 \\
& 1 & -12 & 77
\end{array}\right]
$$

Now the entry in the "Minus Quotients" column is the negative of the quotient when we divide the last integer 77 into the one above it, namely 623 . We already know the quotient to be 8 and so we enter -8 .

$$
\left[\begin{array}{r|r|r|r}
\text { Minus Quotients } & \text { Answer 1 } & \text { Answer 2 } & \text { Integers } \\
\text { Begin } & 1 & 0 & 7553 \\
-12 & 0 & 1 & 623 \\
-8 & 1 & -12 & 77
\end{array}\right]
$$

We now continue this process to make the next row whose entries shall be as follows.

Answer 1: $(-8)(1)+0=-8$.
Answer 2: $(-8)(-12)+1=97$.
Integer: $(-8)(77)+623=7$.
Finally, the new "Minus Quotient" is the negative of the quotient when we divide 7 into 77, i.e. -11 .

Thus the new table is:

## 3.2. $\bar{A} R Y A B H A T C A$ ALGORITHM: EFFICIENT EUCLIDEAN ALGORITHM. 41

$$
\left[\begin{array}{r|r|r|r}
\text { Minus Quotients } & \text { Answer 1 } & \text { Answer 2 } & \text { Integers } \\
\text { Begin } & 1 & 0 & 7553 \\
-12 & 0 & 1 & 623 \\
-8 & 1 & -12 & 77 \\
-11 & -8 & 97 & 7
\end{array}\right]
$$

Now we carry out one more step of similar calculation. We leave it for the reader to verify the calculations.

$$
\left[\begin{array}{r|r|r|r}
\text { Minus Quotients } & \text { Answer 1 } & \text { Answer 2 } & \text { Integers } \\
\text { Begin } & 1 & 0 & 7553 \\
-12 & 0 & 1 & 623 \\
-8 & 1 & -12 & 77 \\
-11 & -8 & 97 & 7 \\
\text { End } & 89 & -1079 & 0
\end{array}\right]
$$

We wrote "End" because we reached a zero in the last column and we don't apply division algorithm by 0 .

We claim that we have all the answers we need. How?
We note the important property that in any of the rows, we have:
$7553($ Answer 1$)+(623)($ Answer 2$)=$ The entry in the Integers column.
The top two numbers in the Answers column give us the final answer:

$$
7=(97) 623-(8) 7553
$$

It is instructive to verify this in each row:

$$
\begin{array}{rrrr}
7553 \cdot(1) & + & 623 \cdot(0) & =7553 \\
7553 \cdot(0) & + & 623 \cdot(1) & =623 \\
7553 \cdot(1) & + & 623 \cdot(-12) & =77 \\
7553 \cdot(-8) & + & 623 \cdot(97) & =7 \\
7553 \cdot(89) & + & 623 \cdot(-1079) & =0
\end{array}
$$

We only need the last two equations here. The desired expression for the GCD is visible in the fourth equation. The fifth equation is used to find all possible expressions of the GCD using the one obtained above.

Thus from the last two equations, we see that for any integer $n$, we have:

$$
7553 \cdot(-8+89 n)+623 \cdot(97-1079 n)=7
$$

Taking $n=0$ we get the original which is often written as:

$$
7=623(97)-7553(8)
$$

By taking $n=1$ we also see that:

$$
7=7553(81)-623(982)
$$

We need both these forms of expressions depending on the intended use.
Note on the algorithm. Even though we have called the above algorithm as "Aryabhata algorithm", it is a changed version of the original.

We explain the changes made for purposes of historical accuracy.
The original algorithm was a two step process. It proceeded as in the original GCD calculations where we had only two columns: Minus quotients and Integers. Actually, it had the quotients rather than their negatives. (The negative integers, though known, were not popular.)

The necessary column of answers was then created by starting with 1,0 at the bottom and lifting the answers up by using the quotients. To read off the final answer, a careful adjustment of sign was needed.

The algorithm also provided for a shortcut. The aim of the algorithm was to write a given number as the combination of the chosen numbers. So, it was not necessary to "find" the GCD. If at some point during the calculations, we can write the given number as a combination of the bottom two entries of the "Integers" column, then we could record the multipliers next to them and then lift them up using the quotients using the algorithm.

If the given number is not a multiple of the GCD and then the problem is unsolvable. This becomes evident when we reach the GCD and the lifting work is no longer needed.

We have chosen to add the two columns of "Answers" to make the process fully automatic and avoid any "lifting work" at the end. This method has more in common with the more modern computer implementations of the Euclidean algorithm (especially the one known as Motzkin algorithm).

In this formulation, the record of the quotients is mostly for double checking the work, since they are not needed once we reach the end of the table.

## Example 2:

Find a smallest positive integer $n$ such that when $n$ is divided by 7553 it gives a remainder of 11 and when divided by 623 it gives a remainder of 4 .

Answer: First, let us note that we want

$$
n=a(623)+4 \text { and } n=b(7553)+11
$$

for some integers $a, b$.
Subtracting the second equation from the first, we get:

$$
a(623)+4-b(7553)-11=0 \text { or } a(623)-b(7553)=7 .
$$

We already know an answer to this, namely $a=97$ and $b=8$.

## 3.2. $\bar{A} R Y A B H A T ̣ A$ ALGORITHM: EFFICIENT EUCLIDEAN ALGORITHM. 43

This gives us:

$$
n=97(623)+4=8(7553)+11=60435 .
$$

Is this indeed the smallest positive integer? Indeed it is! How do we prove this? Suppose, we have another possible answer $n_{1}$ such that:

$$
n_{1}=a_{1}(623)+11 \text { and } n_{1}=b_{1}(7553)+4
$$

Then clearly

$$
n_{1}-n=\left(a_{1}-a\right) 623=\left(b_{1}-b\right) 7553 .
$$

Hence $n_{1}-n$ is divisible by both 623,7553 and hence by their LCM. From the known formula, the LCM is:

$$
\operatorname{LCM}(7553,623)=\frac{(7553)(623)}{7}=672217 .
$$

Thus all other answers $n_{1}$ differ from $n$ by a multiple of 672217 , thus, our $n$ must be the smallest positive answer!

Example 3:
Suppose we try to solve a problem very similar to the above, except we change the two remainders.

We illustrate possible difficulties and their resolution.

## Problem 1:

Find a smallest positive integer $n$ such that when $n$ is divided by 7553 it gives a remainder of 4 and when divided by 623 it gives a remainder of 11 .

## Problem 2:

Find a smallest positive integer $n$ such that when $n$ is divided by 7553 it gives a remainder of 4 and when divided by 623 it gives a remainder of 10 .

Answer: Problem 1. We can work just as before and get:

$$
n=a(623)+11 \text { and } n=b(7553)+4 \text { so } a(623)-b(7553)=-7 .
$$

Multiplying by -1 we get:

$$
7=b(7553)-a(623)
$$

If we recall our alternate answer $7=81(7553)-982(623)$, we see that we could simply take $a=982$ and $b=81$.

Thus, we have, $n=982(623)+11=611797$. We could get the same answer as $n=81(7553)+4 .{ }^{4}$

Answer: Problem 2. Proceeding as before, we get equations:

$$
n=a(623)+10 \text { and } n=b(7553)+4
$$

[^20]so by subtraction
$$
a(623)-b(7553)=-6
$$

Since the GCD 7 does not divide the right hand side -6 , we conclude that there is no such integer $n$.

### 3.3 Division algorithm in polynomials.

Earlier, we have discussed addition, subtraction and multiplication of polynomials. As indicated, we have a problem if we wish to divide one polynomial by another, since the answer may not be a polynomial again.

Let us take an example of $u(x)=x^{3}+x, v(x)=x^{2}+x+1$. It seems obvious that the ratio $\frac{u(x)}{v(x)}$ is not a polynomial.

But we want to really prove this! Here is how we argue. Suppose, if possible, it is equal to some polynomial $w(x)$ and write the equation:

$$
\frac{u(x)}{v(x)}=w(x) \text { or } u(x)=w(x) v(x)
$$

We write this equation out and compare both sides.

$$
x^{3}+x=(w(x))\left(x^{2}+x+1\right) .
$$

By comparing the degrees of both sides, it follows that $w(x)$ must have degree 1 in $x .{ }^{5}$

Let us write $w(x)=a x+b$. Then we get

$$
x^{3}+x=(a x+b)\left(x^{2}+x+1\right)=a x^{3}+(a+b) x^{2}+(a+b) x+b .
$$

Comparing coefficients of $x$ on both sides, we see that we have:

$$
a=1, a+b=0,(a+b)=1, b=0 .
$$

Obviously these equations have no chance of a common solution, they are inconsistent!

This proves that $w(x)$ is not a polynomial!
We therefore make this:
Definition: Divisibility of polynomials. We say that a polynomial $v(x)$ divides a polynomial $u(x)$, if the remainder $r(x)$ obtained by dividing $u(x)$ by $v(x)$ becomes zero.

The zero polynomial can divide only a zero polynomial.
But perhaps we are being too greedy. What if we only try to solve the first two equations?

[^21]Thus we solve $a=1$ and $a+b=0$ to get $a=1, b=-1$. Then we see that using $a=1, b=-1$ :

$$
u(x)=x^{3}+x=(x-1)\left(x^{2}+x+1\right)+x+1 .
$$

Let us name $x-1$ as $q(x)$ and $x+1$ as $r(x)$. Then we get:

$$
u(x)=v(x) q(x)+r(x)
$$

where $r(x)$ has degree 1 which is smaller than the degree of $v(x)$ which is 2 .
Thus, when we try to divide $u(x)$ by $v(x)$, then we get the best possible division as $q(x)=x-1$ and a remainder $x+1$.

We now formalize this idea.
Definition: Division Algorithm. Suppose that $u(x)=a x^{n}+\cdots$ and $v(x)=$ $b x^{m}+\cdots$ are polynomials of degrees $n, m$ respectively and that $v(x)$ is not the zero polynomial. Then there are unique polynomials $q(x)$ and $r(x)$ which satisfy the following conditions:

1. $u(x)=q(x) v(x)+r(x)$.
2. Moreover, either $r(x)=0$ or $\operatorname{deg}_{x}(r(x))<\operatorname{deg}_{x}(v(x))$.

When the above conditions are satisfied, we declare that $q(x)$ is the quotient and $r(x)$ is the remainder when we divide $\mathbf{u}(\mathbf{x})$ by $\mathbf{v}(\mathbf{x})$. Some people use the word division, in place of the word quotient.

The quotient $q(x)$ and the remainder $r(x)$ always exist and are uniquely determined by $u(x)$ and $v(x) .{ }^{6}$

First, we explain how to find $q(x)$ and $r(x)$ systematically.
Let us redo the problem with $u(x)=x^{3}+x$ and $v(x)=x^{2}+x+1$ again. Since $u$ has degree 3 and $v$ has degree 2 , we know that $q$ must have degree $3-2=1$.

For convenience, we often drop the $x$ from our notation and simply write $u, v, q, r$ in place of $u(x), v(x), q(x), r(x)$.

What we want is to arrange $u-q v$ to have a degree as small as possible; this means that either it is the zero polynomial or its degree is less than 2.

Start by guessing $q=a x$. Calculate

$$
u-(a x) v=x^{3}+x-(a x)\left(x^{2}+x+1\right)=(1-a) x^{3}+(-a) x^{2}+(1-a) x .
$$

This means $a=1$ and then we get

$$
u-(x) v=(-1) x^{2}
$$

[^22]Our right hand side has degree 2 which is still not small enough! So we need to improve our $q$. Add a next term to the current $q=x$ and make it $q=x+b$.

## Recalculate: ${ }^{7}$

$$
\begin{aligned}
u-(x+b) v & =u-x v-b v \\
& =(-1) x^{2}-b\left(x^{2}+x+1\right) \\
& =(-1-b) x^{2}+(-b) x+(-b)
\end{aligned}
$$

Thus, if we make $-1-b=0$ by taking $b=-1$, we get $q=x-1$ and

$$
u-q v=(-(-1)) x+(-(-1))=x+1=r .
$$

## Let us summarize this process:

1. Suppose $u(x), v(x)$ have degrees $n, m$ respectively where $n \geq m$. (Note that we are naturally assuming that these are non zero polynomials.)
As above, for convenience we shall drop $x$ from our notations for polynomials.
2. Start with a guess $q=a x^{(n-m)}$ and choose $a$ such that $u-q v$ has degree less than $n$.
3. Note that the whole term $a x^{(n-m)}$ can simply be thought of as:

$$
a x^{(n-m)}=\frac{\text { the leading term of } u}{\text { the leading term of } v} .
$$

This formula is useful, but often it is easier to find the term by inspection.
In our example above, this first term of $q$ came out to be $\frac{x^{3}}{x^{2}}=x$. Thus, the starting guess for $q$ is $q=x$.
4. Using this current value of $q$, if $u-q v$ is zero or if its degree is less than $m$, then stop and set $u-q v$ as the final remainder $r$.
5. If not, add a next term to $q$ to make the degree of $u-q v$ even smaller. This next term can be easily found as:

$$
\frac{\text { leading term of the current } u-q v}{\text { leading term of } v} .
$$

In our example this was $\frac{-x^{2}}{x^{2}}=-1$. Thus, the new guess for $q$ is $q=x-1$.

[^23]6. Continue until $u-q v$ becomes 0 or its degree drops below $m$.

In our example, $q=x-1$ gives the remainder $x+1$ whose degree is clearly less than 2 and we stop.

What we are describing is the process of long division that you learned in high school. We are going to learn more efficient methods for doing it below.

For comparison, we present the long division process for the same polynomials. Compare the steps for better understanding:


There are two lucky situations where we get our $q, r$ without any further work. Let us record these for future use.

1. In case $u$ is the zero polynomial, we take $q(x)=0$ and $r(x)=0$. Check that this has the necessary properties!
2. In case $u$ has degree smaller than that of $v$ (i.e. $n<m$ ), we take $q(x)=0$ and get $r(x)=u(x)$. Check that this is a valid answer as well.
3. One important principle is to "let the definition be with you!!" If you somehow see an answer for $q$ and $r$ which satisfies the conditions, then don't waste time in the long division.

We will see many instances of this later.
We now begin with an extension of the division algorithm.

### 3.4 Repeated Division.

For further discussion, it would help to define some terms.
Definition: Dividend and Divisor. If we divide $u(x)$ by $v(x)$, then we shall call $u(x)$ the dividend and $v(x)$ the divisor. We already know the meaning of the terms quotient(or the division) and the remainder.

We wish to organize the work of division algorithm in a certain way, so we can get more useful information from it.

We assign ourselves some simple tasks:
Task 1. Given

$$
u(x)=x^{3}+2 x-1 \text { and } v(x)=x^{2}+x+2,
$$

calculate the quotient $q(x)$ and the remainder $r(x)$ when you divide $u(x)$ by $v(x)$.
Answer:
As before, we shall drop the $x$ from the notation for convenience.
We need to make $u-q v$ to be zero or have degree less than 2 .
We see $u-(x-1) v=x+1$, so $r=x+1$ and $q=x-1$.
We recommend the following arrangement of the above work, for future use.
We shall make a table:

$$
\left[\begin{array}{l|l|l}
\text { Step no. } & \text { Minus Quotients } & \text { polynomials } \\
0 & \text { Begin } & u \\
1 & -q & v \\
2 & \text { End of step 1 } & r
\end{array}\right]
$$

Thus after writing the entries dividend and divisor $u, v$ in a column, we write the negative of the quotient in the "Minus Quotients" column. The entry below $u, v$ is the remainder $r$ obtained by adding $-q$ times $v$ to $u$.

Thus, our example becomes:

$$
\left[\begin{array}{l|l|l}
\text { Step no. } & \text { Minus Quotients } & \text { polynomials } \\
0 & \text { Begin } & x^{3}+2 x-1 \\
1 & -x+1 & x^{2}+x+2 \\
2 & \text { End of step 1 } & x+1
\end{array}\right]
$$

Note that the actual long division process is not recorded and is to be done on the side.

Task 2. Now repeat the steps of task 1 and continue the table by treating the last two polynomials as the dividend and the divisor.

Thus our new dividend shall be $x^{2}+x+2$ and the new divisor shall be $x+1$.
Answer: We easily see that

$$
x^{2}+x+2-(x)(x+1)=2
$$

i.e. the quotient is $x$ and the remainder is 2 .

Thus our table extends as:

$$
\left[\begin{array}{l|l|l}
\text { Step no. } & \text { Minus Quotients } & \text { polynomials } \\
0 & \text { Begin } & x^{3}+2 x-1 \\
1 & -x+1 & x^{2}+x+2 \\
2 & -x & x+1 \\
3 & \text { End of step 2 } & 2
\end{array}\right]
$$

Task 3. Continue the tasks as far as possible, i.e. until the last remainder is zero.
Answer: Now the last two polynomials give the dividend $x+1$ and the divisor 2 . Thus

$$
x+1-\left(\frac{x+1}{2}\right)(2)=0 .
$$

Thus quotient is $\frac{x+1}{2}$ and the remainder is 0 . The zero remainder says our tasks are over!

Our table extends as:

$$
\left[\begin{array}{l|l|l}
\text { Step no. } & \text { Minus Quotients } & \text { polynomials } \\
0 & \text { Begin } & x^{3}+2 x-1 \\
1 & -x+1 & x^{2}+x+2 \\
2 & -x & x+1 \\
3 & -\frac{x+1}{2} & 2 \\
4 & \text { End of steps! } & 0
\end{array}\right]
$$

## The step numbers are not essential and we shall drop them in future.

The steps carried out above are for a purpose. These steps are used to find the GCD or the greatest common divisor (factor) of the original polynomials $u(x), v(x)$.

The very definition of the division algorithm says that

$$
u(x)-q(x) v(x)=r(x)
$$

Thus, any common factor of $u(x)$ and $v(x)$ must divide the remainder. Using this repeatedly, we see that such a factor must divide each of the polynomials in our column of polynomials in the calculation table.

Since one of these polynomials is 2 , we see that $u(x)$ and $v(x)$ have no non constant common factor!

Indeed, it is not hard to show that the last polynomial in our column, just above the zero polynomial must divide every polynomial above it, and in turn any polynomial which divides the top two, must divide it! Thus, in some sense, it is the largest degree polynomial dividing both the top polynomials - or it is their G (reatest) C(ommon) D(ivisor)- or GCD for short.

### 3.5 The GCD and LCM of two polynomials.

Indeed, this last polynomial (or any constant multiple of it) can be defined as the GCD.

To make the idea of GCD precise, most algebraists would choose a multiple of the last polynomial which makes the highest degree coefficient equal to $\mathbf{1}$.

To make this idea clear, we start with:
Definition: Monic polynomial A nonzero polynomial in one variable is said to be monic if its leading coefficient is 1 . Note that the only monic polynomial of degree 0 is 1 .

Definition: GCD of two polynomials The GCD of two polynomials $u(x), v(x)$ is defined to be a monic polynomial $d(x)$ which has the property that $d(x)$ divides $u(x)$ as well as $v(x)$ and is the largest degree monic polynomial with this property.

We shall use the notation:

$$
d(x)=\operatorname{GCD}(u(x), v(x))
$$

An alert reader will note that the definition gets in trouble if both polynomials are zero. Some people make a definition which produces the zero polynomial as the answer, but then we have to let go of the 'monicness' condition. In our treatment, this case is usually excluded!

Definition: LCM of two polynomials The LCM (least common multiple) of two polynomials $u(x), v(x)$ is defined as the smallest degree monic polynomial $L(x)$ such that each of $u(x)$ and $v(x)$ divides $L(x)$.

As in case of GCD, we shall use the notation:

$$
L(x)=\operatorname{LCM}(u(x), v(x))
$$

If either of $u(x)$ or $v(x)$ is zero, then the only choice for $L(x)$ is the zero polynomial and the degree condition as well as the monicness get in trouble!

As in the case of GCD, we can either agree that the zero polynomial is the answer or exclude such a case!

## Calculation of LCM.

Actually, it is easy to find LCM of two non zero polynomials $u(x), v(x)$ by a simple calculation. The simple formula is:

$$
\operatorname{LCM}(u(x), v(x))=c \frac{u(x) v(x)}{\operatorname{GCD}(u(x), v(x))}
$$

Here $c$ is a constant which is chosen to make the LCM monic.
There are occasional shortcuts to the calculation and these will be presented in special cases.

The most important property of GCD. It is possible to show that the GCD of two polynomials $u(x), v(x)$ is also the smallest degree monic polynomial which can be expressed as a combination $a(x) v(x)-b(x) u(x)$.

Such a presentation of the GCD of two integers was already explained as the Āryabhața algorithm. For polynomials, a similar scheme works, however, the calculation can get messy due to the number of terms in a polynomial. We shall leave it for the reader to investigate further as independent work.

For our polynomials above, the GCD will be the monic multiple of 2 and so the GCD of the given $u(x), v(x)$ is said to be $1 .{ }^{8}$

You may have seen a definition of the GCD of polynomials which tells you to factor both the polynomials and then take the largest common factor.

This is correct, but requires the ability to factor the given polynomials, which is a very difficult task in general.

Our GCD algorithm is very general and useful in finding common factors without finding the factorization of the original polynomials.

Example 4: Let $a$ be a constant. Find the division and remainder when $(x-a)$ divides various powers of $x$, like $x, x^{2}, x^{3}, x^{4}, \cdots$.

Using these results deduce a formula to compute the remainder when a polynomial $f(x)$ is divided by $(x-a)$.

Answer: The main point to note is this:
Suppose that we wish to divide a polynomial $u(x)$ by a non zero polynomial $v(x)$ to find the quotient and remainder.

If we can somehow guess a polynomial $r(x)$ such that

$$
u(x)-r(x)=q(x) v(x)
$$

i.e. $v(x)$ divides $u(x)-r(x)$

- and either $r(x)=0$ or the degree of $r(x)$ is less than that of $v(x)$,
then we have found our remainder $r(x)$. If needed, we can find the quotient by dividing $u(x)-r(x)$ by $v(x)$ if needed!

No long division is actually needed.
For $u(x)=x$, we guess $r(x)=a$ and note that

$$
u(x)-r(x)=x-a=(1)(x-a)
$$

so obviously $q(x)=1$ and $r(x)=a$.
Note that $r(x)=a$ is either 0 or has degree zero in $x$ and hence is a valid remainder.
For $u(x)=x^{2}$ we guess that $r(x)=a^{2}$ and verify it from the identity:

$$
x^{2}-a^{2}=(x+a)(x-a) \text { which means } x^{2}=(x+a)(x-a)+a^{2} .
$$

Thus, dividing $x^{2}$ by $x-a$ gives the quotient $(x+a)$ and the remainder $a^{2}$.

[^24]To divide $x^{3}$ by $(x-a)$, we start with the above relation and multiply it by $x$, so:

$$
x^{3}-a^{2} x=x(x+a)(x-a) \text { or } x^{3}=x(x+a)(x-a)+a^{2} x .
$$

Rearranging, we get:
$x^{3}-a^{3}=x(x+a)(x-a)+a^{2} x-a^{3}=\left(x(x+a)+a^{2}\right)(x-a)=\left(x^{2}+a x+a^{2}\right)(x-a)$.
So $x^{3}$ divided by $x-a$ gives $r(x)=a^{3}$ and $q(x)=x^{2}+a x+a^{2}$.
The reader is invited to repeat this idea and deduce

$$
x^{4}-a^{4}=(x)\left(x^{2}+a x+a^{2}\right)(x-a)+a^{3}(x-a)
$$

or
$x^{4}-a^{4}=\left(x^{3}+a x^{2}+a^{2} x+a^{3}\right)(x-a)$ giving $q(x)=x^{3}+a x^{2}+a^{2} x+a^{3}$ and $r(x)=a^{4}$.
Inspired by this let us guess and prove: $n=1,2, \cdots$ :

$$
x^{n}-a^{n}=\left(x^{n-1}+a x^{n-2}+\cdots+a^{n-1}\right)(x-a) .
$$

Proof. Here is a simple arrangement of multiplication on the right hand side which makes a convincing proof.

$$
\begin{aligned}
& x^{n-1} \quad+a x^{n-2} \quad \cdots \quad+a^{n-2} x \quad a^{n-1} \\
& \begin{array}{llllll} 
& & & \times & (x-a) \\
\hline x^{n} & +a x^{n-1} & \cdots & +a^{n-2} x^{2} & a^{n-1} x &
\end{array} \\
& \begin{array}{lllll} 
& -a x^{n-1} & \cdots & -a^{n-2} x^{2} & -a^{n-1} x
\end{array}-a^{n} \quad \begin{array}{lll} 
& & \\
\hline x^{n} & & -a^{n}
\end{array}
\end{aligned}
$$

We conclude that the remainder of $x^{n}$ is $a^{n}$ with quotient $x^{n-1}+a x^{n-2}+\cdots+a^{n-1}$.
It is easy to see that the remainder of $c x^{n}$ will be $c a^{n}$ for a constant $c$ and generally for any polynomial $f(x)$ the remainder when divided by $(x-a)$ shall be $f(a)$.

For proof, consider this:
The polynomial $f(x)$ is a sum of monomials of the form $c x^{n}$ and we have already seen that each such monomial has remainder $c a^{n}$.

This means that $c x^{n}-c a^{n}$ is divisible by $(x-a)$. Adding up various such differences, we see that $f(x)-f(a)$ is indeed divisible by $(x-a)$. Hence $f(x)$ has remainder $f(a) .{ }^{9}$

We have thus proved the well known

[^25]Remainder theorem The linear expression $(x-a)$ divides a polynomial $f(x)$ if and only if $f(a)=0$.

We have also proved, in view of our concrete steps:
The geometric series formula
For any positive integer $n$ we have

$$
1+x+\cdots+x^{n-1}=\frac{x^{n}-1}{x-1}
$$

For proof, proceed as follows:

- Start with the established identity:

$$
\left(x^{n-1}+a x^{n-2}+\cdots+a^{n-1}\right)(x-a)=x^{n}-a^{n} .
$$

- Set $a=1$ and deduce:

$$
\left(x^{n-1}+x^{n-2}+\cdots+1\right)(x-1)=x^{n}-1
$$

- Rewrite as:

$$
\left(x^{n-1}+x^{n-2}+\cdots+1\right)=\frac{x^{n}-1}{x-1} .
$$

Example 5: Let $a$ be a constant. Find the division and the remainder when $\left(x^{2}-a\right)$ is divided into various powers of $x, x^{2}, x^{3}, x^{4}, \cdots$.

Using these results, deduce a formula for the remainder when a polynomial $f(x)$ is divided by $\left(x^{2}-a\right)$.

Answer: As before we do some initial cases for understanding and experience: We wish to divide various $x^{n}$ by $x^{2}-a$ and find the division and remainder.

Case $n=0$

$$
x^{0}=1=(0)\left(x^{2}-a\right)+1 \text { so } q(x)=0, r(x)=1
$$

Case $n=1$

$$
x^{1}=x=(0)\left(x^{2}-a\right)+x \text { so } q(x)=0, r(x)=x .
$$

so that

$$
x^{n}-a^{n}=q_{n}(x)(x-a) .
$$

Now let $f(x)=c_{0}+c_{1} x+\cdots c_{d} x^{d}$ be a polynomial of degree $d$. Then we get:

$$
f(x)-f(a)=\left(c_{0}-c_{0}\right)+c_{1}(x-a)+c_{2}\left(x^{2}-a^{2}\right)+\cdots+c_{d}\left(x^{d}-a^{d}\right)
$$

and this simplifies to

$$
f(x)-f(a)=\left(c_{1} q_{1}(x)+c_{2} q_{2}(x)+\cdots+c_{d} q_{d}(x)\right)(x-a) .
$$

Hence the remainder of $f(x)$ when divided by $(x-a)$ is $f(a)$.

Case $n=2$

$$
x^{2}=x^{2}=\left(x^{2}-a\right)+a \text { so } q(x)=1, r(x)=a .
$$

Case $n=3$

$$
x^{3}=x\left(x^{2}-a\right)+a x \text { so } q(x)=x, r(x)=a x
$$

Case $n=4$

$$
x^{4}=x^{2}\left(x^{2}-a\right)+a\left(x^{2}-a\right)+a^{2} \text { so } q(x)=x^{2}+a, r(x)=a^{2} .
$$

Case $n=5$

$$
x^{5}=x\left(x^{2}+a\right)\left(x^{2}-a\right)+a^{2} x \text { so } q(x)=x^{3}+a x, r(x)=a^{2} x .
$$

Case $n=6$

$$
x^{6}=x^{2}\left(x^{3}+a x\right)\left(x^{2}-a\right)+a^{2}\left(x^{2}-a\right)+a^{3} \text { so } q(x)=x^{4}+a x^{2}+a^{2}, r(x)=a^{3} .
$$

This is enough experience!
Look at the answers for even powers:

| $n=$ | $q(x)$ | $r(x)$ |
| :--- | :--- | :--- |
| 2 | 1 | $a$ |
| 4 | $x^{2}+a$ | $a^{2}$ |
| 6 | $x^{4}+a x^{2}+a^{2}$ | $a^{3}$ |

We now guess and prove the formula for $x^{8}$. The guess is $q(x)=x^{6}+a x^{4}+a^{2} x^{2}+a^{3}$ and $r(x)=a^{4}$.

The proof is this:

$$
\begin{aligned}
x^{8} & =x^{2}\left(x^{6}\right) \\
& =x^{2}\left(x^{4}+a x^{2}+a^{2}\right)\left(x^{2}-a\right)+x^{2}\left(a^{3}\right) \\
& =x^{2}\left(x^{4}+a x^{2}+a^{2}\right)\left(x^{2}-a\right)+\left(x^{2}-a\right)\left(a^{3}\right)+a^{4} \\
& =\left(x^{6}+a x^{4}+a^{2} x^{2}+a^{3}\right)\left(x^{2}-a\right)+a^{4} .
\end{aligned}
$$

Here is a challenge to your imagination and hence a quick road to the answer!
Name the quantity $x^{2}$ as $y$ and note that then we are trying to divide it by $(y-a)$. We already learned how to find the answers as polynomials in $y$. use them and the plug back $y=x^{2}$ to get our current answers!

But what do we do for an odd power? Suppose that we know the answer for an even power:

$$
x^{2 m}=q(x)\left(x^{2}-a\right)+r(x) .
$$

We now know that $r(x)=a^{m}$.

Then the answer for $x^{2 m+1}$ shall be given by:

$$
x^{2 m+1}=x\left(q(x)\left(x^{2}-a\right)+r(x)\right)=x q(x)\left(x^{2}-a\right)+a^{m} x .
$$

Since $a^{m} x$ is zero or has degree less than 2 , it is the remainder.
Combining the above work, we see that we at least have the complete formula for the remainders of $x^{n}$. It is this: Write $n=2 m+s$ where $s=0$ or 1 . (In other words, $s$ is the remainder when we divide $n$ by 2.)

Then the remainder of $x^{n}$ is $a^{m} x^{s}$ !
For a general polynomial, rather than writing a general formula, let us illustrate the method of calculation.

Say $f(x)=x^{5}+2 x^{4}+x^{3}-x+3$. Find the remainder when it is divided by $x^{2}-2$.
The remainders of the five terms are as follows:

| term | Even or odd | remainder |
| :--- | :--- | :--- |
| $x^{5}$ | odd | $2^{2} x=4 x$ |
| $2 x^{4}$ | even | $2\left(2^{2}\right)=8$ |
| $x^{3}$ | odd | $2 x$ |
| $-x$ | odd | $(-1) x=-x$ |
| 3 | even | 3 |

So finally the remainder of $f(x)$ is

$$
4 x+2 x-x+8+3=5 x+11
$$

Note that we have not kept track of $q(x)$. If we need it, we can use the definition and calculate by dividing $f(x)-r(x)$ by $x^{2}-2$.

## Enhanced remainder theorem!

Here is an even simpler idea to calculate the remainder. Note that we ultimately have $f(x)=q(x)\left(x^{2}-a\right)+r(x)$. If we agree to replace every " $x^{2}$ " by $a$ in $f(x)$, then we notice that on the right hand side the part $q(x)\left(x^{2}-a\right)$ will become zero. So, $f(x)$ will eventually reduce to the remainder!

Here is how this wonderful idea works for our problem solved above:

$$
\begin{array}{ll}
\text { Current polynomial } & \text { Explanation } \\
x^{5}+2 x^{4}+x^{3}-x+3 & \text { Start } \\
2 x^{3}+4 x^{2}+2 x-x+3 & x^{2} \text { in each term is replaced by } 2 \\
4 x+8+2 x-x+3 & \text { two more reductions done } \\
5 x+11 & \text { collected terms }
\end{array}
$$

Naturally, this is much easier than the actual division.

## Chapter 4

## Introduction to analytic geometry.

While algebra is a powerful tool for calculations, it is not so helpful for visualization and intuition for a beginning student. Also, many important motivations for algebra originally came from geometric ideas. Hence, we next introduce the algebraic version of geometric concepts.

Here is a quick summary of our plan. We think of the geometric objects called lines, planes, three space, etc. These are made up of points. We associate numbers (single or multiple) to each points so that we can express various geometric concepts and calculations as algebraic concepts and calculations.

In geometry, we first study points in a line, then a plane and then a three (or higher) dimensional space.

### 4.1 Coordinate systems.

To work with a line, we agree to choose a special point called the "origin" and a scale so that a certain other point is at a distance of 1 unit from the origin - to be called a unit point. This sets up a system of coordinates (associated real numbers) to every point of the line and we can handle the points by using the algebraic operations of real numbers.

If $P$ is a point with coordinate $x$, then we may also call it the point $P(x)$.


To work with a plane, we agree to set up a special point called the origin and a pair of perpendicular lines called the $x$ and $y$ axes.

On each of the axes, we agree to a scale so that every axis has a coordinate system of its own. Then every point in the plane can be associated with a pair of real numbers called coordinates and these are described in the following way:

Given any point $P$ in the plane, draw a line from it parallel to the $y$ axis. Suppose that it hits the $x$-axis at a point $Q$. Then the coordinate $Q$ on the $x$-axis is called
the $x$ coordinate of $P$. Similarly a line is drawn through $P$ parallel to the $x$-axis. Suppose that it hits the $y$-axis at a point $R$. Then the coordinate of $R$ on the $y$-axis is called the $y$ coordinate of $P$.

If these coordinates are $(a, b)$ respectively, then we simply write $P(a, b)$ or say that $P$ is the point (with coordinates) $(a, b)$.


Thus the chosen origin has coordinates $(0,0)$, the $x$-axis consists of all points with coordinates of the form $(a, 0)$ with $(1,0)$ being the unit point on the $x$ axis. Similarly, the $y$-axis consists of all points with coordinates $(0, b)$ with its unit point at $(0,1)$.

An algebraically minded person can turn this all around and simply declare a plane as a set of pairs of real numbers $(a, b)$ and declare what is meant by the origin and lines and other various geometric objects contained in the plane.

This has the advantage of conveniently defining the three space as a set of all triples of real numbers $(a, b, c)$, where each triple denotes a point. The $x$-axis is now all points of the form $(a, 0,0)$, the $y$-axis is full of points of the form $(0, b, 0)$ and the $z$-axis is full of points of the form $(0,0, c)$.

Evidently, now it is very easy to think of $4,5,6$ or even higher dimensions!

### 4.2 Geometry: Distance formulas.

## Distance on a line.

For points $A(a), B(b)$ on a line, the distance between them is given by $|b-a|$ and we may write $d(A, B)$ to denote it.

For example, given points $A(5), B(3), C(-1)$, the distance between $A$ and $B$ is

$$
d(A, B)=|3-5|=|-2|=2
$$

The distance between $A$ and $C$ is

$$
d(A, C)=|(-1)-(5)|=|-6|=6
$$

Finally, the distance between $C$ and $B$ is


We also have reason to use a signed distance, which we shall call the shift from $A$ to $B$ as $b-a$. Sometimes, we may wish do denote this formally as $\overrightarrow{A B} .{ }^{1}$

Warning: The notation $\overrightarrow{A B}$ is often replaced by a simpler $\overline{A B}$ where the direction from $A$ to $B$ is understood by convention.

There are also some books on Geometry which use the notation $\overrightarrow{A B}$ to denote the ray from $A$ towards $B$, so it includes all points in the direction to $B$, both between $A, B$ and beyond $B$. So, when you see the notation, carefully look up its definition.

Our convention for a ray. We shall explicitly write "ray $A B$ " when we mean to discuss all points of the line from $A$ to $B$ extended further out from $B$.

Thus, if you consider our picture above, the ray $A B$ will refer to all points on the line with coordinates less than or equal to 5 , the coordinate of $A$ (i.e. to the left of $A)$. The ray $B A$ instead, will be all points with coordinate bigger than or equal to 3 , the coordinate of $B$, or points to the right of $B$.

This shift gives the distance when we take its absolute value but it tells more. If the shift is positive, it says $B$ is to the right of $A$ and when negative the opposite holds.

Thus, for our points $A, B, C$, we get

$$
\overrightarrow{A B}=3-5=-2, \overrightarrow{A C}=-1-5=-6 \text { and } \overrightarrow{C B}=3-(-1)=4
$$

Thus we see $\overrightarrow{A B}$ is negative, so $A$ is to the right of $B$. On the other hand, $\overrightarrow{C B}$ is positive and hence $B$ is to the right of $C$. This is clear from the picture as well.

Distance in the plane.
For points $P_{1}\left(a_{1}, b_{1}\right), P_{2}\left(a_{2}, b_{2}\right)$ in a plane, we have two shifts which we conveniently write as a pair $\left(a_{2}-a_{1}, b_{2}-b_{1}\right)$ and call it the shift from $P_{1}$ to $P_{2}$. As before, we may use the notation $\overrightarrow{P_{1} P_{2}}$ which is clearly very convenient now, since it has both the components built in it!

Alert! When we write $\overrightarrow{P_{1} P_{2}}$, be sure to subtract the coordinates of the first point from the second. It is a common mistake to subtract the second point instead, leading to a wrong answer.

[^26]Now, the distance between the two points has to be computed by a more complicated formula using the Pythagorean Theorem ${ }^{2}$ and we have:

The distance formula $d\left(P_{1}, P_{2}\right)=\sqrt{\left(a_{2}-a_{1}\right)^{2}+\left(b_{2}-b_{1}\right)^{2}}=\left|\overrightarrow{P_{1} P_{2}}\right|$.
To make it look similar to a line, we have added a more suggestive notation for it: $\left|\overrightarrow{P_{1} P_{2}}\right|$.

The signs of the two components of the shift also tell us of the relative position of $P_{2}$ from $P_{1}$. Basically it says which quadrant the point $P_{2}$ lives in when viewed from $P_{1}$. Because of this, we shall also call this shift by the term direction numbers from the first point to the second.

For example, now consider the points $P(3,2), Q(2,3), R(-1,-1)$. Study their display in the plane:


The shift from $P$ to $Q$ is

$$
\overrightarrow{P Q}=(2-3,3-2)=(-1,1)
$$

and the distance from $P$ to $Q$ is

$$
d(P, Q)=\sqrt{(-1)^{2}+(1)^{2}}=\sqrt{2} .
$$

Notice that from the position $P$, the point $Q$ does appear to be one unit to the left and one unit up!

Verify that for the points $Q, R$, we get $\overrightarrow{Q R}=(-1-2,-1-3)=(-3,-4)$ and $d(Q, R)=\sqrt{(-3)^{2}+(-4)^{2}}=\sqrt{25}=5$.

[^27]Similarly for the points $P, R$, we get $\overrightarrow{P R}=(-1-3,-1-2)=(-4,-3)$ and hence the distance $d(P, R)=5$ also.

We can similarly handle points in a three dimensional space and set up shifts and distance formulas. The higher dimensions can be handled as needed.

### 4.3 Change of coordinates on a line.

In the case of the coordinates of a line, we had to choose an origin and then a scale and indeed we inadvertently chose a direction as well, by deciding where the unit point was.

What happens if we choose a different origin and a different scale?
Clearly the coordinates of a point will change. We would like to create an algebraic recipe for the resulting change.

We first illustrate this by an example.
Let us say that we have a coordinate system so that each point has an assigned coordinate. We shall call these the original $x$-coordinates.

We would like to choose a new origin at the point $Q(3)$ - the point with original $x$-coordinate 3 . Let us choose the new unit point to the right of $Q$ at distance 1 , so it would be at $x=4$.

Let $x$ be the value of the original coordinate of some point and let $x^{\prime}$ be its new coordinate. Then we claim the formula:

$$
x^{\prime}=x-3
$$

We shall say that under this coordinate change the point $P(x)$ goes to the point $P^{\prime}(x-3)$.

Thus the point $Q(3)$ goes to the point $Q^{\prime}(3-3)=Q^{\prime}(0)$. This is the new origin!
Also, it is easy to see that the distance between any two points stays unchanged whether we use the $x$ or the $x^{\prime}$ coordinates. ${ }^{3}$

A most general change of coordinates can be described by a substitution $x^{\prime}=a x+b$ where $a$ is non zero. Its effect can be analyzed as described below.

1. Case $a>0$ : The origin is shifted to the point with $x$-coordinate $-b / a$. All shifts are multiplied by $a$ and hence the direction is preserved. The distances are also scaled (multiplied) by $a$.
For example, let $a=3, b=-1$. Consider the points $P(2), Q(5)$. The new coordinate of $P$ shall be $3(2)-1=5$. Thus $P(2)$ becomes $P^{\prime}(5)$.

Similarly for the point $Q$ we get $3(5)-1=14$ and thus $Q(5)$ becomes $Q^{\prime}(14)$.

[^28]The original shift $\overrightarrow{P Q}=5-2=3$. The new shift is $14-5=9$ which is 3 times the old shift.
The distance 3 becomes $3(3)=9$.
2. Case $a<0$ : The origin is still shifted to the point with $x$-coordinate $-b / a$. All shifts are multiplied by $a<0$ and hence the direction is reversed. The distances are scaled (multiplied) by $|a|=-a$.
For example, let $a=-2, b=-5$.
For the same points $P(2), Q(5)$, the new coordinates become $(-2) 2-5=-9$ and $(-2) 5-5=-15$. Thus $P(2)$ goes to $P^{\prime}(-9)$ and $Q(5)$ goes to $Q^{\prime}(-15)$.
Thus the new shift $\overrightarrow{P^{\prime} Q^{\prime}}$ is $-15-(-9)=-6$ which is indeed the old shift $\overrightarrow{P Q}=3$ multiplied by $a=-2$. The distance changes from 3 to 6 .

The reader should investigate what happens to the unit points or other specific given points.

### 4.4 Change of coordinates in the plane.

We now generalize the above formulas and get similar results. Thus we have:

1. Translation or Change of origin only. A transformation $x^{\prime}=x-p, y^{\prime}=y-q$ changes the origin to the point with original $(x, y)$ coordinates $(p, q)$. All shifts and distances are preserved.
2. Change of origin with axes flips. A transformation $x^{\prime}=u x-p, y^{\prime}=v y-q$ where $u, v$ are $\pm 1$ changes the origin to the point with original $(x, y)$ coordinates $(u p, v q) .{ }^{4}$
All distances are preserved. However, shifts can change and we list their properties in the form of a table. It is recommended that the reader verifies these by making a picture similar to the worked out problem below.

| Value of $u$ | Value of $v$ | Effect |
| ---: | ---: | :--- |
| 1 | -1 | $y$ axis is flipped. A reflection about the $x$ axis. |
| -1 | 1 | $x$ axis is flipped. A reflection about the $y$ axis. |
| -1 | -1 | Both axes are flipped. A reflection about the origin. |

Thus, the transformation consists of first a translation to make (up,uq) as the new origin followed by the indicated flip(s).

[^29]The process of identifying the new origin and describing the flips is analyzing the images of a few well chosen points.

As an illustration, let us see what a transformation $x^{\prime}=-x-2, y^{\prime}=y-1$ does. The point $P(-2,1)$ becomes $P^{\prime}(0,0)$ or the new origin.

All points of the form $(a, 1)$ get transformed to $(-a-2,0)$ i.e. become the new $x^{\prime}$ axis. The point $Q(-2+1,1)=(-1,1)$ which has a shift of $(1,0)$ from $P$ goes to $(-1,0)$ showing that there is a flip of the $x$ axis direction.

Similarly the points $R(-2, b)$ will map to the $(0, b-1)$ to become the new $y^{\prime}$ axis. There is no flip involved, since the point $R(-2,2)$ with shift $(0,1)$ from $P$ goes to $R(0,1)$.


The general idea consists of three steps.
(a) First determine what becomes the new origin - say it is $P(p, q)$.
(b) Then find the images of $Q(p+1, q)$ and $R(p, q+1)$.
(c) Compare the original triangle $P Q R$ with its image for any possible flips!

### 4.5 General change of coordinates.

We can indeed stretch these ideas to analyze a general change of coordinates in the plane. We will, however, leave the analysis for higher courses. First we state the most important facts for information and inducement to study further.

The most general coordinate change in the plane can be written as

$$
x^{\prime}=a(x-p)+b(y-q), y^{\prime}=c(x-p)+d(y-q) .
$$

where $a, b, c, d$ are constants such that the determinant $\left|\begin{array}{ll}a & b \\ c & d\end{array}\right|=a d-b c \neq 0$.
It is easy to see that the origin is shifted to the point with original coordinates $(p, q)$, but shifts and distances change in a complicated manner. If you wish to study geometric properties like distances and angles, you must investigate changes which preserve distances. These are the so called isometries ("iso" means same, "metry" is related to measure)!

The changes of coordinates which preserve distances can be described nicely using trigonometry. We, however, choose to postpone these discussions to a higher course. We will give a brief sample of this in a later chapter. ${ }^{5}$

## Examples of changes of coordinates:

We illustrate the above discussion coordinate changes in the line and plane by various examples.

Example 1 Find a transformation on the line which changes the point $A(3)$ to $A^{\prime}(5)$ and $B(-1)$ to $B^{\prime}(1)$.

Use your answer to find out the new coordinate of the point $C(3)$. Also, if $D(d)$ goes to $D^{\prime}(-5)$ what is $d$ ?

## Answer:

- We assume that the transformation is $x^{\prime}=p x+q$ and try to find $p, q$. From the information about the point $A$, we see that

$$
5=p(3)+q=3 p+q
$$

The information about $B$ says:

$$
1=p(-1)+q=-p+q
$$

[^30]We need to solve these two equations together. Of course, we can use any of the techniques that we know, but here, it is easiest to subtract the second from the first to get:

$$
4=3 p+q-(-p+q)=4 p
$$

So $p=1$. Plugging the value of $p$ into the first equation, we see that:

$$
5=3+q \text { so } q=2
$$

Thus the transformation is:

$$
x^{\prime}=x+2
$$

- To find the image of $C(3)$, we set $x=3$ and deduce that $x^{\prime}=3+2=5$. So, $C(3)$ becomes $C^{\prime}(5)$.
- To find the point $D$, we set $x^{\prime}=-5$ and solve $-5=x+2$ to get $x=-7$. So $d=-7$.

Example 2 Find a plane transformation using only translations and flips which sends $P(1,1)$ to $P^{\prime}(4,1), Q(2,3)$ to $Q^{\prime}(5,-1)$.
Determine the new image of the origin. Also determine the point which goes to the new origin.
What point $R$ goes to $R^{\prime}(3,3)$ ?

## Answer:

- Assume that the transformation is given by

$$
x^{\prime}=u x-p, y^{\prime}=v y-q .
$$

We have the equations for point $P$ :

$$
4=u-p, 1=v-q
$$

Similarly for the point $Q$, we get:

$$
5=2 u-p, \quad-1=3 v-q
$$

Solve the equations

$$
u-p=4 \text { and } 2 u-p=5
$$

to deduce $u=1, p=-3$.
Similarly, solve the equations

$$
v-q=1 \text { and } 3 v-q=-1
$$

to deduce $v=-1, q=-2$.
Thus the transformation is:

$$
x^{\prime}=x+3, y^{\prime}=-y+2 .
$$

- Clearly, the origin $O(0,0)$ goes to $O^{\prime}(3,2)$.
- If $x^{\prime}=0$ and $y^{\prime}=0$ then clearly, $x=-3$ and $y=2$, so the point $S(-3,2)$ becomes the new origin!
- If $x^{\prime}=3$ and $y^{\prime}=3$, then $x=0$ and $y=-1$, so $R(0,-1)$ is the desired point.


## Chapter 5

## Equations of lines in the plane.

You are familiar with the equations of lines in the plane of the form $y=m x+c$. In this chapter, we introduce a different set of equations to describe a line, called the parametric equations. These have distinct advantages over the usual equations in many situations and the reader is advised to try and master them for speed of calculation and better understanding of concepts.

### 5.1 Parametric equations of lines.

Consider a point $P(a, b)$ in the plane. If $t$ is any real number, we can see that the point $(a t, b t)$ lies on the line joining $P$ to the origin. Define $Q(t)$ to be the point ( $a t, b t$ ). We may simply write $Q$ for $Q(t)$ to shorten our notation.

It is easy to see that the line joining $P$ to the origin is filled by points $Q(t)$ as $t$ takes all possible real values.

Thus, if we write $O$ for the origin, then we can describe our formula as

$$
\overrightarrow{O Q}=(a t, b t)=t(a, b)=(t)(\overrightarrow{O P}) \text { for some real } t
$$



More generally, a line joining a point $A\left(a_{1}, b_{1}\right)$ to $B\left(a_{2}, b_{2}\right)$ can then be described as all points $Q(x, y)$ such that $\overrightarrow{A Q}=(t)(\overrightarrow{A B})$, i.e. $\left(x-a_{1}, y-b_{1}\right)=t\left(a_{2}-a_{1}, b_{2}-b_{1}\right)$ and this gives us the:

Parametric two point form: $x=a_{1}+t\left(a_{2}-a_{1}\right), y=b_{1}+t\left(b_{2}-b_{1}\right)$.


Here is a concrete example where we take points $A(-1,1)$ and $B(0,2)$. (See the picture above.)

The parametric equation then comes out as

$$
x=-1+t(0-(-1))=-1+t \text { and } y=1+t(2-1)=1+t
$$

We have marked this point as $Q$ in the picture.
Even though, we have a good formula in hand, we find it convenient to rewrite it in different forms for better understanding. Here are some variations:

- The shift $\overrightarrow{A B}$ can be conveniently denoted as $B-A$, where we think of the points as pairs of coordinates and perform natural term by term operations on them. Now, our formula can be conveniently written as:


## Compact parametric two point form:

$$
Q(x, y)=A+t(B-A) \text { or simply }(x, y)=A+t(B-A)
$$

For example, for our $A(-1,1)$ and $B(0,2)$ we get:
$A+t(B-A)=(-1,1)+t((0,2)-(-1,1))=(-1,1)+t(1,1)=(-1+t, 1+t)$ as before.

- Note that this formula can also be rewritten as: $Q(x, y)=Q=(1-t) A+t(B)$.

Here, the calculation gets a bit simpler:

$$
(1-t)(-1,1)+t(0,2)=(-1+t, 1-t)+(0,2 t)=(-1+t, 1+t)
$$

In this form, the parameter $t$ carries more useful information which we investigate next.

## Examples of parametric lines:

1. Find a parametric form of the equations of a line passing thru $A(2,-3)$ and $B(1,5)$.
Answer: The two point form gives:

$$
x=2+t(1-2), y=-3+t(5-(-3)) \text { or } x=2-t, y=-3+8 t .
$$

The compact form gives the same equations:

$$
(x, y)=(2,-3)+t((1,5)-(2,-3))=(2,-3)+t(-1,8)=(2-t,-3+8 t)
$$

2. Verify that the points $P(1,5)$ and $Q(4,-19)$ lie on the line given above.

Answer: We want to see if some value of $t$ gives the point $P$. This means:

$$
\text { Can we solve? } 1=2-t, 5=-3+8 t
$$

A clear answer is yes, $t=1$ does the job. Similarly, $t=-2$ solves:

$$
4=2-t,-19=-3+8 t
$$

Thus $P$ and $Q$ are points of the above line thru $A, B$.
3. Determine the parametric form of the line joining $P, Q$. Note that we must get the same line! Compare with the original form.
Answer: The compact form would be given by:

$$
(x, y)=P+t(Q-P)=(1,5)+t((4,-19)-(1,5))=(1,5)+t(3,-24) .
$$

The original compact form of the line was:

$$
(x, y)=(2,-3)+t(-1,8)
$$

## An observation.

Note that the pair of coefficients multiplying $t$ were $(-1,8)$ for the original equation and are $(3,-24)$ for the new one. Thus the second pair is -3 times the first pair.
This is always the case! Given two parametric forms of equations of the same line, the pair of coefficients of $t$ in one set is always a non zero multiple of the pair of coefficients in the other set.

Thus, it makes sense to make a:
Definition: Direction numbers of a line. The pair of multipliers of $t$ in a parametric form of the equation of a line are called (a pair of ) direction numbers of that line. Any non zero multiple of the pair of direction numbers is also a pair of direction numbers for the same line.
The pair of direction numbers can also be understood as the shift $\overrightarrow{P Q}$ for some two points on the line!

In addition, if we set the parameter $t=0$, then we must get some point on the line.

In turn, if we have two lines with one common point and proportional direction numbers, then they must be the same line!
Thus we can easily manufacture many other parametric forms of the same line by starting with any one point on the line and using any convenient direction numbers.

For example, the above line can be also conveniently written as:

$$
(x, y)=(2,-3)+t(-2,16)=(2-2 t,-3+16 t)
$$

Here, we chose a point $(2,-3)$ on the line and chose the direction numbers as $(-2,16)$ - a multiple by 2 of our original direction numbers $(-1,8)$.
If we choose the point $(4,-19)$ instead and take $(5,-40)=(-5)(1,-8)$ as direction numbers, then we get:

$$
(x, y)=(4,-19)+t(5,-40)=(4+5 t,-19-40 t)
$$

You can make many other examples! Thus, the parametric form is good for generating many points on a chosen line (by taking many values of $t$ ) but is not so good to decide if it describes the same line that we started with.

### 5.2 Meaning of the parameter $t$ :

Consider the parametric description of points on a line joining $A\left(a_{1}, b_{1}\right)$ and $B\left(a_{2}, b_{2}\right)$ given by

$$
(x, y)=(1-t) A+t B=\left(a_{1}+t\left(a_{2}-a_{1}\right), b_{1}+t\left(b_{2}-b_{1}\right)\right) .
$$

We have the following information about the parameter $t$.

| Value of $t$ | Interpretation or conclusion |
| :--- | :--- |
| $t=0$ | Point $A$ |
| $0<t<1$ | Points from $A$ to $B$ |
| $t=1$ | Point $B$ |
| $t>1$ | Points past $B$ away from $A$ |
| $t<0$ | Points past $A$ away from $B$ |

In other words, the parameter $t$ can be thought as a coordinate system on the line joining $A, B$ chosen such that $A$ is the origin and $B$ the unit point.

We can obviously get different coordinate systems, if we choose some other two points on the line for the origin and the unit point.

The above facts can be proved by working out the following precise formula.
For any point $Q=(1-t) A+t B$ on the line, let us note that

$$
Q-A=(1-t) A+t B-A=t(B-A)
$$

and

$$
B-Q=B-((1-t) A+t B)=(1-t)(B-A)
$$

Thus the distances $|\overrightarrow{A Q}|$ and $|\overrightarrow{Q B}|$ are indeed in the proportion $|t|:|(1-t)|$.
This says that to divide the interval $A B$ in a ratio $t:(1-t)$ we simply take the point $Q=(1-t) A+t B$. For $0<t<1$ this has a natural meaning and we note the following:

## Useful Formulas

Consider the points $A\left(a_{1}, b_{1}\right), B\left(a_{2}, b_{2}\right)$.

1. Midpoint formula. The midpoint of $A, B$ is given by

$$
\frac{(A+B)}{2}=\left(\frac{a_{1}+a_{2}}{2}, \frac{b_{1}+b_{2}}{2}\right)
$$

2. Division formula. For $0<t<1$ the point $Q=(1-t) A+t B$ divides the interval in the ratio $t:(1-t)$. Thus the two trisection points between $A, B$ are given as follows:

Take $t=1 / 3$ and then $Q=(2 / 3) A+(1 / 3) B$ is the trisection point near $A$.
Take $t=2 / 3$ and then $Q=(1 / 3) A+(2 / 3) B$ for the point near $B$.
3. External divisions formulas. If $t>1$ then the point $Q=(1-t) A+t B$ gives an external division point in the ratio $t:(t-1)$ on the $B$ side. If $t<0$ then $1-t>1$ and we get an external division point in the ratio $-t: 1-t$ on the $A$ side.

Thus, for example, the point $Q=-A+2 B$ with $t=2$ has the property that the distance $|\overrightarrow{A Q}|=2|\overrightarrow{Q B}|$.
4. The distance formula If $Q=(1-t) A+t B$, then the distance $|\overrightarrow{A Q}|$ is simply $|t| \cdot|\overrightarrow{A B}|$ and consequently, for any other point $R=(1-s) A+s B$, we can calculate the distance

$$
d(Q, R)=|\overrightarrow{Q R}|=|s-t| \cdot|\overrightarrow{A B}|=|s-t| \cdot d(A, B)
$$

## Examples of special points on parametric lines:

1. Let $L$ be the line (discussed above) joining $A(2,-3), B(1,5)$.

Let us use the parametric equations found earlier:

$$
x=2-t, y=-3+8 t
$$

What values of the parameter $t$ give the points $A, B$ and their midpoint $M$ ?
Answer: At $t=0$ we get the point $A$ and at $t=1$ we get the point $B$. At $t=\frac{1}{2}$ we get

$$
\left(2-\frac{1}{2},-3+8\left(\frac{1}{2}\right)\right)=\left(\frac{3}{2}, 1\right)
$$

This is the midpoint $M$.
2. Find the two trisection points of the segment from $A$ to $B$.

Answer: We simply take $t=\frac{1}{3}$ and $t=\frac{2}{3}$ in the parameterization.
So, the two points are:

$$
\left(2-\frac{1}{3},-3+8 \cdot \frac{1}{3}\right)=\left(\frac{5}{3},-\frac{1}{3}\right)
$$

and

$$
\left(2-\frac{2}{3},-3+8 \cdot \frac{2}{3}\right)=\left(\frac{4}{3}, \frac{7}{3}\right)
$$

Warning! Don't forget that for this formula to work, we must take the parameterization for which the original two points are given by $t=0$ and $t=1$ respectively.
3. Find out where the point $Q(4,-19)$ is situated relative to $A, B$.

Answer: First we decide that value of $t$ which gives the point $Q$.
Thus we solve:

$$
4=2-t \text { and }-19=-3+8 t
$$

Clearly (as we already know) this is given by $t=-2$. For understanding, let us make a picture of the line and mark the values of $t$ next to our points.


The fact that $t=-2<0$ says that we have a point on the $A$ side giving an external division in the ratio $\frac{-t}{1-t}=\frac{2}{3}$. Thus it confirms that the point $Q$ is on the $A$-side and the ratio of the distances $d(Q, A)$ and $d(Q, B)$ is indeed $\frac{2}{3}$.
4. Let $U$ be a point on the same line $L$ joining the points $A(2,-3)$ and $B(1,5)$. Assume that in the usual parameterization which gives $t=0$ at $A$ and $t=1$ at $B$, we have $t=3$ at $U$. Assume that $V$ is another point on the same line at which $t=-5$. Find the distance $d(U, V)$.
Answer: We could, of course, calculate $U$ by plugging in $t=3$ in the parametric equations $(x, y)=(2-t,-3+8 t)$ and get $U(-1,21)$. Similarly, we get $V(7,-43)$. We can apply the distance formula and declare:

$$
\sqrt{(7-(-1))^{2}+(-43-21)^{2}}=\sqrt{8^{2}+64^{2}}=\sqrt{4160}
$$

The final answer comes out to be $8 \sqrt{65}$.

But we recommend using a more intelligent approach! Note that we know:

$$
d(A, B)=\sqrt{(1-2)^{2}+(5-(-3))^{2}}=\sqrt{1+64}=\sqrt{65} .
$$

Since we are using parameter values 3 and -5 for our two points, we know that their distance is simply:

$$
d(U, V)=|3-(-5)| d(A, B)=8 \sqrt{65} .
$$

Clearly, this is far more efficient! ${ }^{1}$
5. Use the parametric form of the above line to verify that the point $S(-1,20)$ is not on it.

## Answer:

We try to solve:

$$
-1=2-t, 20=-3+8 t
$$

The first equation gives $t=3$ and when substituted in the second equation, we get:

$$
20=-3+8(3)=21 \text { a clearly wrong equation!. }
$$

This shows that the point $S$ is outside the line!

### 5.3 Comparison with the usual equation of a line.

The parametric description of a line given above is very useful to generate lots of convenient points on a line, but if we wish to understand the position and the orientation of a line as well as its intersections with other lines and curves, we also need the usual equation of a line as a relation between its $x, y$ coordinates.

We start with the parametric equations of a line thru points $A\left(a_{1}, b_{1}\right), B\left(a_{2}, b_{2}\right)$

$$
\text { Equation 1: } x=a_{1}+t\left(a_{2}-a_{1}\right), \text { Equation 2: } y=b_{1}+t\left(b_{2}-b_{1}\right)
$$

We would like to get rid of this parameter $t$ and we can eliminate it by the following simple trick.

[^31]Construct a new equation $\left(a_{2}-a_{1}\right)$ (Equation 2) $-\left(b_{2}-b_{1}\right)($ Equation 1) and we get:

$$
\left(a_{2}-a_{1}\right) y-\left(b_{2}-b_{1}\right) x=\left(a_{2}-a_{1}\right) b_{1}-\left(b_{2}-b_{1}\right) a_{1}=\left(a_{2} b_{1}-a_{1} b_{2}\right)
$$

This clearly describes the precise condition that a point $(x, y)$ lies on the line. We record this as:

The two point form of the equation of a line thru $A\left(a_{1}, b_{1}\right), B\left(a_{2}, b_{2}\right)$ :

$$
\left(a_{2}-a_{1}\right) y-\left(b_{2}-b_{1}\right) x=\left(a_{2} b_{1}-a_{1} b_{2}\right)
$$

If $a_{2}=a_{1}$ then it is easy to see that the equation reduces to $x=a_{1}$ a vertical line. Similarly, if $b_{2}=b_{1}$, then we get a horizontal line $y=b_{1}$.

Example.
Thus for our points $A(2,-3), B(1,5)$ we get:

$$
(1-(2)) y-(5-(-3)) x=((1)(-3)-(2)(5) \text { or }-y-8 x=-13 \text { or } y+8 x=13
$$

You can also verify this by the elimination of $t$ from the equations

$$
x=2-t, y=-3+8 t
$$

We now define the slope of a line joining any two points $A\left(a_{1}, b_{1}\right), B\left(a_{2}, b_{2}\right)$ as the ratio $m=\frac{b_{2}-b_{1}}{a_{2}-a_{1}}$ where we agree that it is $\infty$ in case $a_{1}=a_{2}$ i.e. when we have a vertical line.

We also define the intercepts of a line as follows:
We declare:

- The x-intercept is a number $a$ if $(a, 0)$ is on the line. From the above equation of the line, the $x$-intercept is clearly $\frac{a_{2} b_{1}-a_{1} b_{2}}{b_{1}-b_{2}}$.
- This may fail if $b_{2}=b_{1}$ but $a_{1} b_{2} \neq a_{2} b_{1}$, i.e. the line is horizontal but not the $x$-axis.

In this case the intercept is declared $\infty$.

- For the $x$-axis itself, it is considered undefined!
- Thus, in general, the $x$-intercept is $\frac{a_{2} b_{1}-a_{1} b_{2}}{b_{1}-b_{2}}$ with a suitable agreement when the denominator is zero.
- The y-intercept is similarly defined as $b$, when the point $(0, b)$ is on the line. From the above equation of the line, the $y$-intercept is clearly $\frac{a_{2} b_{1}-a_{1} b_{2}}{a_{2}-a_{1}}$.
- This may fail if $a_{2}=a_{1}$ but $a_{1} b_{2} \neq a_{2} b_{1}$, i.e. if the line is vertical but not the $y$ axis then the intercept is declared to be $\infty$.
- For the $y$-axis itself, it is considered undefined!
- In general, the formula is: $\frac{a_{2} b_{1}-a_{1} b_{2}}{a_{2}-a_{1}}$ with a suitable agreement when the denominator is zero.
- Warning. It is tempting to memorize the formulas for the intercepts, but notice that there is a subtle change in the form of the denominator. We recommend a fresh calculation rather than a memorized formula.
- For a non vertical line, it is customary to divide the equation by $a_{2}-a_{1}$ and rewrite it as:

$$
y-\frac{b_{2}-b_{1}}{a_{2}-a_{1}} x=\frac{a_{2} b_{1}-a_{1} b_{2}}{a_{2}-a_{1}}
$$

This formula is worth memorizing.

The last form above is often further restated as: Slope intercept form of a line:

$$
y=m x+c \text { where } m=\frac{b_{2}-b_{1}}{a_{2}-a_{1}} \text { is the slope and } c=\frac{a_{2} b_{1}-a_{1} b_{2}}{a_{2}-a_{1}} y \text {-intercept. }
$$

Thus, for our line thru $A(2,-3), B(1,5)$, we have:

$$
-y-8 x=-13 \text { becomes } y+8 x=13 \text { and finally } y=-8 x+13
$$

Thus, the slope is -8 and intercept is 13 . We can calculate each of these quantities from the formulas and then write the equation directly.

It is convenient to understand the orientation of such a line $y=m x+c$.
If we fix $c$, then we get the family of lines with a common point $(0, c)$. When $m=0$ the line is horizontal and as $m$ increases, the line starts to twist upwards (in a counter clockwise direction). Higher and higher values of $m$ bring it closer to vertical, the true vertical being reserved for $m=\infty$ by convention. All these lines appear to rise as the $x$-coordinates increase.

If we, in turn start to run $m$ thru negative values to $-\infty$, then the line twists clockwise with $m=-\infty$ corresponding to a vertical line. These lines appear to fall as $x$-coordinate increases.


While the above two point formula is good in all situations, sometimes it is useful to find fast clever ways to find the equation of a line.

Consider an equation of the form $a x+b y=c$ where $a, b, c$ are constants. This is clearly the equation of some line provided at least one of $a, b$ is non zero.

Hence, if we can somehow construct such an equation satisfying the given conditions, then it must be the right answer. In fact, if we construct an equation which can be evidently manipulated to such a form, then it is a correct answer as well. ${ }^{2}$

Here are some of the samples of this simple idea.
Another two point form.

$$
\left(y-b_{1}\right)\left(a_{2}-a_{1}\right)=\left(x-a_{1}\right)\left(b_{2}-b_{1}\right)
$$

Proof: It has the right form and is clearly satisfied by the two points, just substitute and see!

Point ( $\mathrm{P}(\mathrm{a}, \mathrm{b})$ )-slope $(\mathrm{m})$ form.

$$
(y-b)=m(x-a) .
$$

Proof: When rearranged, it becomes $y=m x+(b-m a)$.
So, $m$ is the slope. Also, the equation is clearly satisfied, if we plug in $x=a, y=b$.
So, we are done by the "duck principle".
By the way, this work also gives us the formula for the $y$-intercept as $b-m a$.
The $x$-intercept, if needed is seen to be $a-\frac{1}{m} \cdot b$ and the evident symmetry in these two formulas is noteworthy!

Example. Find the equation of a line with slope -6 and passing thru the point $P(3,4)$.

Answer: Direct application of the formula gives:

$$
(y-4)=-6(x-3)
$$

[^32]This simplifies to:

$$
y=-6 x+4+18=-6 x+22
$$

It has slope -6 and intercept 22 .
Intercept form. Suppose the $x, y$ intercepts of a line are $p, q$ respectively and these are non zero. Then the equation of the line can be written as:

$$
\frac{x}{p}+\frac{y}{q}=1
$$

This form can be interpreted suitably if one of $p, q$ is zero, but fails completely if both are zero.

Proof. Just check that the two intercept points $(p, 0),(0, q)$ satisfy the equation and apply the duck principle!

Example. Find the equation of a line with $x$ intercept 3 and $y$ intercept 4.
Answer: Apply the formula:

$$
\frac{x}{3}+\frac{y}{4}=1
$$

This simplifies to:

$$
4 x+3 y=12 \text { or } y=-\frac{4}{3} x+4
$$

Actually, it is debatable if this is really a simplification! One should not get too attached to a single standard form.

Parallel lines. It is easy to see that parallel lines have the same slope. Thus a line parallel to a given $y=m x+c$ is given by $y=m x+k$ where $k$ can be chosen to satisfy any further conditions. More generally, the line parallel to any given $u x+v y=w$ can also be written as $u x+v y=k$ for a suitable $k$. The point is that we don't have to put the equation in any standard form to get it right!

Example. Find a line parallel to $y+8 x=13$ which passes thru the point $(3,10)$.
Answer: We know that the answer can be of the form $y+8 x=k$. Impose the condition of passing thru $(3,10)$, i.e.

$$
10+8(3)=k
$$

so we deduce that $k=34$ and hence the answer is

$$
y+8 x=34
$$

Perpendicular lines. It is possible to prove that a line perpendicular to a line with slope $m$ has slope $-1 / m$, where by convention we take $1 / \infty=0$ and $1 / 0=\infty$. Thus, to make a perpendicular line to a given line of slope $m$, we can simply choose any line with slope $-\mathbf{1} / \mathbf{m}$.

But should we find the slope first? We give an alternative way using the duck principle.

A line perpendicular to $u x+v y=w$ is given by $v x-u y=k$ where, as before, $k$ is to be chosen by any further conditions. For proof, it is easy to verify the slopes for the two equations ( $-\frac{u}{v}$ and $\frac{v}{u}$ respectively) and verify that their product is indeed $-1 .^{3}$

Example. Find a line perpendicular to $8 x+y=13$ which passes thru the point $(3,10)$.

Answer: We know that the answer can be of the form $x-8 y=k$. Impose the condition of passing thru $(3,10)$, i.e.

$$
3-8(10)=k
$$

So we deduce that $k=-77$ and hence the answer is

$$
x-8 y=-77 \text { or } 8 y-x=77 .
$$

### 5.4 Examples of equations of lines.

1. Intersecting Lines. Find the point of intersection of the lines $L$ and $M$ as described below.

The line $L$ is given by parametric equations:

$$
L: x=2+3 t, y=-1+2 t
$$

and the line $M$ is given by a usual equation:

$$
M: 4 x-5 y=7
$$

Answer: Note that at a common point, we can substitute the parametric equations of $L$ into the usual equation of $M$ to get:

$$
4(2+3 t)-5(-1+2 t)=7 \text { or }(12-10) t+8+5=7 .
$$

Simplification leads to:

$$
2 t=7-13=-6 \text { or } t=-3 .
$$

Plugging back into the equations for $L$, we get:
The common point is: $x=2+3(-3)=-7, y=-1+2(-3)=-7$.
Observation. You will find that for intersecting two lines, it is best to have one in parametric form and one in usual form.

[^33]2. Conversion between the parametric and the usual form of a line. Find the point of intersection of the lines given by parametric equations:
$$
L: x=2+3 t, y=-1+2 t
$$
and
$$
N: x=3+5 t, y=1+4 t
$$

Answer. We could try to equate the corresponding $x, y$ parameterizations, but it would be a mistake to equate them using the same parameter $t$. After all, at a common point, we could have two different values of the parameter, depending on the line being used!

So one way is to solve the two equations:

$$
2+3 t=3+5 s \text { and }-1+2 t=1+4 s
$$

These are two linear equations in two variables, and are easily solved. Using the values of the parameter, we can read of the points from the parametric equations of either line.
But we recommend the following trick! Convert the second parametric equations to the usual form by eliminating $t$.
Thus, from the equations $x=3+5 t, y=1+4 t$ we see that

$$
4 x-5 y=4(3+5 t)-5(1+4 t)=12-5=7
$$

So we discover that the second line $N$ is the same as the old line $M$ of the above problem.
Now intersect $L, N=M$ as before to find the common point $(-7,-7)$ with the parameter $t=-3$ on line $L$. We did not find the parameter value on the line $N$, but can find it if challenged! On the line $N$ at the common point, we have $-7=3+5 t$ so $t=\frac{-10}{5}=-2$.
The moral is that when intersecting two lines, try to put one in parametric form while the other is in usual form. In fact, this is exactly what we do when we solve one equation for $y$ and plug into the other; we just did not mention the word parameter when we learned this.
For example, if we solve $y+2 x=3$ for $y$, we are making a parametric form by setting $x=x, y=3-2 x$, i.e. we are using the letter $x$ itself as a parameter name!
3. Points equidistant from two given points. Suppose that you are given two points $A\left(a_{1}, b_{1}\right)$ and $B\left(a_{2}, b_{2}\right)$. Determine the equation satisfied by the coordinates of a point $P(x, y)$ equidistant from both.

Answer: We use the distance formula to write the equality of the square of distances from $P$ to each of $A, B$.

$$
\left(x-a_{1}\right)^{2}+\left(y-b_{1}\right)^{2}=\left(x-a_{2}\right)^{2}+\left(y-b_{2}\right)^{2} .
$$

Now expand to get:

$$
x^{2}+y^{2}-\left(2 a_{1} x+2 b_{1} y\right)+a_{1}^{2}+b_{1}^{2}=x^{2}+y^{2}-\left(2 a_{2} x+2 b_{2} y\right)+a_{2}^{2}+b_{2}^{2} .
$$

Simplification leads to:

$$
x\left(a_{1}-a_{2}\right)+y\left(b_{1}-b_{2}\right)=(1 / 2)\left(\left(a_{1}^{2}-a_{2}^{2}\right)+\left(b_{1}^{2}-b_{1}^{2}\right)\right),
$$

where we have cancelled the terms $x^{2}, y^{2}$ and then thrown away a factor of -2 before collecting terms.
This is clearly a line and it is easy to check that it passes thru the midpoint of $A, B$, namely, $\left(\frac{a_{1}+a_{2}}{2}, \frac{b_{1}+b_{2}}{2}\right)$.
The reader can also see that the slope of the line is $-\frac{a_{1}-a_{2}}{b_{1}-b_{2}}$.
Since the slope of the line joining $A, B$ is $\frac{b_{2}-b_{1}}{a_{2}-a_{1}}$, it is obviously perpendicular to the line joining $A, B$.
Thus, we have proved that the set of points equidistant from two given points $A, B$ is the perpendicular bisector of the segment joining them.

Often, it is easy to use this description rather than develop the equation directly. Thus the locus of points equidistant from $A(1,3), B(3,9)$ is easily found thus: The slope of the given line is $\frac{9-3}{3-1}=6 / 2=3$. The midpoint is

$$
\left(\frac{(1+3)}{2}, \frac{(3+9)}{2}\right)=(2,6) .
$$

Therefore, perpendicular line has slope $-1 / 3$ and hence the answer using the point-slope form is:

$$
(y-6)=-(1 / 3)(x-2) \text { or } y=-x / 3+20 / 3
$$

4. Right angle triangles. Given three points $A, B, C$, they form a right angle triangle if the Pythagorean theorem holds for the distances or equivalently two of the sides are perpendicular.

Say, we are given points $A(2,5), B(6,4)$ and wish to choose a third point $C$ somewhere on the line $x=4$ such that $A B C$ is a right angle triangle.
How shall we find it?
Answer: We can assume that the point $C$ has coordinates $(4, t)$ for some $t$. The right angle can occur at $A$ or $B$ or $C$. We handle each case separately.

Case A Suppose that we have a right angle at A. Then then the slope of the line $A C$ times the slope of the line $A B$ must be $-1 .{ }^{4}$ This gives:

$$
\frac{t-5}{4-2} \cdot \frac{4-5}{6-2}=-1 \text { i.e. } t-5=(-1)(4)(2) /(-1)=8
$$

This gives $t=13$ and $C$ is at $(4,13)$. This is marked as $C$ in the picture below.
Case B Suppose that we have a right angle at B. Then our equation becomes:

$$
\frac{t-4}{4-6} \cdot \frac{4-5}{6-2}=-1 \text { i.e. } t-4=(-1)(4)(-2) /(-1)=-8
$$

So we get $C$ at $(4,-4)$. This is noted as $P$ in the picture below.

## Case C Suppose that we have a right angle at C.

Then we have the equation:

$$
\frac{t-5}{4-2} \cdot \frac{t-4}{4-6}=-1 \text { or }(t-5)(t-4)=(-1)(2)(-2) \text { i.e. } t^{2}-9 t+20=4
$$

This leads to two solutions $\frac{9 \pm \sqrt{17}}{2}$.
To avoid cluttering our picture, we have shown only one of these and have marked it as a $Q$ and we have noted a decimal approximation to its coordinate.
You should figure out where the other solution lies.


[^34]Food for thought. Should we always get four possible solutions, no matter what line we choose to lay our point $C$ on? It is interesting to experiment. Can you think of examples where no solution is possible?

## Chapter 6

## Special study of Linear and Quadratic Polynomials.

We now pay special attention to the polynomials which occur most often in applications, namely the
linear polynomials $(a x+b$ with $a \neq 0)$ and the
quadratic polynomials $\left(a x^{2}+b x+c\right.$ with $\left.a \neq 0\right)$.

### 6.1 Linear Polynomials.

First, as an example, consider $L(x)=3 x-5$. This can be rearranged as

$$
L(x)=3(x-5 / 3)
$$

At $\mathbf{x}=5 / 3$ we have $L(x)=0$, so the expression is zero! If $\mathbf{x}>5 / 3$ then $L(x)=$ $3(x-5 / 3)>0$, so the expression is positive, since it is a product of positive factors. Similarly, if $\mathbf{x}<\mathbf{5 / 3}$, then $L(x)=3(x-5 / 3)$ is negative since it is a product of a positive and a negative number.

Thus, we see that the linear expression $L(x)$ is zero at a single point, positive on one side of it and negative on the other.

If the expression were $-3 x+5$ then the argument will be similar, except now it will be positive when $x<5 / 3$ and negative when $x>5 / 3$.

We summarize these arguments below.
The linear polynomials $L(x)=a x+b$ can be written as $a(x+b / a)$ and their behavior is very easy to describe.

1. Case $a>0$

If $x$ has value less than $-b / a$ then $x+b / a$ is negative and hence $L(x)=a(x+b / a)$ is also negative. At $x=-b / a$, the expression becomes 0 and it becomes positive thereafter. The expression steadily increases from large negative values to large positive values, hitting the zero exactly once!
2. Case $a<0$

This is just like above, except for $x<-b / a$ the value of $a(x+b / a)$ now becomes positive, it becomes 0 at $x=-b / a$ just like before and then becomes negative for $x>-b / a$. Thus, the expression steadily decreases from large positive values to large negative values, hitting zero exactly once!

## Recommended method.

In spite of the theory above, the best way to handle a concrete expression goes something like this.

Say, we have to analyze the behavior of $L(x)=-4 x+8$. Find out where it is zero, i.e. solve $-4 x+8=0$. This gives $x=8 / 4=2$. Thus it will have a certain sign for $x<2$ and a certain sign for $x>2$.

How do we find these signs? Just test any convenient points.
To analyze the cases $x<2$, we try $x=0$.
The evaluation $L(0)=-4(0)+8=8$ gives a positive number. So we conclude that

$$
L(x) \text { is positive for } x<2 \text {. }
$$

Similarly, to test $x>2$, try $x=3$.
Note that $L(3)=-4(3)+8=-4<0$. So we conclude that

$$
L(x) \text { is negative for } x>2 .
$$

You may be wondering about why we bothered developing the theory if all we have to do is to check a few points.

The reason is that without the theory, we would not know which and how many points to try and why should we trust the conclusion. If we understand the theory, we can handle more complicated expressions as well. We illustrate this with the quadratic polynomials next.

### 6.2 Factored Quadratic Polynomial.

Suppose that our quadratic expression $Q(x)=a x^{2}+b x+c$ is written in ${ }^{1}$

$$
\text { factored form } a(x-p)(x-q)
$$

where $p<q$.
We describe how to make a similar analysis depending on the sign of $a$. First, we need a good notation to report our answers efficiently.

[^35]
## Interval notation.

Given real numbers $a<b$ we shall use the following standard convention. ${ }^{2}$

| Values of $x$ | Notation |
| :---: | :---: |
| $a<x$ and $x<b$ | $(a, b)$ |
| $a \leq x$ and $x<b$ | $[a, b)$ |
| $a<x$ and $x \leq b$ | $(a, b]$ |
| $a \leq x$ and $x \leq b$ | $[a, b]$ |

A simple way to remember these is to note that we use open parenthesis when the end point is not included, but use a closed bracket if it is!

The notation is often extended to allow $a$ to be $-\infty$ or $b$ to be $\infty$.
Thus $(-\infty, 4)$ denotes all real numbers $x$ such that $-\infty<x<4$ and it clearly has the same meaning as $x<4$, since the first inequality is automatically true!

As an extreme example, the interval $(-\infty, \infty)$ describes the set $\Re$ of all real numbers!

The reader should verify the following:

1. Case $a>0$ Remember that

$$
Q(x)=a(x-p)(x-q) \text { with } p<q .
$$

Then we have:

$$
\begin{array}{cl}
\text { Values of } x & \text { behavior of } Q(x) \\
(-\infty, p) & Q(x)>0 \\
x=p & Q(x)=0 \\
(p, q) & Q(x)<0 \\
x=q & Q(x)=0 \\
(q, \infty) & Q(x)>0
\end{array}
$$

See the graphic display below showing the results when $a>0$.

| Positive |  |
| :---: | :---: |
| Zero at p | Negative $\quad$ Positive |
| Zero at q |  |

How do you verify something like this? Here is a sample verification.
Consider the third case of the interval $(p, q)$. Since $p<x<q$ in this interval, we note that:
$x-p>0$ and $q-x>0$ which means the same as $x-q<0$.

[^36]Since $Q(x)=(a)(x-p)(x-q)$ and since $a>0$, we see that $Q(x)$ is a product of two positive terms $a$ and $(x-p)$ and one negative term $(x-q)$.

Hence it must be negative!
In general, if our expression is a product of several pieces and we can figure out the sign for each piece, then we know the sign for the product!
2. Case $a<0$

Everything is as in the above case, except the signs of $Q(x)$ are reversed!
To simplify our discussion, let us make a few formal definitions.
Definition: Absolute Maximum value Given an expression $F(x)$ we say that it has an absolute maximum at $x=t$ if $F(t) \geq F(x)$ for every $x \in \Re$. We will say that $F(t)$ is the absolute maximum value of $F(x)$ and that it occurs at $x=t$.

We may alternatively say that $x=t$ is an absolute maximum for $F(x)$ and the absolute maximum value of $F(x)$ is $F(t)$.

We similarly make a Definition: Absolute Minimum value Given an expression $F(x)$ we say that it has an absolute minimum at $x=t$ if $F(t) \leq F(x)$ for every $x \in \Re$. We will say that $F(t)$ is the absolute minimum value of $F(x)$ and that it occurs at $x=t$.

We may alternatively say that $x=t$ is an absolute minimum for $F(x)$ and the absolute minimum value of $F(x)$ is $F(t)$.

Definition: Absolute Extremum value A value $x=t$ is said to be an absolute extremum for $F(x)$, if $F(x)$ has either an absolute maximum or an absolute minimum at $x=t$.

Set

$$
m=\frac{(p+q)}{2} \text { and } u=x-m
$$

Note that $m$ is the average value of $p$ and $q$ and also $x=u+m$. Note the following algebraic manipulation:

$$
\begin{aligned}
Q(x) & =a\left(u+\frac{p+q}{2}-p\right)\left(u+\frac{p+q}{2}-q\right) \\
& =a\left(u-\frac{p-q}{2}\right)\left(u+\frac{p-q}{2}\right) \\
& =a\left(u^{2}-T^{2}\right) \text { where } T=\frac{p-q}{2}
\end{aligned}
$$

Since $T^{2}$ is a positive number and $u^{2}$ is always greater than or equal to zero, the smallest value that $u^{2}-T^{2}$ can possibly have is when $u=0$. However this happens exactly when $x=m$. This shows that $u^{2}-T^{2}=(x-m)^{2}-T^{2}$ has an absolute minimum value at $x=m$. If we take $x$ large enough then $u^{2}$ can be made as large as we want. This means that there is no upper bound to the values of $u^{2}-T^{2}$

It follows that if $Q(x)$ that can be put in factored form $Q=(a)(x-p)(x-q)$ with positive $a$, then $Q(x)$ has an absolute minimum at $x=m$, where $m$ is the average of $p, q$.

In case $Q(x)=a(x-p)(x-q)$ with a negative $a$, the situation reverses and $Q(x)$ has an absolute maximum (and no absolute minimum) at $x=m$, the average of $p, q$.

Thus, in either case, we can say that $Q(x)$ has an extremum at $x=m$ the average of $p, q$.

We shall show below that indeed, this value $x=m$ is actually $x=-b /(2 a)$.
Thus the important point is that we need not find the values $p, q$ to find this extremum value!

Remark. Note that for a general expression, the absolute maximum or minimum value may not exist and may occur at several values of $x$.

Here are some examples to clarify these ideas.

1. As already observed, a linear expression like $3 x-5$ does not have a absolute maximum or minimum; its values can be arbitrarily small or large!
2. Consider the expression

$$
F(x)=\left(x^{2}-1\right)\left(2 x^{2}-7\right) .
$$

We invite the reader to use a graphing calculator or a computer to sketch a graph and observe that it has value $-25 / 8=-3.125$ at $x=-1.5$ as well as $x=1.5$.

It is easy to see from the graph that it has an absolute minimum value at these values of $x$. It is also clear that there are no absolute maximum values.

The complete argument to prove this without relying on the picture requires a further development of calculus and has to be postponed to a higher level course.


### 6.3 The General Quadratic Polynomial.

Recall that in the example 6 of (1.5), we proved that:

$$
Q\left(u-\frac{b}{2 a}\right)=a u^{2}-\left(\frac{b^{2}-4 a c}{4 a}\right)=a\left(u^{2}-\frac{b^{2}-4 a c}{4 a^{2}}\right) .
$$

For convenience, let us set

$$
u=x+\frac{b}{2 a} \text { and } M=\frac{b^{2}-4 a c}{4 a^{2}}
$$

Our equation then becomes:

$$
Q(x)=a\left(u^{2}-M\right)
$$

It is now very easy to describe its behavior.
We have exactly three cases:

1. Case of $M<0$ or No real roots. The quantity $u^{2}-M$ is always positive and hence there are no real values of $x$ which make $Q(x)=0$.

In this case either $a>0$ and $Q(x)$ is always positive or $a<0$ and $Q(x)$ is always negative.
In case $a>0$, the value $Q(x)=a\left(u^{2}-M\right)$ is going to be the least exactly when $u=0$, i.e. $x=u-\frac{b}{2 a}=-\frac{b}{2 a}$.
Thus the absolute minimum value is $-a M=\frac{4 a c-b^{2}}{4 a}$ and it is reached at $x=-\frac{b}{2 a}$.
In case $a<0$, a similar analysis says that the absolute maximum value is $-a M=\frac{4 a c-b^{2}}{4 a}$ and it is reached at $x=-\frac{b}{2 a}$.
Thus we have established that $x=-\frac{b}{2 a}$ gives an extremum value regardless of the sign of $a$. Moreover, the extremum value is always $\frac{4 \mathrm{ac}-\mathrm{b}^{2}}{4 \mathrm{a}}$
2. Case of $M=0$ or a double root. In this case $Q(x)=a u^{2}=a\left(x+\frac{b}{2 a}\right)^{2}$ is $a$ times a complete square and becomes zero at $x=-\frac{b}{2 a}$.
By an analysis similar to the above, we see that $x=-\frac{b}{2 a}$ again gives an extremum value and the value as before is $-\mathbf{a M}=0$. Indeed, it is a minimum for $a>0$ and maximum for $a<0$.
3. Case of $M>0$ or two real roots In this case $Q(x)=0=a\left(u^{2}-M\right)$ leads to a pair of solutions ${ }^{3}$

$$
u= \pm \sqrt{M}= \pm \frac{\sqrt{b^{2}-4 a c}}{\sqrt{4 a^{2}}}= \pm \frac{\sqrt{b^{2}-4 a c}}{2|a|}
$$

Watch the absolute value sign in $|a|$. The square root of $4 a^{2}$ is $2 a$ if $a>0$ and $-2 a$ if $a<0$. Failure to note this can lead to serious errors! However, in this case, since we have $\pm$ as a multiplier, we can replace $|2 a|$ by simply $2 a$.
So, the solutions for $u$ are simply:

$$
u= \pm \frac{\sqrt{b^{2}-4 a c}}{2 a}
$$

This leads to the two roots of the original equation, namely:

$$
x=u-\frac{b}{2 a}= \pm \frac{\sqrt{b^{2}-4 a c}}{2 a}-\frac{b}{2 a} .
$$

To put it in the more familiar form, we write it with a common denominator $2 a$ as: The quadratic formula for solutions of $\mathbf{a x}^{2}+\mathbf{b x}+\mathbf{c}=\mathbf{0}, \mathbf{a} \neq \mathbf{0}$

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

If we call the two roots $p, q$ for convenience, then they are

$$
\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

in some order and their average $\frac{p+q}{2}$ is clearly

$$
\frac{1}{2} \cdot\left(\frac{-b+\sqrt{b^{2}-4 a c}}{2 a}+\frac{-b-\sqrt{b^{2}-4 a c}}{2 a}\right)=\frac{1}{2} \cdot\left(\frac{-2 b}{2 a}\right)=-\frac{b}{2 a}
$$

So the extremum value $-b /(2 a)$ is indeed the average of the two roots $p, q .{ }^{4}$

[^37]4. As a final conclusion of the above analysis is that for a quadratic expression $Q(x)=a x^{2}+b x+c$ with $a \neq 0$, we get its extremum value at $x=-\frac{b}{2 a}$ and that this extremum value is always $\frac{4 a c-b^{2}}{4 a}$.

This can be described geometrically.
If we were to plot the graph of $y=a x^{2}+b x+c$ then we always get a parabola and the point $x=-\frac{b}{2 a}, y=\frac{4 a c-b^{2}}{4 a}$ is seen to be its vertex.
This gives the Vertex Theorem which says that the extremum value of the $y$ coordinates on a parabola is always at its vertex.

Moreover, this extremum is an absolute maximum if $\mathbf{a}<\mathbf{0}$ and absolute minimum if $\mathrm{a}>0$.

### 6.4 Examples of quadratic polynomials.

We discuss a few exercises based on the above analysis.

1. Find the extremum values for $Q(x)=2 x^{2}+4 x-6$ and determine interval(s) on which $Q(x)$ is negative.

Answer: We have $a=2, b=4, c=-6$. Thus $-\frac{b}{2 a}=-\frac{4}{4}=-1$. So, if we replace $x=u-\frac{b}{2 a}=u-1$, then we have:

$$
\begin{aligned}
Q(x) & =2(u-1)^{2}+4(u-1)-6 \\
& =2 u^{2}-4 u+2+4 u-4-6 \\
& =2 u^{2}-8=2\left(u^{2}-4\right)
\end{aligned}
$$

We have shown the simplification by hand, but if you remember the theory, you could just write down the last line from $Q(x)=a\left(u^{2}-M\right)$ thus:

$$
a=2 \text { and } M=\frac{b^{2}-4 a c}{4 a^{2}}=\frac{16-4(2)(-6)}{4\left(2^{2}\right)}=\frac{64}{16}=4 .
$$

At any rate, the value of $Q(x)$ is clearly bigger than or equal to $2(-4)=-8$ and this
absolute minimum -8 is attained when $u=0$ or $x=u-1=-1$.
Finally, the quadratic formula (or straight factorization) will yield two roots $p=-3, q=1$.

If you plot these two values on the number line you get three intervals. To check the sign of $Q(x)$, the easiest way is to test a few strategic points as we did in the linear case.


| Intervals | $(-\infty,-3)$ | -3 | $(-3,1)$ | 1 | $(1, \infty)$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Test points | -4 | -3 | 0 | 1 | 4 |
| Values | 10 | 0 | -6 | 0 | 42 |
| Conclusion | Positive | Zero | Negative | Zero | Positive |

Thus, the expression is negative exactly on the interval $(-3,1)$.
2. Find the extremum values for $Q(x)=2 x^{2}+4 x+6$ and determine interval(s) on which $Q(x)$ is negative.
Answer: Now we have $a=2, b=4, c=6$, so

$$
M=\frac{b^{2}-4 a c}{4 a^{2}}=\frac{16-4(2)(6)}{4(4)}=-2 .
$$

This is negative, so the theory tells us that $Q(x)$ will have its extremum at $x=-\frac{b}{2 a}=-1$ and this extremum value of $Q(x)$ is $2(-1)^{2}+4(-1)+6=4$.
The theory says that the expression will always keep the same positive sign and so 4 is actually the minimum value and there is no maximum.
Hence there are no points where $Q(x)<0$.

94CHAPTER 6. SPECIAL STUDY OF LINEAR AND QUADRATIC POLYNOMIALS.

## Chapter 7

## Functions

### 7.1 Plane algebraic curves

The simplest notion of a plane algebraic curve is the set of all points satisfying a given polynomial equation $f(x, y)=0$.

We have seen that a parametric form of the description of a curve is useful for generating lots of points of the curve and to understand its nature.

We have already learned that all points of a line can be alternately described by linear parameterization $x=a+u t, y=b+v t$ for some suitable constants $a, b, u, v$ where at least one of $u, v$ is non zero.

In case of a circle (or, more generally a conic) we will find such a parameterization, but with rational functions of the parameter $t$, rather than polynomial functions.

Definition: A rational Curve. We say that a plane algebraic curve $f(x, y)=0$ is rational if it can be parameterized by rational functions.

This means that we can find two rational functions $x=u(t), y=v(t)$ such that at least one of $u(t), v(t)$ is not a constant and $f(u(t), v(t))$ is identically zero.

Actually, to be precise, we can only demand that $f(u(t), v(t))$ is zero for all values of $t$ for which $u(t), v(t)$ are both defined.

Examples. Here are some examples of rational algebraic curves. You are advised to verify the definition as needed.

1. A line. The line $y=3 x+5$ can be parameterized as $x=t-2, y=3 t-1$. Check:

$$
(3 t-1)=3(t-2)+5
$$

2. A Parabola. The parabola $y=4 x^{2}+1$ can be parameterized by $x=t, y=$ $4 t^{2}+1$.
3. A Circle. The circle $x^{2}+y^{2}=1$ can be parameterized by

$$
x=\frac{1-t^{2}}{1+t^{2}}, y=\frac{2 t}{1+t^{2}}
$$

You need to verify the identity:

$$
\left(\frac{1-t^{2}}{1+t^{2}}\right)^{2}+\left(\frac{2 t}{1+t^{2}}\right)^{2}=1
$$

for all values of $t$ for which $1+t^{2} \neq 0$. Since we are working with real numbers as values, this includes all real numbers.
4. A Singular Curve. The curve $y^{2}=x^{3}$ is easily parameterized by $x=t^{2}, y=$ $t^{3}$. Note that a very similar curve $y^{2}=x^{3}-1$ is declared not to be parameterizable below.
5. A Complicated Curve. Verify that the curve:

$$
x y^{3}-x^{2} y^{2}+2 x y=1
$$

can be parameterized by:

$$
x=\frac{t^{3}}{t^{2}-1}, y=\frac{t^{2}-1}{t} .
$$

This is simply an exercise in substitution. The process of deciding when a curve can be parameterized and then the finding of the parameterization, both belong to higher level courses.

It is tempting to think that all plane algebraic curves could be parameterized by rational functions, but alas, this is not the case. The simplest example of a curve which cannot be parameterized by rational functions is the curve given by $y^{2}=x^{3}-1 .{ }^{1}$

Since we cannot hope to have all algebraic curves parameterized, we do the next best thing; restrict our attention to the ones which are! In fact, to get a more useful theory, we need to generalize the idea of an acceptable function to be used for the parameterization. We may see some instances of this later.

[^38]
### 7.2 What is a function?

As we have seen above, a line $y=3 x+4$ sets up a relation between $x, y$ so that for every value of $x$ there is an associated well defined value of $y$, namely $3 x+4$. We may give a name to this expression and write $y=f(x)$ where $f(x)=3 x+4$.

We wish to generalize this idea.
Definition of a function. Given two variables $x, y$ we say that $y$ is a function of $x$ on a chosen domain $D$ if for every value of $x \epsilon D$, there is a well defined value of $y$. Sometimes we can give a name to the procedure or formula which sets up the value of $y$ and write $y=f(x)$.

We often describe $x$ as the independent variable and $y$ as the dependent variable related to it by the function.

The chosen set of values of $x$, namely the set $D$ is called the domain of the function. ${ }^{2}$

The range of the function is the set of all "values" of the function. Thus for the the function $y=x^{2}$ with domain $\Re$, the range is the set of all non negative real numbers or the interval $[0, \infty)$. Some books use the word "image" for the range.

Be aware that some books declare this as the intended set where the $y$-values live; thus, for our $y=x^{2}$ function, the whole set of reals could be called the range. We prefer to call such a set "target" and reserve the term "range" for the actual values obtained.

In other words, a target of a function is like a wish list, while the range is the actual success, the values taken on by the function as all points of the domain are used. It is often difficult to evaluate the range.

We will not, however, worry about these technicalities and explain what we mean if a confusion is possible.

## Examples of Mathematical Functions.

1. Polynomial functions. Let $p(x)$ be a polynomial in $x$ and let $D$ be the set of all real numbers $\Re$. Then $y=p(x)$ is a function of $x$ on $D=\Re$.
2. Rational functions. Let $h(x)=\frac{x-1}{(x-2)(x-3)}$. Then $y=h(x)$ has a well defined value for every real value of $x$ other than $x=2,3$. There is a convenient notation for this set, namely $\Re \backslash\{2,3\}$, which is read as "reals minus the set of 2,3 ". Let $D=\Re \backslash\{2,3\}$. Then $y=h(x)$ is a function of $x$ on $D$.
3. Piecewise or step functions. Suppose that we define $f(x)$ to be the integer $n$ where $n \leq x<n+1$. It needs some thought to figure out the meaning of this

[^39]function. The reader should verify that $f(x)=0$ if $0 \leq x<1 ; f(x)=1$ if $1 \leq x<2$ and so on. For added understanding, verify that $f(-5.5)=-6$, $f(5.5)=5, f(-23 / 7)=-4$ and so on.
This particular function is useful and is denoted by the word floor, so instead of $f(x)$ we may write floor $(x)$.
There is a similar function called ceiling or ceil for short. It is defined as $\operatorname{ceil}(x)=n$ if $n-1<x \leq n$.
In general, a step function is a function whose domain is split into various intervals over which we can have different definitions of the function.
A natural practical example of a step function can be something like the following:

## Shipping Charges Example.

A company charges for shipping based on the total purchase.
For purchases of up to $\$ 50$, there is a flat charge of $\$ 10$. For every additional purchase price of $\$ 10$ the charge increases by $\$ 1$ until the net charge is less than $\$ 25$. If the charge calculation becomes $\$ 25$ or higher, then shipping is free!

Describe the shipping charge function $S(x)$ in terms of the purchace price $x$.

| Purchase price | Shipping charge | Comments |
| :--- | :--- | :---: |
| $x \leq 0$ | 0 | No sale! |
| $0<x \leq 50$ | 10 | Sale up to $\$ 50$ |
| $50<x<200$ | $10+\operatorname{ceil}\left(\frac{x-50}{10}\right)$ | Explanation below. |
| $200 \leq x$ | 0 | Free shipping for big customers! |

The only explanation needed is that for $x>50$ the calculation of shipping charges is 10 dollars plus a dollar for each additional purchase of up to 10 dollars.
This can be analyzed thus: For $x>50$ our calculation must be $\$ 10$ plus the extra charge for $(x-50)$ dollars. This charge seems to be a 10 -th of the extra amount, but rounded up to the next dollar. This is exactly the ceil function evaluated for $(x-50) / 10$.
This calculation becomes $25=10+15$ when $x=200$ and $\operatorname{ceil}((200-50) / 10)=$ 15.

Finally, according to the given rule we set $S(x)=0$ if $x \geq 200$.
About graphing. It is often recommended and useful to make a sketch of the graph of a function. If this can be done easily, then we recommend it. The trouble with the graphical representation is that it is prone to errors of graphing
as well as calculations. The above example can easily be graphed, but would need at least a dozen different pieces and then reading it would not be so easy or useful. On the other hand, curves like lines and circles and other conics can be drawn fairly easily, but if one relies on the graphs, their intersections and relative positions are easy to misinterpret.

In short, we recommend relying more on calculations and less on graphing!

## Example of Real life functions.

Suppose that we define $T(t)$ to be the temperature of some chemical mixture at time $t$ counted in minutes. It is understood that there is some sensor inserted in the chemical and we have recorded readings at, say 10 minute intervals for a total of 24 hours.

Clearly, there was a definite function with well defined values for the domain $0 \leq t \leq(24)(60)=1440$. The domain might well be valid for a bigger set.

We have, however, no way of knowing the actual function values, without access to more readings and there is no chance of getting more readings afterwards.

What do we do?
Usually, we try to find a model, meaning an intelligent guess of a mathematical formula (or several formulas in steps), based on the known data and perhaps known chemical theories. We could then use Statistics to make an assertion about our model being good or bad, acceptable or unacceptable and so on.

In this elementary course, we have no intention (or tools) to go into any such analysis. In the next few sections, we explain the beginning techniques of such an analysis; at least the ones which can be analyzed by purely algebraic methods.

### 7.3 Modeling a function.

We assume that we are given a small number of points in the plane and we try to find a function $f(x)$ of a chosen type so that the given points are on the graph of $y=f(x)$. Then the function $f(x)$ is said to be a fit for the given (data) points.

Here are the simplest techniques:

1. Linear fit. Given two points $P_{1}\left(a_{1}, b_{1}\right), P_{2}\left(a_{2}, b_{2}\right)$ we already know how to find the equation of a line joining them, namely $\left(y-b_{1}\right)\left(a_{2}-a_{1}\right)=\left(x-a_{1}\right)\left(b_{2}-b_{1}\right)$, thus

$$
y=\frac{b_{2}-b_{1}}{a_{2}-a_{1}}(x)+\frac{b_{1} a_{2}-a_{1} b_{2}}{a_{2}-a_{1}} .
$$

is the desired fit.
Of course, we know the problem when $a_{1}=a_{2}$. Thus, if our data points are $(2,5)$ and $(2,7)$, then we simply have to admit that finding a function $f(x)$ with $f(2)=5$ and $f(2)=7$ is an impossible task and no exact fitting is possible.

Now, if there are more than two points given and we still want to make a best possible linear function fit, then there is a well known formula. A full development will take us afar, but here is a brief summary:
Optional Explanation of General Linear Fit. For the inquisitive, here are the details, without proof. Suppose that you are given a sequence of distinct data points $\left(a_{1}, b_{1}\right), \cdots\left(a_{n}, b_{n}\right)$ where $n \geq 2$, then the best fit linear function is given by $y=m x+c$ where $m$ and $c$ are calculated by the following procedure.
Define and evaluate:

$$
\begin{aligned}
p & =\sum a^{2}=a_{1}^{2}+\cdots+a_{n}^{2}, q=\sum a=a_{1}+\cdots+a_{n} \\
r & =\sum a b=a_{1} b_{1}+\cdots+a_{n} b_{n} \\
s & =\sum b=b_{1}+\cdots b_{n} .
\end{aligned}
$$

Here the suggestive notations like $\sum a^{2}$ are presented as an aid to memory, the explicit formulas follow them. The greek symbol $\sum$ indicates a "sum" and the expression describes a typical term.

If we take a typical pair of values $x=a_{i}, y=b_{i}$ for $i=1,2, \cdots, n$ then we get a "wished for" equation:

$$
b_{i}=m a_{i}+c
$$

and if we add this for all values of $i$ then we get the equation $s=m q+c n$.
If we multiply our starting equation by $a_{i}$, we get

$$
a_{i} b_{i}=m a_{i}^{2}+c a_{i}
$$

and adding these for all values of $i$ we get $r=m p+c q$.
Thus, we may think of the equations $r=m p+c q$ and $s=m q+c n$ as the average "wish" for all the data points together.
Then we solve $m p+c q=r$ and $m q+c n=s$ for $m, c$. Note that the Cramer's Rule or a direct verification will give the answer:

$$
m=\frac{n r-s q}{n p-q^{2}}, c=\frac{s p-q r}{n p-q^{2}} .
$$

It is instructive to try this out for some concrete set of points and observe that sometimes, none of the points may lie on the resulting line and the answer can be somewhat different from your intuition. However, it has a statistical and mathematical justification of being the best possible fit!
Example. Suppose that in a certain experiment we have collected the following ten data points:
$(1,1),(2,3),(3,1),(4,2),(5,4),(6,3),(7,4),(8,7),(9,10)$, and $(10,8)$. Find the best fit linear function for this data.

Answer: Noting that we have a collection of points $\left(a_{1}, b_{1}\right), \ldots,\left(a_{10}, b_{10}\right)$, let us calculate the quantities $p, q, r$, and $s$ as explained in the theory above.

$$
\begin{array}{lll}
p=1+4+9+16+25+36+49+64+81+100 & =385 \\
q=1+2+3+4+5+6+7+8+9+10 & =55 \\
r=1+6+3+8+20+18+28+56+90+80 & =310 \\
s=1+3+1+2+4+3+4+7+10+8 & =43
\end{array}
$$

Now we write a proposed linear equation $y=m x+c$ using the facts that $n=10$, $m=\frac{n r-s q}{n p-q^{2}}$, and $c=\frac{s p-q r}{n p-q^{2}}$.
So we have that

$$
\begin{aligned}
& m=\frac{(10)(310)-(43)(55)}{(10)(385)-55^{2}}=\frac{49}{55} \\
& c=\frac{(43)(385)-(55)(310)}{(10)(385)-55^{2}}=-\frac{3}{5}
\end{aligned}
$$

The linear equation of best fit is $y=\frac{49}{55} x-\frac{3}{5}$. The reader should plot the ten points along with the line and see how best the fit looks. This ends the optional explanation of the General Linear Fit.
2. Quadratic fit. In general three given points will not lie on any common line. So, for a true fit, we need to use a more general function. We use a quadratic fit or a quadratic function $y=q(x)=p x^{2}+q x+r$. We simply plug in the given three points and get three equations in the three unknowns $p, q, r$. We then proceed to solve them.
Example: Find the quadratic fit for the three points $(2,5),(-1,1)$ and $(3,7)$.
Answer: Let the quadratic function be

$$
y=f(x)=p x^{2}+q x+r .
$$

Don't confuse these $p, q, r$ with the expressions used above! They are new notations, just for this example.
Since we know that $f(2)=5, f(-1)=1$, and $f(3)=7$, we can write a system of three equations with three unknowns $p, q, r$ as follows:

$$
\begin{aligned}
& 5=4 p+2 q+r \\
& 1=p-q+r \\
& 7=9 p+3 q+r .
\end{aligned}
$$

Solving this system of equations, we get:

$$
p=\frac{1}{6}, q=\frac{7}{6} \text { and } r=2 .
$$

So the quadraric function that fits the three given points is

$$
y=f(x)=\frac{1}{6} x^{2}+\frac{7}{6} x+2
$$

It is good to double check that its graph indeed passes thru the three points. Thus verify that $f(2)=5, f(-1)=1$, and $f(3)=7$.
Don't forget that if someone gives four or more points for such a problem, then usually the precise answer may not exist. We can have a statistical fitting answer like above, but we will not developing such general formulas here. Look up the topic of Linear Regression in appropriate sources for Statistics.

## Chapter 8

## The Circle

### 8.1 Circle Basics.

A circle is one of the most studied geometric figures. It is defined as the locus of all points which keep a fixed distance $r$ (called the radius) from a given point $P(h, k)$ called the center. ${ }^{1}$

The distance formula leads to a simple equation: $\sqrt{(x-h)^{2}+(y-k)^{2}}=r$, but since it is not easy to work with the square root, we square the equation and simplify:

$$
\text { Basic Circle }(x-h)^{2}+(y-k)^{2}=r^{2}
$$

and after expansion, it becomes

$$
x^{2}+y^{2}-2 h x-2 k y=r^{2}-h^{2}-k^{2} .
$$

For example, the circle with center $(2,3)$ and radius 5 is given as:

$$
(x-2)^{2}+(y-3)^{2}=5^{2} \text { or } x^{2}-4 x+4+y^{2}-6 y+9=25 .
$$

The equation is usually rearranged as:

$$
x^{2}+y^{2}-4 x-6 y=12
$$

## Recognizing a circle.

We now show how to guess the center and the radius if we see the rearranged form of the equation.

Assume that the equation of a circle appears as: $x^{2}+y^{2}+u x+v y=w$. Compare with the basic circle equation and note:

$$
u=-2 h, v=-2 k, w=r^{2}-\left(h^{2}+k^{2}\right)
$$

[^40]This gives us that the center must be $(h, k)=(-u / 2,-v / 2)$.
Also, $r^{2}=w+\left(h^{2}+k^{2}\right)$, so $r^{2}=w+u^{2} / 4+v^{2} / 4$ and hence the radius must be $r=\sqrt{w+u^{2} / 4+v^{2} / 4}$.

Thus, for example, if we have a circle given as:

$$
x^{2}+y^{2}+4 x-10 y=20
$$

then we have:

$$
u=4, v=-10, w=20
$$

Thus, we get:

$$
\text { Center is: }(-u / 2,-v / 2)=\left(-\frac{4}{2},-\frac{-10}{2}\right)=(-2,5) .
$$

Also,
The radius is: $\sqrt{w+u^{2} / 4+v^{2} / 4}=\sqrt{20+4+25}=\sqrt{49}=7$.
What happens if the quantity $w+u^{2} / 4+v^{2} / 4$ is negative or zero? We get a circle which has no points (i.e. is imaginary) or reduces to a single point $(-u / 2,-v / 2)$.

Consider, for example the equation $x^{2}+y^{2}+2 x+2 y=s$ and argue that for $s<-2$ we get a circle with no point, for $s=-2$ we have just the point $C(-1,-1)$ and for $s>-2$ we get a circle centered at $C(-1,-1)$ with radius $\sqrt{s+2}$.


### 8.2 Parametric form of a circle.

For the equation of a line, say something like $y=m x+c$, it is easy to get a parametric form, say $x=t, y=m t+c$ and thus it is easy to create many points on the line by choosing different values of $t$.

This is not so easy for the complicated equation of a circle, say something like

$$
x^{2}+y^{2}=12 .
$$

Here, if we try to set $x=t$, the value of $y$ is not unique and we need to solve a quadratic equation $y^{2}=12-t^{2}$ to finish the task.

We now construct a parametric form for the circle which also illustrates a new technique of analyzing an equation.

Consider the general equation of a circle centered at $(0,0)$ and having radius $r>0$, given by

$$
x^{2}+y^{2}=r^{2} .
$$

We shall show below that every point $P(x, y)$ of this circle can be uniquely described as

$$
x=r \frac{1-m^{2}}{1+m^{2}} \text { and } y=r \frac{2 m}{1+m^{2}} .
$$

Here $m$ is a parameter which is allowed to be any real number as well as $\infty$ and has a well defined geometric meaning as the slope of the line joining joining the point $(x, y)$ to the special point $A(-r, 0)$.

## Proof of the parameterization of a circle.

## The following derivation of this parameterization can be safely omitted in a first reading.

Note that $A(-r, 0)$ is clearly a point on our circle. Let us take a line of slope $m$ through $A$ in a parametric form, say $x=-r+t, y=0+m t$. Let us find where it hits the circle by plugging into the equation of the circle.

We get the following sequence of simplifications, to be verified by the reader.

$$
\begin{array}{lll}
(t-r)^{2}+m^{2} t^{2} & = & r^{2} \\
t^{2}-2 r t+r^{2}+m^{2} t^{2} & = & r^{2} \\
\left(1+m^{2}\right) t^{2}-2 r t & = & 0 \\
t\left(\left(1+m^{2}\right) t-2 r\right) & =0
\end{array}
$$

This gives two easy solutions, $t=0$ or $t=\frac{2 r}{\left(1+m^{2}\right)}$.
Clearly $t=0$ gives the point $A$ and the other value gives the other intersection of the circle with the line.

This other point has

$$
x=-r+t=-r+\frac{2 r}{\left(1+m^{2}\right)}=r \frac{1-m^{2}}{1+m^{2}}
$$

and

$$
y=m t=m \frac{2 r}{1+m^{2}}=r \frac{2 m}{1+m^{2}} .
$$

Thus:

$$
(x, y)=\left(r \frac{1-m^{2}}{1+m^{2}}, r \frac{2 m}{1+m^{2}}\right)
$$

This gives a nice parametric form for the circle with parameter $m$. As we take different values of $m$ we get various points of the circle.

What happens to the limiting case when $m$ is sent to infinity. Indeed for this case, the line should be replaced by the vertical line $x=-r$ and the equation of substitution gives us:

$$
r^{2}+y^{2}=r^{2} \text { which leads to } y^{2}=0
$$

This says that the only point of intersection is $A(-r, 0)$, so it makes sense to declare that the limiting value of the parameterization gives $A(-r, 0)$.

There is an interesting algebraic manipulation of the formula which helps in such limiting calculations often. We present it for a better understanding.

We rewrite the expression $r \frac{1-m^{2}}{1+m^{2}}$ by dividing both the numerator and the denominator by $m^{2}$ :

$$
r \frac{1-m^{2}}{1+m^{2}}=r \frac{\frac{1}{m^{2}}-1}{\frac{1}{m^{2}}+1}
$$

Now with the understanding that $1 / \infty$ goes to 0 , we can deduce the result to be

$$
r \frac{-1}{1}=-r
$$

A similar calculation leads to:

$$
r \frac{2 m}{1+m^{2}}=r \frac{\frac{2}{m}}{\frac{1}{m^{2}}+1} \rightarrow r \frac{0}{1}=0
$$

## Remark.

The above rational parameterization of a circle is indeed nice and useful, but not as well known as the trigonometric one that we present later.

Even though rational functions are easier to work with, the trigonometric functions have a certain intrinsic beauty and have been well understood over a period of well over two thousand years.

### 8.3 Application to Pythagorean Triples.

## The Simple Identity: <br> $$
(a-b)^{2}+4 a b=(a+b)^{2}
$$

has been used for many clever applications.

We now show how a clever manipulation of it gives us yet another proof of the circle parameterization.

First we divide it by $(a+b)^{2}$ to get

$$
\frac{(a-b)^{2}}{(a+b)^{2}}+\frac{4 a b}{(a+b)^{2}}=1
$$

Replacing $a$ by 1 and $b$ by $m^{2}$ we see that

$$
\left(\frac{1-m^{2}}{1+m^{2}}\right)^{2}+\left(\frac{2 m}{1+m^{2}}\right)^{2}=1
$$

We now show how it can lead to a new explanation of the parameterization of a circle.

Consider, as above the circle

$$
x^{2}+y^{2}=r^{2} .
$$

Rewrite it after dividing both sides by $r^{2}$ to get:

$$
\left(\frac{x}{r}\right)^{2}+\left(\frac{y}{r}\right)^{2}=1
$$

Using our identity, we could simply set:

$$
\frac{x}{r}=\frac{1-m^{2}}{1+m^{2}} \text { and } \frac{y}{r}=\frac{2 m}{1+m^{2}}
$$

and clearly our circle equation is satisfied!
It follows that

$$
x=r \frac{1-m^{2}}{1+m^{2}}, y=r \frac{2 m}{1+m^{2}}
$$

is a parameterization!
We now use the above to discuss:

## Pythagorean triples.

A Pythagorean triple is a triple of integers $x, y, z$ such that

$$
x^{2}+y^{2}=z^{2} .
$$

For convenience, one often requires all numbers to be positive. If $x, y, z$ don't have a common factor, then the triple is said to be primitive.

The same original (simple) identity can also be used to generate lots of examples of Pythagorean triples.

We divide the original identity by 4 and plug in $a=s^{2}, b=t^{2}$. Thus, we have:

$$
\frac{\left(s^{2}-t^{2}\right)^{2}}{4}+s^{2} t^{2}=\frac{\left(s^{2}+t^{2}\right)^{2}}{4}
$$

and this simplifies to:

$$
\left(\frac{s^{2}-t^{2}}{2}\right)^{2}+(s t)^{2}=\left(\frac{s^{2}+t^{2}}{2}\right)^{2}
$$

Then we easily see that

$$
x=\frac{s^{2}-t^{2}}{2}, y=s t, z=\frac{s^{2}+t^{2}}{2}
$$

forms a Pythagorean triple!
What is more exciting is that with a little number theoretic argument, this can be shown to give all possible primitive positive integer solutions $(x, y, z)$.

These primitive Pythagorean triples have been rediscovered often, in spite of their knowledge for almost 2000 years. It is apparent that they have been known around the world for a long time.

Try generating some triples by taking suitable values for $s, t$. (For best results, take odd values $s>t$ without common factors!) For instance, we can take:

| $s$ | $t$ | $x$ | $y$ | $z$ |
| :--- | :--- | ---: | ---: | ---: |
| 3 | 1 | 4 | 3 | 5 |
| 5 | 1 | 12 | 5 | 13 |
| 5 | 3 | 8 | 15 | 17 |
| 7 | 5 | 12 | 35 | 37 |
| 7 | 3 | 20 | 21 | 29 |
| 7 | 1 | 24 | 7 | 25 |

What is the moral? A good identity goes a long way!

## Optional section: A brief discussion of the General conic.

A conic is a plane curve which is obtained by intersecting a cone with planes in different position. Study of conic sections was an important part of Greek Geometry and several interesting and intriguing properties of conics have been developed over two thousand years.

In our algebraic viewpoint, a conic can be described as any degree two curve, i.e. a curve of the form

$$
\text { General conic: } a x^{2}+b y^{2}+2 h x y+2 f x+2 g y+c=0 .
$$

We invite the reader to show that the above idea of parameterizing a circle works just as well for a conic. We simply take a point $A(p, q)$ on the conic and consider a line $x=p+t, y=q+m t$. The intersection of this line with the conic gives two value of $t$, namely

$$
t=0, t=\frac{-2(b q m+f+g m+a p+h p m+h q)}{a+2 h m+b m^{2}} .
$$

The reader is invited to carry out this easy but messy calculation as an exercise in algebra.

It is interesting to work out the different types of the conics: circle/ellipse, parabola, hyperbola and a pair of lines. We can work out how the equations can tell us the type of the conic and its properties. But a detailed analysis will take too long. Here we simply illustrate how the idea of parameterization can be worked out for a given equation.

Example of parameterization. Parameterize the parabola

$$
y^{2}=2 x+2
$$

using the point $A(1,2)$.
Answer: We shall first change coordinates to bring the point $A$ to the origin. This is not needed, but does simplify our work.

Thus we set $x=u+1, y=v+2$, then the point $A(1,2)$ gets new coordinates $u=0, v=0$. Our equation becomes:

$$
(v+2)^{2}=2(u+1)+2 \text { or } v^{2}+4 v=2 u .
$$

Now a line through the origin in $(u, v)$ coordinates is given by $u=t, v=m t$ where $m$ is the slope.

Substitution gives

$$
m^{2} t^{2}+4 m t=2 t \text { or, by simplifying } t\left(m^{2} t+(4 m-2)\right)=0 .
$$

Thus there are two solutions $t=0, t=\frac{2-4 m}{m^{2}}$. We ignore the known solution $t=0$ corresponding to our starting point $A$.

Note that when $m=0$, we end up with a single solution $t=0$.
We use the second solution for the parameterization:

$$
u=t=\frac{2-4 m}{m^{2}} \text { and } v=m t=\frac{2 m-4 m^{2}}{m^{2}}
$$

When we go back to the original coordinates, we get:

$$
x=1+\frac{2-4 m}{m^{2}}=\frac{m^{2}-4 m+2}{m^{2}}
$$

and

$$
y=2+\frac{2 m-4 m^{2}}{m^{2}}=\frac{2 m-2 m^{2}}{m^{2}} .
$$

It is an excellent idea to plug this into the original equation and verify that the equation is satisfied for all values of $m$. This means that the final equation should be free of $m$ and always true!

Actually, the value $m=0$ is a problem, since it appears in the denominator. However, when plugged into the equation of the curve, the $m$ just cancels! When we
take the value $m=0$ the other point of intersection of the line simply runs off to infinity. Indeed, this is a situation where a graph is very useful to discover that the line corresponding to $m=0$ will be parallel to the axis of the parabola!


### 8.4 Examples of equations of a circle.

1. Equation of a circle with given properties.

Problem 1. Find the equation of a circle with center $(2,3)$ and radius 6.
Answer: Recall that the equation of a circle with center $(h, k)$ and radius $r$ is:

$$
(x-h)^{2}+(y-k)^{2}=r^{2} .
$$

Thus our equation must be:

$$
(x-2)^{2}+(y-3)^{2}=6^{2} \text { or } x^{2}-4 x+4+y^{2}-6 y+9=36
$$

It is customary to rearrange the final equation as:

$$
x^{2}+y^{2}-4 x-6 y=23 .
$$

Problem 2. Find the equation of a circle with $A(1,5)$ and $B(7,-3)$ as the ends of a diameter.

Answer. First the long way:
We know that the center must be the midpoint of $A B$ so it is:

$$
\frac{A+B}{2}=\left(\frac{1+7}{2}, \frac{5-3}{2}\right)=(4,1) .
$$

Also, the radius must be half the diameter $(=d(A, B))$, i.e.

$$
\text { radius is } \frac{1}{2} \sqrt{(7-1)^{2}+(-3-5)^{2}}=\frac{1}{2} \sqrt{36+64}=\frac{1}{2} \sqrt{100}=5 \text {. }
$$

So the answer is

$$
(x-4)^{2}+(y-1)^{2}=25 .
$$

Simplified, we get:

$$
x^{2}+y^{2}-8 x-2 y=8 .
$$

Now here is an elegant short cut. We shall prove that given two points $A\left(a_{1}, b_{1}\right)$ and $B\left(a_{2}, b_{2}\right)$ the equation of the circle with diameter $A B$ is:

$$
\text { Diameter form of circle }\left(x-a_{1}\right)\left(x-a_{2}\right)+\left(y-b_{1}\right)\left(y-b_{2}\right)=0 .
$$

This formula will make a short work of the whole problem, since it gives the answer:

$$
(x-1)(x-7)+(y-5)(y+3)=0 .
$$

The reader should simplify and compare it with the earlier answer.
We give the following hints, so the reader can prove this formula by the "duck principle".

- First expand and simplify the equation and argue that it is indeed is the equation of a circle.
- Notice the coefficients of $x, y$ and argue that the center of the circle is indeed the midpoint of line joining $A, B$.
- Observe that both the points satisfy the equation trivially!
- So, it must the desired equation by the duck principle.

Extra observation. This particular form of the equation of a circle is developed for yet another purpose, which we now explain.

It is a well known property of circles that if $A, B$ are the ends of a diameter and $P(x, y)$ is any point of the circle, then $A P$ and $B P$ are perpendicular to each other. Conversely, if $P$ is any point which satisfies this condition, then it is on the circle.

In words, this is briefly described as the angle subtended by a circle in a semi circle is a right angle.
We claim that the diameter form of the equation of a circle proves all this by a simple rearrangement. Divide the equation by $\left(x-a_{1}\right)\left(x-a_{2}\right)$ to get:

$$
1+\frac{\left(y-b_{1}\right)\left(y-b_{2}\right)}{\left(x-a_{1}\right)\left(x-a_{2}\right)}=0
$$

or

$$
\left(\frac{y-b_{1}}{x-a_{1}}\right)\left(\frac{y-b_{2}}{x-a_{2}}\right)=-1
$$

Note that $\frac{y-b_{1}}{x-a_{1}}$ is the slope of $A P$ and similarly $\frac{y-b_{2}}{x-a_{2}}$ is the slope of $B P$. So, the new equation is simply saying that the product of these slopes is -1 , i.e. the lines $A P$ and $B P$ are perpendicular!
2. Intersection of two circles. Find all the points of intersection of the two circles:

$$
x^{2}+y^{2}=5, x^{2}+y^{2}-3 x-y=6
$$

Answer: Remember our basic idea of solving two equations in two variables $x, y$ was:

- Solve one equation for one variable, say $y$.
- Plug the value into the second equation to get an equation in $x$.
- Solve the resulting equation in $x$ and plug this answer in the first solution to get $y$.

But here, both equations are quadratic in $x, y$ and we will run into somewhat unpleasant calculations with the square roots.
A better strategy is to try the elimination philosophy. It would be nice to get rid of the quadratic terms which make us solve a quadratic. We try this next:
Let us name $f=x^{2}+y^{2}-5$ and $g=x^{2}+y^{2}-3 x-y-6$.
Note that the quadratic $y$ term is $y^{2}$ for both, so we get:

$$
f-g=\left(x^{2}+y^{2}-5\right)-\left(x^{2}+y^{2}-3 x-y+6\right)=3 x+y+1 .
$$

Let us name this linear expression:

$$
h=3 x+y+1 .
$$

We have at least gotten rid of the quadratic terms. Also any common points satisfying $f=g=0$ also satisfy $h=0$. So, we could solve this $h=0$ for $y$ and use it to eliminate $y$ from $f=0$ as well as $g=0$.

$$
h=0 \text { gives us } y=-3 x-1 .
$$

Plugging this into the equation $f=0$ we get

$$
x^{2}+(-1-3 x)^{2}-5=0 .
$$

This simplifies to:

$$
x^{2}+1+6 x+9 x^{2}-5=0 \text { or } 10 x^{2}+6 x-4=0 .
$$

Further, it is easy to verify that it has two roots:

$$
x=-1, x=\frac{2}{5} .
$$

Plugging back into the expression for $y$, we get corresponding $y$ values and hence the intersection points are: ${ }^{2}$

$$
P(-1,2) \text { and } Q\left(\frac{2}{5},-\frac{11}{5}\right)
$$

Actually, for intersecting two circles whose equations have the same quadratic terms ( $x^{2}+y^{2}$ for example) the process of subtracting one from the other always gives a nice linear condition.

Indeed, this is a short cut to the solution of the next problem.
3. Line joining the intersection points of the two circles. Find the line joining the common points of the two circles:

$$
x^{2}+y^{2}=5, x^{2}+y^{2}-3 x-y=6
$$

Answer: We use the "duck principle". As before, we set $f=x^{2}+y^{2}-5$ and $g=x^{2}+y^{2}-3 x-y-6$ and note that all the common points satisfy $f-g=3 x+y+1=0$.

Thus $3 x+y+1=0$ contains all the common points and is evidently a line, in view of its linear form; so we have found the answer!

Note that we do not need to solve for the common points at all. Moral: Avoid work if you can get the right answer without it!
4. Circle through three given points. Find the equation of a circle through the given points $A(0,0), B(3,1), C(1,3)$.
Answer: We begin by assuming that the desired equation is

$$
x^{2}+y^{2}+u x+v y=w
$$

and write down the three conditions obtained by plugging in the given points:

$$
0+0+0+0=w, 9+1+3 u+v=w, 1+9+u+3 v=w
$$

These simplify to:

$$
w=0,3 u+v-w+10=0, \quad \text { and } u+3 v-w+10=0 .
$$

[^41]We note that $w=0$ and hence the last two equations reduce to:

$$
3 u+v+10=0, \quad \text { and } u+3 v+10=0
$$

By known techniques (Cramer's Rule for example) we get their solution and conclude that

$$
u=-5 / 2, v=-5 / 2, w=0
$$

So the equation of the desired circle is:

$$
x^{2}+y^{2}-5 x / 2-5 y / 2=0 .
$$

From what we already know, we get that this is a circle with

$$
\text { center: }\left(-\frac{u}{2},-\frac{v}{2}\right)=\left(\frac{5}{4}, \frac{5}{4}\right)
$$

and

$$
\text { radius: } \sqrt{w+\frac{u^{2}}{4}+\frac{v^{2}}{4}}=\sqrt{0+\frac{25}{16}+\frac{25}{16}} \sqrt{50 / 16}=(5 / 4) \sqrt{2} .
$$

## Done!

5. Failure to find a circle through three given points. The three equations that we obtained above can fail to have a solution, exactly when the three chosen points all lie on a common line. The reader is invited to attempt to find a circle through the points:

$$
(0,0),(3,1),(6,2)
$$

and notice the contradiction in the resulting equations:

$$
0+0+0+0=w, 9+1+3 u+v=w, 36+4+6 u+2 v=w
$$

which lead to:

$$
w=0,3 u+v=-10,6 u+2 v=-40
$$

6. Smallest circle with a given center meeting a given line. Find the circle with the smallest radius which has center at $A(1,1)$ and which meets the line $2 x+3 y=6$.
Answer: We assume that the circle has the equation:

$$
(x-1)^{2}+(y-1)^{2}=r^{2} \text { with } r \text { to be determined. }
$$

We solve the line equation for $y$ :

$$
y=(6-2 x) / 3=2-(2 / 3) x
$$

and plug it into the equation of the circle:

$$
(x-1)^{2}+(2-(2 / 3) y-1)^{2}=r^{2} \text { which simplifies to: } \frac{13}{9} x^{2}-\frac{10}{3} x+2=r^{2}
$$

Our aim is to find the smallest possible value for the left hand side, so the radius on the right hand side will become the smallest!
From our study of quadratic polynomials, we already know that $a x^{2}+b x+c$ with $a>0$ has the smallest possible value when $x=-\frac{b}{2 a}$.
For the quadratic in $x$ on our left hand side, we see $a=\frac{13}{9}$ and $b=-\frac{10}{3}$ so

$$
-\frac{b}{2 a}=-\frac{-\frac{10}{3}}{(2)\left(\frac{13}{9}\right)}=\frac{(10)(9)}{(3)(2)(13)}=\frac{15}{13} .
$$

Thus, $x=\frac{15}{13}$ for the desired minimum value. Taking this value for the variable $x$, our left hand side becomes

$$
\frac{13}{9}\left(\frac{15}{13}\right)^{2}-\frac{10}{3}\left(\frac{15}{13}\right)+2=\frac{1}{13}
$$

Thus $\frac{1}{13}=r^{2}$ and hence the smallest radius is $\sqrt{1 / 13}$. Note that if we take $r=\sqrt{1 / 13}$ then we get a single point of intersection:

$$
x=\frac{15}{13} \text { and } y=2-\left(\frac{2}{3}\right) x=\frac{16}{13} .
$$

The resulting point $P\left(\frac{15}{13}, \frac{16}{13}\right)$ is a single common point of the circle with radius $\sqrt{\frac{1}{13}}$ and the given line. In other words, our circle is tangent to the given line. Finding a circle with a given center and tangent to a given line is a problem of interest in itself, which we discuss next.
7. Circle with a given center and tangent to a given line.

We can use the same example as above and ask for a circle centered at $A(1,1)$ and tangent to the line

$$
L: 2 x+3 y-6=0 .
$$

If $P$ is the point of tangency, then we simply note that the segment $\overline{A P}$ must be perpendicular to the line $L$ and the point $P$ is nothing but the intersection point of $L$ with a line passing through $A$ and perpendicular to $L$.
Since the line $L$ is

$$
2 x+3 y=6
$$

we already know that any perpendicular line must look like: ${ }^{3}$

$$
L^{\prime}:-3 x+2 y=k \text { for some } k
$$

Since we want it to pass through $A(1,1)$ we get:

$$
-3(1)+2(1)=k
$$

so $k=-1$ and the line is:

$$
L^{\prime}:-3 x+2 y=-1
$$

We use our known methods to determine that the common point of $L$ and $L^{\prime}$ is $P(15 / 13,16 / 13)$. The distance $d(A, P)$ is then seen to be $\sqrt{\frac{1}{13}}$ as before.
Note that this calculation was much easier and we solved a very different looking problem, yet got the same answer. This is an important problem solving principle! Solve an equivalent but different problem which has a simpler formula.
There is yet another way to look at our work above.
8. The distance between a point and a line. Given a line $L: a x+b y+c=0$ and a point $A(p, q)$, show that the point $P$ on the line $L$ which is closest to $A$ is given by the formula

$$
P\left(p-a \frac{w}{a^{2}+b^{2}}, q-b \frac{w}{a^{2}+b^{2}}\right) \text { where } w=a p+b q+c .
$$

The shortest distance, therefore, is given by

$$
\sqrt{\frac{w^{2}}{a^{2}+b^{2}}}=\frac{|w|}{\sqrt{a^{2}+b^{2}}}
$$

This is often remembered as:

$$
\text { The distance from }(p, q) \text { to } a x+b y+c=0 \text { is } \frac{|a p+b q+c|}{\sqrt{a^{2}+b^{2}}} \text {. }
$$

Proof of the formula. We take a point $P$ on $L$ to be the intersection of the line $L$ and a line $L^{\prime}$ through $A$ perpendicular to $L$. We claim that this point $P$ must give the shortest distance between any point on $L$ and $A$.

[^42]

To see this, take any other point $B$ on $L$ and consider the triangle $\triangle A P B$. This has a right angle at $P$ and hence its hypotenuse $\overline{A B}$ is longer than the side $\overline{A P}$; in other words

$$
d(A, B)>d(A, P) \text { for any other point } B \text { on } L
$$

We leave it as an exercise to the reader to show that the above formula works. Here are the suggested steps to take: ${ }^{4}$

- The perpendicular line $L^{\prime}$ should have parametric equations:

$$
x=p+a t, y=q+b t .
$$

Just verify that this line must be perpendicular to the given line and it obviously passes through the point $A(p, q)$ (at $t=0$.)

- The common point $P$ of $L, L^{\prime}$ is obtained by substituting the parametric equations into $a x+b y+c=0$. Thus for the point $P$, we have:

$$
a(p+a t)+b(q+b t)+c=0 \text { or } t\left(a^{2}+b^{2}\right)=-(a p+b q+c) .
$$

In other words:

$$
t=-\frac{w}{a^{2}+b^{2}}
$$

- The point $P$, therefore is:

$$
x=p-a \frac{w}{a^{2}+b^{2}}, y=q-b \frac{w}{a^{2}+b^{2}} .
$$

[^43]- Now the distance between $A(p, q)$ and $P$ is clearly

$$
d(A, P)=\sqrt{(a t)^{2}+(b t)^{2}}=\sqrt{\left(a^{2}+b^{2}\right) t^{2}}=\sqrt{\left(a^{2}+b^{2}\right) \cdot \frac{w^{2}}{\left(a^{2}+b^{2}\right)^{2}}}
$$

- The final answer simplifies to:

$$
\sqrt{\frac{w^{2}}{a^{2}+b^{2}}}=\frac{|w|}{\sqrt{a^{2}+b^{2}}}
$$

as claimed!
Example of using the above result. Find the point on the line $L: 2 x-$ $3 y-6=0$ closest to $A(1,-1)$.
Answer: We have

$$
a=2, b=-3, c=-6, p=2, q=-1 .
$$

So $w=a p+b q+c=2(1)-3(-1)-6=-1$.
Thus the point $P$ is given by:

$$
P\left(1-2\left(\frac{-1}{4+9}\right),-1-(-3)\left(\frac{-1}{4+9}\right)\right)=P\left(\frac{15}{13},-\frac{16}{13}\right) .
$$

Moreover, the distance is:

$$
\left|\frac{w}{a^{2}+b^{2}}\right|=\left|\frac{-1}{4+9}\right|=\frac{1}{13} .
$$

9. Half plane defined by a line.

Consider a line given by $3 x-4 y+5=0$.
Consider the points $A(3,4)$ and $B(4,4)$. Determine if they are on the same or the opposite side of the line, without relying on a graph!
For any point $P(x, y)$ we can evaluate the expression $3 x-4 y+5$. Obviously, it evaluates to 0 on the line itself. It can be shown that it keeps a constant sign on one side of the line.
Thus at the point $A$, the expression evaluates to $3(3)-4(4)+5=-2$, while at $B$ we get $3(4)-4(4)+5=1$. This tells us that the two points lie on the opposite sides of the line. If we consider the origin $(0,0)$ which gives the value 5 for the expression, we see that $B$ and the origin lie on the same side, while $A$ is on the other side.

It is instructive to verify this by a careful graph and also use this idea as a double check while working with graphs.

How should one prove the above analysis?

## Here is a formal proof.

This can be safely skipped in a first reading.
Suppose we are given a linear expression $E(x, y)=a x+b y+c$ in variables $x, y$. Consider the line defined by $L: E(x, y)=0$.
Suppose we have two points $P, Q$ which give values $p, q$ for the expression respectively.
Then any point on the segment from $P$ to $Q$ is of the form $t P+(1-t) Q$ where $0 \leq t \leq 1$ and it is not hard the see that the expression at such a point evaluates to $t p+(1-t) q$.
Now suppose that $p, q$ are both positive, then clearly $t p+(1-t) q$ is also positive for all $0 \leq t \leq 1$.

We claim that both the points must lie on the same side of the line $L$.
Suppose, if possible that the points lie on opposite sides of the line; we shall show a contradiction, proving our claim!
Since some point in between $P, Q$ must lie on the line, the expression $t p+(1-t) q$ must be zero at such point. As already observed the expression, however, always stays positive, a contradiction!

We can similarly argue that when both $p, q$ are negative, the expression $t p+$ $(1-t) q$ always stays negative for $0 \leq t \leq 1$ and hence there is no point of the line between $P, Q$.
When $p, q$ have opposite signs, then we can see that that the quantity $\frac{q}{q-p}$ is between 0 and 1 and this value of $t$ makes the expression evaluate to: ${ }^{5}$

$$
\frac{q}{q-p}(p)+\left(1-\frac{q}{q-p}\right) q=\frac{p q}{q-p}+\frac{-p q}{q-p}=0 .
$$

Thus, the line joining $P, Q$ must intersect the line given by $E(x, y)=0$. This proves that the points lie on opposite sides of the line!
We record the above information for future use.
Let $L$ be the line given by the equation $a x+b y+c=0$ as above. We naturally have that $a, b$ cannot be both zero and hence $\sqrt{a^{2}+b^{2}}$ is a positive quantity.
For any point $P(x, y)$ define the expression $f_{L}(P)=a x+b y+c$.

[^44]Assume that we have points $P, Q$ outside the line, i.e. $f_{L}(P) \neq 0 \neq f_{L}(Q)$.
These points $P, Q$ lie on the same side of the line if $f_{L}(P)$ and $f_{L}(Q)$ have the same sign, or, equivalently, $f_{L}(P) f_{L}(Q)>0$.
On the other hand, they lie on the opposite sides if the expressions have opposite signs, i.e $f_{L}(P) f_{L}(Q)<0$.
Moreover as we travel along the segment $\overline{P Q}$ we get every value of the expression from $f_{L}(P)$ to $f_{L}(Q)$. In particular, the expression $f_{L}(A)$ is either a constant or strictly increases or strictly decreases.
This last fact is very important in the subject called Linear Programming which discusses behavior of linear expressions on higher dimensional sets.

### 8.5 Trigonometric parameterization of a circle.

There is another well known parameterization of a circle which is used in the definition of the trigonometric functions $\sin (t), \cos (t)$. Suppose we take a circle of radius 1 centered at the origin:

$$
x^{2}+y^{2}=1
$$

A circle of radius 1 is called a unit circle.
We wish to arrange a parameter $t$ such that given any point $P$ on the unit circle, there is a certain value of $t$ associated with it and given any value of the parameter $t$, there is a unique point $P(t)$ associated with it.

Here is a very simple idea. Start with the point $U(1,0)$ on our circle. Given any real number $t$, walk $t$ units along the circle in the counter clockwise direction and stop. Call the resulting point $P(t)$. Let us see what this means.

There is a famous number called $\pi$ which is defined as the ratio of the circumference of a circle to its diameter. Since our circle is a unit circle, its diameter must be 2 and hence by definition, its circumference is $2 \pi .^{6}$

- Thus, if we take $t=\pi / 2$ then we have gone a quarter circle and reach the point $(0,1)$; we say $P(\pi / 2)=(0,1)$.
- If we go half the circle, we take $t=\pi$ and $P(\pi)=(-1,0)$.
- If we further go another quarter circle then we reach the point $P(3 \pi / 2)=(0,-1)$ and finally, if we go a complete circle, we come back to the starting point $P(2 \pi)=(1,0)$.

This means that if we intend to travel a length $t$ around the circle and $t=2 \pi+s$ then the position $P(t)$ at which we ultimately arrive will be the same as if we travel $s=t-2 \pi$. We express this symbolically as $P(t)=P(s)=P(t-2 \pi)$

Thus we conclude that to determine the position $P(t)$ we can always throw away multiples of $2 \pi$ until we get a "remainder" which lies between 0 and $2 \pi$. Formally, we can describe this as follows:

We write our

$$
t=2 \pi(n)+s \text { where } 0 \leq s<2 \pi
$$

and $n$ is some integer.
Indeed, this is very much like our old division, except we have now graduated to dividing real numbers by real numbers!

[^45]Then we can march $s$ units along the unit circle counter clockwise from the point $U(1,0)$ and reach a point $P(s)$ which, by definition, is also the point denoted as $P(t)$.

So for any number $t$ we have an associated point $P(t)$ on the unit circle. As the figure indicates there is uniquely associated to $P(t)$ an angle made by the ray $\overrightarrow{O P}$ with the starting ray $\overrightarrow{O U}$. We denote this angle by the symbol $\angle$ UOP.

- This allows us to think of the number $t$ as the measure of an angle $\angle \mathrm{UOP}$; namely the resulting angle made by the ray $\overrightarrow{O P}$ with the ray $\overrightarrow{O U}$.
- This measure is said to be in units called "radians". The explanation for using this term is probably as follows.

If we take a circle of radius $r$ then the circumference of the circle, by definition, is $2 \pi r$, so the ratio $\frac{\text { distance along the arc }}{\text { radius }}$ is $2 \pi$. Thus the measure $t$ can be thought of as a measure in units of "radiuses" (or should it be radii?).

- By an abuse of the notation, we may simply call $t$ to be the angle $\angle U O P$, in radians. Since it is the number $t$ that is used in our formulas, this should not cause confusion.

However, if you are working with geometry itself, it is customary to declare $t=m \angle U O P$, i.e. $t$ is the measure of the angle $\angle U O P$.

- The possible "numerical" angles (numbers of radians) for the same locator point are infinitely many, so for a given angle of measure $t$ we usually use the remainder $s$ of $t$ after throwing away enough multiples of $2 \pi$ as the radian measure of the angle. Thus, we take the liberty to say things like "This angle is $y$ radians." when of course we know that the angle itself is a geometric figure and " $y$ radians" is simply its measure.
Thus for the point $B(0,1)$ the angle of the ray $\overrightarrow{O B}$ is said to be $\pi / 2$ radians.
- We reemphasize that we can either march off the $t$ units directly or march off the $s$ units where $s$ is the remainder of $t$ after dividing by $2 \pi$ as explained above.

If we choose to march off $t$ units directly, then a negative $t$ requires us to march in the clockwise direction instead of the counter clockwise direction (also called anti clockwise in some books.) ${ }^{7}$

- We formally define $P(t)$ to be the locator point of the "angle" $t$ (in radians).

[^46]

## - A more familiar unit: degrees.

Even though this description is pleasing, it is difficult to work with, since the number $\pi$ itself is such a mystery. Even a fancy calculator or a great big computer can only approximate the number $\pi$ and hence any such calculations are basically imprecise! Then why do we still want to mention and use this number? Because, in mathematical work and especially Calculus, it is important to work with radians.

However, for our simpler application, it is more convenient to use the usual measure called degrees.

- We simply divide the circle into 360 equal parts and call the angle corresponding to each part a degree. Thus our point $B(0,1)$ corresponds to the angle of $\frac{360}{4}=90$ degrees and it is often written as $90^{\circ}$.
Here is a picture of the circle showing the degree measures for some important points.

- It is well worth making a simple rule to convert between radian and degree measures.
To convert radians to degrees, simply multiply by $\frac{180}{\pi}$.
To convert from degrees to radians, reverse the procedure and multiply by $\frac{\pi}{180}$.
We now review how the locator point $P(t)$ is determined $t$ is given in degrees.

We write:

$$
t=360 n+s \text { where } 0 \leq s<360 \text { and } n \text { is some integer. }
$$

Thus $s$ is the remainder of $t$ after division by 360 and we mark off $s$ degrees in our circle (in the counter clockwise manner) to locate the point $P(t)=P(s)$.
Thus, for $t=765$ we see that $t=360(2)+45$ so $s=45$ and the corresponding point is the point at the tip of the segment making the 45 degree angle.
Note that for $t=-1035$, we get the same $s$ since $-1035=360(-3)+45$.
For the reader's convenience we present a table of the corresponding angle measures of the marked points in our picture.

| degrees | 0 | 30 | 45 | 90 | 180 | 270 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| radians | 0 | $\pi / 6$ | $\pi / 4$ | $\pi / 2$ | $\pi$ | $3 \pi / 2$ |

- In the remaining work, we shall use the degree notation ${ }^{\circ}$ when we use degrees, but when we mean to use radians, we may skip any notation or mention of units!


## Definition of the Trigonometric Functions.

Given an angle $t$, let $P(t)$ be its locator point. We know its $x$ and $y$ coordinates. Write

$$
P(t)=(x(t), y(t))
$$

Then the definition of the trigonometric functions is simply:

$$
\sin (t)=y(t) \text { and } \cos (t)=x(t)
$$

Before proceeding, let us calculate a bunch of these values.

## Some trigonometric values.

1. Find the values of $\cos (0), \sin (0)$.

Answer: When the angle $\angle U O P$ is 0 , our point $P$ is the same as $U(1,0)$. So by definition,

$$
\cos (0)=1, \sin (0)=0
$$

2. Find the values of $\cos \left(90^{\circ}\right), \sin \left(90^{\circ}\right)$.

Answer: We already know this point $P\left(90^{\circ}\right)$ to be $(0,1)$, so we get:

$$
\cos \left(90^{\circ}\right)=0, \sin \left(90^{\circ}\right)=1
$$

The reader can verify the values for $180^{\circ}$ and $270^{\circ}$ just as easily. They are:

$$
\cos \left(180^{\circ}\right)=-1, \quad \sin \left(180^{\circ}\right)=0
$$

and

$$
\cos \left(270^{\circ}\right)=0, \quad \sin \left(270^{\circ}\right)=-1
$$

3. Find the values of $\cos \left(30^{\circ}\right), \sin \left(30^{\circ}\right)$.

Answer: This needs some real work! We shall give a shortcut later.
Let us make a triangle $\triangle O M P$ where $P$ is the point at $30^{\circ}$ and $M$ is the foot of the perpendicular from $P$ onto the segment $\overline{O U}$.
We show a picture below where we have added the midpoint $H$ of $O P$ and the segment $\overline{H M}$ joining it with $M$.


A little geometric argument can be used to deduce that ${ }^{8}$

$$
d(H, O)=d(H, M)=d(H, P)
$$

and then the triangle $\triangle H M P$ is equilateral. This says that

$$
d(M, P)=d(H, P)=\frac{1}{2} d(O, P)=\frac{1}{2} .
$$

Thus, we have given a brief idea of how to figure out that

$$
\sin \left(30^{\circ}\right)=\frac{1}{2}
$$

[^47]How to find $\cos \left(30^{\circ}\right)$ ? We can now use simple algebra using the fact that the point $P\left(30^{\circ}\right)$ is on the circle $x^{2}+y^{2}=1$. So,

$$
\cos \left(30^{\circ}\right)^{2}+\left(\frac{1}{2}\right)^{2}=1
$$

and it is easy to deduce that

$$
\cos \left(30^{\circ}\right)^{2}=1-\frac{1}{4}=\frac{3}{4} .
$$

It follows that

$$
\cos \left(30^{\circ}\right)= \pm \sqrt{\frac{3}{4}}= \pm \frac{\sqrt{3}}{2}
$$

Which sign shall we take? Clearly, from the picture, the $x$-coordinate is positive, so we get

$$
\cos \left(30^{\circ}\right)=\frac{\sqrt{3}}{2}
$$

Bonus thought! What do we get if we take the minus sign? After a little thought, you can see that the angle of $180-30=150$ degrees will have the same sin, namely $\frac{1}{2}$ and its cos will be indeed $-\frac{\sqrt{3}}{2}$.
Similarly, for the angle $-30^{\circ}$ which is also the angle $330^{\circ}$, you can see that

$$
\sin \left(-30^{\circ}\right)=-\frac{1}{2} \text { and } \cos \left(-30^{\circ}\right)=\frac{\sqrt{3}}{2}
$$

4. Find the value of $\sin \left(13^{\circ}\right), \cos \left(13^{\circ}\right)$.

Answer: There is no clever way for this calculation. All we can hope to do is to use a calculator or computer, or develop a formula which can give us better and better approximations.
A calculator will say:

$$
\sin \left(13^{\circ}\right)=0.2249510544 \text { and } \cos \left(13^{\circ}\right)=0.9743700648
$$

Indeed, for most concrete values of angles, you may end up just using calculator for evaluating these functions.

### 8.6 Basic Formulas for the Trigonometric Functions.

Introduction. The aim of this section is to list the basic formulas for the various trigonometric functions and learn their use.

The actual theoretical development of these will be done later.
These functions, though easy to define, don't have simple formulas and evaluating their values requires careful approximations. You can use a calculator and get decimal numbers, but that is not our goal.

We shall try to understand how these functions can be evaluated precisely for many special values of the argument and how they can be manipulated formally using algebraic techniques. They form the basis of many mathematical theories and are, in some sense, the most important functions of modern mathematics, next to polynomials. Indeed, the theory of Fourier analysis lets us approximate all practical functions in terms of suitable trigonometric functions and it is crucial to learn how to manipulate them. It is customary to also learn about exponential and logarithmic functions, but we have chosen to postpone it to a higher level course. Part of the reason is that their theoretical treatment requires a lot more sophistication.

As far as evaluation at specific points is concerned, trigonometric, exponential or logarithmic functions are equally easy - they are keys on the calculator! You probably have already seen and used them in concrete problems anyway.

We have already described connection between the angle $t$, its corresponding locator point $P(t)$ and the corresponding trigonometric functions

$$
\sin (t)=\text { the } y \text {-coordinate of } P(t) \text { and } \cos (t)=\text { the } x \text {-coordinate of } P(t) .
$$

Unless otherwise stated, our angles will always be in radian measure, so a right angle has measure $\pi / 2$.

## Negative angles.

Sometimes it reduces our work if we use negative values of angles, rather that converting them to positive angles. After all, $-30^{\circ}$ is easier to visualize than $330^{\circ}$. We can give a natural meaning to negative angles, they represent arc length measure in the negative or clockwise direction! Check out that going $30^{\circ}$ clockwise or $330^{\circ}$ counter clockwise leads to the same locator point!

Thus, sometimes it is considered more convenient to take the angle measure between $-180^{\circ}$ to $180^{\circ}$.

Definition: Other trigonometric functions. Given an angle $t \in \Re$, we know that its locator point $P(t)$ has coordinates $(\cos (t), \sin (t))$.

We now define four more functions which are simply related to the above two functions:

$$
\tan (t)=\frac{\sin (t)}{\cos (t)}, \quad \cot (t)=\frac{\cos (t)}{\sin (t)}
$$

Also:

$$
\csc (t)=\frac{1}{\sin (t)}, \quad \sec (t)=\frac{1}{\cos (t)}
$$

These functions have full names which are:

| Notation | sin | cos | tan | cot | csc | sec |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Full name | Sine | Cosine | Tangent | Cotangent | Cosecant | Secant |

We have the following basic identities derived from the fact that $P(t)$ is on the unit circle.

## Fundamental Identity 1.

$$
\sin ^{2}(t)+\cos ^{2}(t)=1
$$

We can divide this identity by $\sin ^{2}(t)$ or $\cos ^{2}(t)$ to derive two new identities for the other functions. The reader should verify these:

Fundamental Identity 2.

$$
1+\cot ^{2}(t)=\csc ^{2}(t)
$$

Fundamental Identity 3.

$$
\tan ^{2}(t)+1=\sec ^{2}(t)
$$

Remark on notation. It is customary to write $\sin ^{2}(t)$ in place of $(\sin (t))^{2}$. This is not done for other functions. Indeed, many times the notation $f^{2}(x)$ is use in indicate applying the function twice - i.e. $f(f(x))$. It is wise to verify what this notation means before making guesses!

For trigonometric functions, the convention is as we explained above! If you have any confusion in your mind, it is safer to use the explicit $(\sin (t))^{2}$ rather than $\sin ^{2}(t)$.

This convention works for other powers too. This indeed leads to a confusion later on if you try to use negative powers. We will explain it when it comes up.

### 8.7 Connection with the usual Trigonometric Functions.

While the above definitions are precise and general, it is important to connect them with the classical definition.

Given a triangle $\triangle A B C$ as in the picture below, we denote the angle at $A$, by the same symbol $A$. In the modern conventions explained above, it corresponds to the angle $\angle B A C$.

Consider the following picture, where $\overline{C H}$ is the perpendicular from $C$ onto $\overline{A B}$. The old definition used to be:

$$
\sin (A)=\frac{d(C, H)}{d(A, C)} \text { and } \cos (A)=\frac{d(A, H)}{d(A, C)}
$$

In our picture, the points $B^{\prime}, C^{\prime}$ are constructed by making the unit circle centered at $A$. The point $H^{\prime}$ is such that $\overline{C^{\prime} H^{\prime}}$ is perpendicular from $C^{\prime}$ onto $\overline{A B}$.

Now, clearly the two triangles $\triangle A H C$ and $\triangle A H^{\prime} C^{\prime}$ are similar with proportional sides.


Hence we have:

$$
\frac{d(C, H)}{d(A, C)}=\frac{d\left(C^{\prime}, H^{\prime}\right)}{d\left(A, C^{\prime}\right)}=\frac{d\left(C^{\prime}, H^{\prime}\right)}{1}=\sin (A) .
$$

This proves that the old and the new sines are the same.
Similarly we get the cosine matched!
Thus, we can continue to use the old definitions when needed. The major problem is when the angle $A$ becomes larger than $90^{\circ}$. Then the perpendicular $H$ falls to the left of $A$ and the distance $d(A, H)$ will have to be interpreted as negative to match our new definition!

The old definitions did not provide for this at all. They usually assumed the angles to be between $0^{\circ}$ and $90^{\circ}$ and for bigger angles (obtuse angles) you had to make special cases for calculations!

Our new definitions take care of all the cases naturally!

### 8.8 Important formulas described.

In a later section, we shall give proofs for various identities connecting the trigonometric functions. These are very important in working out useful values of these functions and in making convenient formulas for practical problems.

It is useful to understand what these identities are saying and learn to use them, before you understand all the details of their proof.

1. Addition Formulas We describe how to find the trigonometric functions for the sum or difference of two angles.

Fundamental Identity 4. $\quad \cos (t-s)=\cos (t) \cos (s)+\sin (t) \sin (s)$

Fundamental Identity 4-variant. $\cos (t+s)=\cos (t) \cos (s)-\sin (t) \sin (s)$
It is important to note the changes in sign when we go from cos formulas to the sin formulas.

Fundamental Identity 5. $\quad \sin (t+s)=\sin (t) \cos (s)+\cos (t) \sin (s)$

Fundamental Identity 5-variant. $\sin (t-s)=\sin (t) \cos (s)-\cos (t) \sin (s)$

## Examples of use.

Given the locator point $P\left(20^{\circ}\right)=(0.9397,0.3420)$ and the locator point $P\left(30^{\circ}\right)=$ ( $0.8660,0.5000$ ), answer the following:

- What is $\cos \left(50^{\circ}\right)$ ?

Answer: Note that $50=20+30$. Using Fundamental Identity 4-variant:

$$
\cos \left(50^{\circ}\right)=\cos \left(20^{\circ}+30^{\circ}\right)=(0.9397)(0.8660)-(0.3420)(0.5000)=0.6427
$$

- What is $\sin \left(10^{\circ}\right)$ ?

Answer: Now we write $10=30-20$. Using Fundamental Identity 5variant:

$$
\sin \left(10^{\circ}\right)=\sin \left(30^{\circ}-20^{\circ}\right)=(0.5000)(0.9397)-(0.8660)(0.3420)=0.1737
$$

2. General results from the above identities.

## Complementary Angle Identity.

$$
\cos (\pi / 2-s)=\sin (s), \quad \sin (\pi / 2-s)=\cos (s)
$$

Use $t=\pi / 2$ in Fundamental Identities 4 and 5-variant.

Supplementary Angle Identity. $\cos (\pi-s)=-\cos (s), \sin (\pi-s)=\sin (s)$
Use $t=\pi$ in Fundamental Identities \& and 5-variant.

Evenness of the cosine function.

$$
\cos (-s)=\cos (s)
$$

Oddness of the sine function. $\sin (-s)=-\sin (s)$

Use $t=0$ in Fundamental Identities 4 and 5-variant.

## Examples of use.

Given the locator point $P\left(20^{\circ}\right)=(0.9397,0.3420)$ and the locator point $P\left(30^{\circ}\right)=$ ( $0.8660,0.5000$ ), answer the following:

- What is $\cos \left(70^{\circ}\right)$ ?

Answer: Using Complementary Angle Identity:

$$
\cos \left(70^{\circ}\right)=\cos \left(90^{\circ}-20^{\circ}\right)=\sin \left(20^{\circ}\right)=0.3420
$$

- What is $\cos \left(250^{\circ}\right)$ ?

Answer: Note that $250^{\circ}=180^{\circ}+70^{\circ}$. So from the addition formulas, we get:

$$
\cos \left(250^{\circ}\right)=\cos \left(180^{\circ}\right) \cos \left(70^{\circ}\right)-\sin \left(180^{\circ}\right) \sin \left(70^{\circ}\right)=0-\cos \left(70^{\circ}\right)=-0.3420
$$

We have, of course used the fact that $P\left(180^{\circ}\right)=(-1,0)$.

- What is $\sin \left(60^{\circ}\right)$ ?

Answer: Using Complementary Angle Identity:

$$
\sin \left(60^{\circ}\right)=\sin \left(90^{\circ}-30^{\circ}\right)=\cos \left(30^{\circ}\right)=0.8660
$$

- What is $\cos \left(-30^{\circ}\right)$ ?

Answer: Using Evenness of the cosine function:

$$
\cos \left(-30^{\circ}\right)=\cos \left(30^{\circ}\right)=0.5000
$$

- What is $\sin \left(-30^{\circ}\right)$ ?

Answer: Using Oddness of the sine function:

$$
\sin \left(-20^{\circ}\right)=-\sin \left(20^{\circ}\right)=-0.3420
$$

## 3. Double angle formulas.

Fundamental Identity 6.

$$
\cos (2 t)=\cos ^{2}(t)-\sin ^{2}(t)
$$

Fundamental Identity 7.

$$
\sin (2 t)=2 \sin (t) \cos (t)
$$

The fundamental identity 6 has two variations and they both are very useful, so we record them individually.

Fundamental Identity 6-variant 1. $\quad \cos (2 t)=2 \cos ^{2}(t)-1$

Fundamental Identity 6-variant 2. $\quad \cos (2 t)=1-2 \sin ^{2}(t)$

## Examples of use.

Given the locator point $P\left(20^{\circ}\right)=(0.9397,0.3420)$ and the locator point $P\left(30^{\circ}\right)=$ ( $0.8660,0.5000$ ), answer the following:

- What is $\cos \left(40^{\circ}\right)$ ?

Answer: Using Fundamental Identity 6 (or variations) :

$$
\cos \left(40^{\circ}\right)=\cos \left(2 \cdot 20^{\circ}\right)=0.9397^{2}-0.3420^{2}=2\left(0.9397^{2}\right)-1=1-2\left(0.3420^{2}\right)
$$

These evaluate to: 0.7660 to a four digit accuracy.

- What is $\sin \left(40^{\circ}\right)$ ?

Answer: Using Fundamental Identity 7:

$$
\sin \left(40^{\circ}\right)=\sin \left(2 \cdot 20^{\circ}\right)=2(0.9397)(0.3420)=0.6428 .
$$

- What is $\sin \left(10^{\circ}\right)$ ?

Answer: Later, we shall use a half angle formula for this, but now we can think $10^{\circ}=40^{\circ}-30^{\circ}$.
So, from Fundamental Identity 5 -variant:

$$
\sin \left(10^{\circ}\right)=\sin \left(40^{\circ}\right) \cos \left(30^{\circ}\right)-\cos \left(40^{\circ}\right) \sin \left(30^{\circ}\right)
$$

Using computed values, we get:

$$
(0.6428)(0.8660)-(0.7660)(0.5000)=0.1737
$$

- What is $\cos \left(10^{\circ}\right)$ ?

Answer: We can use the Fundamental Identity 4 as above, or we may use the Fundamental Identity 1 as

$$
\cos (t)= \pm \sqrt{1-\sin ^{2}(t)}
$$

Note that this has a plus or minus sign and it needs to be decided by the quadrant that our angle is in.
Here $10^{\circ}$ is in the first quadrant, so we have the plus sign and we get:

$$
\cos \left(10^{\circ}\right)=\sqrt{1-\sin ^{2}\left(10^{\circ}\right)}=\sqrt{1-0.1737^{2}}=0.9848
$$

## 4. Half Angle formulas.

## Fundamental Identity 8.

$$
\cos \left(\frac{t}{2}\right)= \pm \sqrt{\frac{1+\cos (t)}{2}}
$$

## Fundamental Identity 9.

$$
\sin \left(\frac{t}{2}\right)= \pm \sqrt{\frac{1-\cos (t)}{2}}
$$

## Examples of use.

Given the locator point $P\left(20^{\circ}\right)=(0.9397,0.3420)$, answer the following:

- What is $\cos \left(10^{\circ}\right)$ ?

Answer: Using Fundamental Identity 8 :

$$
\cos \left(10^{\circ}\right)=\sqrt{\frac{1+0.9397}{2}}=0.9848 \text { again. }
$$

Note that we used the plus sign since the angle is in the first quadrant.

- What is $\sin \left(10^{\circ}\right)$ ?

Answer: Using Fundamental Identity 9 :

$$
\sin \left(10^{\circ}\right)=\sqrt{\frac{1-0.9397}{2}}=0.1736
$$

Note that we used the plus since the angle is in the first quadrant.
Also note that the accuracy is lost a bit. A more precise value of $\cos \left(20^{\circ}\right)$ is .9396926208 and a more accurate result for our $\sin \left(10^{\circ}\right)$ comes out 0.1736481776 . This should explain why the 4 -digit answer can come out 0.1736 or 0.1737 depending on how it is worked out. You have to worry about this when supplying a numerical answer.

## Recalculation of a known value.

What is $\sin \left(\frac{\pi}{4}\right)$ and $\cos \left(\frac{\pi}{4}\right)$ ?
Answer:
Since $2 \frac{\pi}{4}=\frac{\pi}{2}$, we see two conclusions:

$$
\sin \left(\frac{\pi}{4}\right)=\cos \left(\frac{\pi}{4}\right)
$$

due to the Complementary Angle Identity and

$$
\sin \left(\frac{\pi}{4}\right)= \pm \sqrt{\frac{1-\cos \left(\frac{\pi}{2}\right)}{2}}
$$

by the Fundamental Identity 9. Using the fact that $\cos \left(\frac{\pi}{2}\right)=0$, we deduce:

$$
\sin \left(\frac{\pi}{4}\right)=\cos \left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} .
$$

5. Functions for related angles. Given one angle $t$, there are some angles naturally related to it. These are:

- Supplementary angle $\pi-t$.
- Complementary angle $\pi / 2-t$.
- Reflection at the origin $\pi+t$ which is equivalent to $t-\pi$.
- Reflection in the x axis $-t$.

The trigonometric functions of all these related angles can be easily calculated in terms of $\sin (t), \cos (t)$. We have shown some of the calculations and we leave the rest as an exercise.
6. What about the other functions? You might be wondering about the other four named functions we introduced, namely secant, cosecant, tangent and cotangent. We have not seen them except in the identities 2,3 .
The above results for the sine and cosine functions lead to corresponding results for them. Such results are best derived as needed, rather than memorized.
Thus, for example, you can deduce for yourself that sec is an even function and that tan, cot and csc are odd functions. As a sample proof, see that:

$$
\tan (-t)=\frac{\sin (-t)}{\cos (-t)}=\frac{-\sin (t)}{\cos (t)}=-\tan (t)
$$

where we have used the known properties of sine and cosine functions in the middle.

There is one formula worth noting:

$$
\text { Fundamental Identity } \mathbf{1 0 .} \quad \tan (s+t)=\frac{\tan (s)+\tan (t)}{1-\tan (s) \tan (t)}
$$

The proof consists of using the identities 5 and the variant of the identity 4, together with some easy simplifications. Actually the easiest proof comes by starting from the right hand side and simplifying. We present the steps without explanation:

$$
\begin{aligned}
\frac{\tan (s)+\tan (t)}{1-\tan (s) \tan (t)} & =\frac{\frac{\sin (s)}{\cos (s)}+\frac{\sin (t)}{\cos (t)}}{1-\frac{\sin (s) \sin (t)}{\cos (s) \cos (t)}} \\
& =\frac{\sin (s) \cos (t)+\sin (t) \cos (s)}{\cos (s) \cos (t)-\sin (s) \sin (t)} \\
& =\frac{\sin (s+t)}{\cos (s+t)} \\
& =\tan (s+t)
\end{aligned}
$$

As a consequence, we also have:
Fundamental Identity 11.

$$
\tan (2 t)=\frac{2 \tan (t)}{1-\tan ^{2}(t)}
$$

The proof is trivial, just take $s=t$ in the identity 10. However, it is very useful in many theoretical calculations.
7. A special application to circles. Consider two points, the origin: $O(0,0)$ and the unit point: $U(1,0)$. Consider the locus of all points $P(x, y)$ such that the angle $\angle O P U$ is fixed, say equal to $\alpha$.
What does this locus look like?


Let us use a variation of the Fundamental Identity 10 derived by changing $t$ to $-t$ :
Fundamental Identity 10-variant.. $\quad \tan (s-t)=\frac{\tan (s)-\tan (t)}{1+\tan (s) \tan (t)}$
Let $s$ be the angle made by line $U P$ with the $x$-axis, so we know that

$$
\tan (s)=\text { slope of the line UP }=\frac{y}{x-1}
$$

Let $t$ be the angle made by line $O P$ with the $x$-axis, so we know that

$$
\tan (t)=\text { slope of the line } \mathrm{OP}=\frac{y}{x}
$$

We know that the angle $\angle O P U$ can be calculated as $s-t$. Thus, by our assumption, $s-t=\alpha$. Now by the above variant of the Fundamental Identity 10 we get:

$$
\tan (\alpha)=\tan (s-t)=\frac{\tan (s)-\tan (t)}{1+\tan (s) \tan (t)}=\frac{\frac{y}{x-1}-\frac{y}{x}}{1+\frac{y}{x-1} \cdot \frac{y}{x}}
$$

This simplifies as follows:
The numerator simplifies thus:

$$
\frac{y}{x-1}-\frac{y}{x}=\frac{y(x-(x-1))}{x(x-1)}=\frac{y}{x(x-1)} .
$$

The denominator simplifies thus:

$$
1+\frac{y}{x-1} \cdot \frac{y}{x}=1+\frac{y^{2}}{x(x-1)}=\frac{x(x-1)+y^{2}}{x(x-1)}=\frac{x^{2}+y^{2}-x}{x(x-1)}
$$

Finally:

$$
\tan (\alpha)=\frac{y}{x^{2}+y^{2}-x}
$$

and thus we can rearrange it in a convenient form: ${ }^{9}$

$$
x^{2}+y^{2}-x=\cot (\alpha) y .
$$

This looks like a circle! Indeed, what we have essentially proved is the following well known theorem from Geometry:
Consider two end points $A, B$ of the chord of a circle and let $P$ be any point on the circle. Then the angle $\angle A P B$ has a constant tangent.
In fact, noticing that the points $A, B$ split the circle into two sectors, the angle is constant on each of the sectors and the angle in one sector is supplementary to the angle in the other sector. ${ }^{10}$

[^48]But tangent of an angle is the negative of the tangent of the supplementary angle. How does our equation handle both these cases? Luckily, we don't have to worry. If you take the point $P$ below the $x$-axis, it is easy to see that we are working with the supplementary angles of each of $s, t,(s-t)$ and our equation simply gets multiplied by -1 on both sides. ${ }^{11}$

This is a situation when Algebra is more powerful than the Geometry! We shall make a good use of this formula in our applications.
Observation. Recall that we have done a special case of this earlier in the chapter on circles. There we handled the case of the angle in a semi circle, which, in our current notation corresponds to $\alpha=\pi / 2$. In other words, $\cot (\alpha)=0$. At that time, we did not assume special position for our circle and got a more general formula than our current formula. If necessary, we can redo our current calculations without assuming special positions for the points, but the notation will get messy!
8. Working with a triangle. There are two very important formulas for the calculations with a triangle. These are well worth memorizing.

Notation. For convenience, we assume that the vertices of our triangle are named $A, B, C$ and the corresponding lengths of opposite sides are named by the corresponding small letters $a, b, c$.
Convention. We shall use the letters $A, B, C$ to also denote the corresponding angles themselves, so instead of $\sin (\angle A)$ we shall simply write $\sin (A)$ and similarly for others.


Our picture also shows the foot of the perpendicular from $A$ onto the side $\overline{B C}$ and its length is marked by $h$.

[^49]Name the part $d(B, M)$ as $a_{1}$ and the part $d(M, C)$ as $a_{2}$ so $a=a_{1}+a_{2}$.
It is easy to note:

$$
c \sin (B)=h=b \sin (C)
$$

We note two facts from this:

Fact 1

$$
\frac{c}{\sin (C)}=\frac{b}{\sin (B)}
$$

and
Fact 2.

$$
h^{2}=b c \sin (B) \sin (C)
$$

By obvious symmetry in the symbols, we claim from Fact 1, the
The sine law for a triangle. $\quad \frac{a}{\sin (A)}=\frac{b}{\sin (B)}=\frac{c}{\sin (C)}$.

Now we develop a formula for $a^{2}$.
Note that

$$
a^{2}=\left(a_{1}+a_{2}\right)^{2}=a_{1}^{2}+a_{2}^{2}+2 a_{1} a_{2} .
$$

Also note that
Fact 3

$$
a_{1}=c \cos (B), a_{2}=b \cos (C)
$$

Now using the right angle triangles $\triangle A B M$ and $\triangle A M C$, we note:
Fact 4

$$
a_{1}^{2}=c^{2}-h^{2}, a_{2}^{2}=b^{2}-h^{2} .
$$

Using Facts 2,3 and 4 above, we get:

$$
\begin{array}{rl|l}
a^{2} & =a_{1}^{2}+a_{2}^{2}+2 a_{1} a_{2} & \\
& =c^{2}-h^{2}+b^{2}-h^{2}+2 a_{1} a_{2} & \text { Fact 4 } \\
& =b^{2}+c^{2}-2 h^{2}+2 b c \cos (B) \cos (C) & \text { Fact 3 } \\
& =b^{2}+c^{2}-2 b c \sin (B) \sin (C)+2 b c \cos (B) \cos (C) & \text { Fact 2 } \\
& =b^{2}+c^{2}+2 b c \cos (B+C) & \text { Fundamental Identity 4-variant. }
\end{array}
$$

Since $B+C=\pi-A$, we see from our supplementary angle identity, $\cos (B+C)=$ $-\cos (A)$ and hence we have:

The cosine law for a triangle. $\quad a^{2}=b^{2}+c^{2}-2 b c \cos (A)$.

### 8.9 Using trigonometry.

1. Calculating exact values. So far, we have succeeded in calculating the trigonometric functions for a few special angles $0^{\circ}, 30^{\circ}, 45^{\circ}, 60^{\circ}, 90^{\circ}$ and their related angles.

The half angle formula lets us evaluate many more angles.
Thus

$$
\sin \left(15^{\circ}\right)=\sqrt{\frac{1-\cos \left(30^{\circ}\right)}{2}}
$$

where, we have chosen the plus sign since we know the angle is in the first quadrant. The right hand side is

$$
\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}=\sqrt{\frac{2-\sqrt{3}}{4}}=\frac{\sqrt{6}-\sqrt{2}}{4} .
$$

The last answer may be mysterious. When it is given to you, it is not hard to check it; just square it and see that you get the desired $\frac{2-\sqrt{3}}{4}$.

There is a systematic theory of finding square roots of algebraic expressions like this, called surds. It used to be a standard part of old algebra books; alas, no more!

It is clear that from this $15^{\circ}$ angle, you can keep on halving the angle and get down to a small angle. For several hundred years, people did their astronomy using multiples of $7.5^{\circ}$ angle.

You may ask why one should bother with this. Your calculator will happily report the value 0.2588190451 for $\sin \left(15^{\circ}\right)$. The calculator will also report the same value for our theoretical calculation $\frac{\sqrt{6}-\sqrt{2}}{4}$. You can and should use the calculator for ordinary practical problems, but not when developing the theory. You should also have a healthy suspicion about the accuracy of the reported answer, especially if it has gone through several steps of calculations.
2. Folding Paper. Consider a piece of paper 8 inches wide and 10 inches long.

Fold it so that the top right corner lands on the left edge and the crease starts at the bottom right corner.


We want to determine the position where the crease is made.
Note the names of the corners and observe that we must have:

$$
d(L, D)=d(A, D)=10 \text { and } d(C, D)=8
$$

By the Pythagorean theorem, we have:

$$
d(L, C)=\sqrt{10^{2}-8^{2}}=\sqrt{36}=6 .
$$

Thus, we get:

$$
d(L, B)=10-6=4
$$

Now a little argument shows that: ${ }^{12}$

$$
\angle B L M=\angle C D L
$$

Since $\tan (\angle C D L)=\frac{6}{8}=\frac{3}{4}$ we get that $\tan (\angle B L M)=\frac{3}{4}$. Hence

$$
d(B, M)=d(B, L) \tan (\angle B L M)=4 \frac{3}{4}=3 .
$$

Thus the crease is made at 3 inches from the top left corner!

[^50]Since $\triangle L C D$ is clearly a right angle triangle, we get:

$$
\angle C D L+\angle D L C=90^{\circ} .
$$

The conclusion follows from these two displayed equations!

Note: We used the given dimensions. It is instructive to make similar calculations using variable names for the width and length, say $w$ and $l$ respectively. The reader should attempt to prove the formula using similar calculations!

$$
d(B, M)=d(B, L) \tan (\angle B L M)=\left(l-\sqrt{l^{2}-w^{2}}\right) \frac{\sqrt{l^{2}-w^{2}}}{w}
$$

3. Estimating heights. Consult the picture below for reference points.

A tall building $B C$ is observed from the position $A$ on the ground. The top $C$ of the building appears to have an elevation of $26.56^{\circ}$. This means the segment $\overline{A C}$ makes an angle of $26.56^{\circ}$ with the horizontal $\overline{A B}$.

If we observe the top of the building from point $M 200$ feet closer to the building () then it has an elevation of $32^{\circ}$. Estimate the height of the building.


Answer: Even though, we are given some concrete values, the recommended procedure is to name everything, finish calculations and then plug in values.

Let the angles at $A$ and $M$ be denoted by $s$ and $t$ respectively. Thus $s=26.55^{\circ}$ and $t=32^{\circ}$. Let $h$ be the height of the building $d(B, C)$ and let $k$ be the unknown distance $d(A, B)$. We have two equations:

E1

$$
\tan (s)=\frac{h}{k}
$$

and

$$
\tan (t)=\frac{h}{k-200}
$$

We wish to solve for $h, k$ from these two equations.
Rearrangement gives:
E3. $\quad h-\tan (s) k=0, h-\tan (t) k=-200 \tan (t)$
Subtracting the second equation in E3 from the first, we get
$E 4$.

$$
(\tan (t)-\tan (s)) k=200 \tan (t)
$$

Thus

$$
k=\frac{200 \tan (t)}{(\tan (t)-\tan (s))}
$$

Since we really need $h$ and not $k$, we use the first equation of E3 to deduce:

$$
h=k \tan (s)=\frac{200 \tan (s) \tan (t)}{(\tan (t)-\tan (s))} .
$$

It remains to plug in the known angles and evaluate in a calculator. You should verify that it gives an answer of 499.88 feet. ${ }^{13}$

### 8.10 Proof of the Addition Formula.

Optional section. As we have already seen, there is yet no nice way to calculate sine and cosine values for random angles except for what the calculator announces. But these functions have been in use for well over 2000 years. What did people do before the calculators were born?

The technique was to develop an addition formula - a way of computing the functions for a sum of two angles if we know them for each angle. This single formula when used cleverly, lets us create a table of precise values for several angles and then we can interpolate (intelligently guess) approximate values for other angles. Indeed a table of 96 values was extensively used for several hundred years of astronomy and mathematics.

So, let us work on this proof. You may find some of the arguments repeated from the earlier section, to make this section self contained.

First, let us recall the

## The Fundamental Identity

$$
\sin ^{2}(t)+\cos ^{2}(t)=1
$$

which is evident from the definition of the trigonometric functions.
One main idea is this. Given two points $P(s)$ and $P(t)$ on the unit circle, it is clear that the distance between them depends only on the difference of their angles, i.e. $(t-s)$. This is clear since the circle is such a symmetric figure that the distance traveled along the circle should create the same separation, no matter where we take the walk.

Thus, we can safely conclude that

$$
\begin{equation*}
d(P(s), P(t))=d(P(0), P(t-s)) \tag{*}
\end{equation*}
$$

[^51]This is the only main point, the only thing that needs some thought!
The rest is pure algebraic manipulation, as we now show. We shall next evaluate the distance, simplify the expression and get what we want. Before we begin, let us agree to square both sides, since we want to avoid handling the square root in the distance formula.

Note that by definition $P(t)=(\cos (t), \sin (t))$. Also note that

$$
P(0)=(\cos (0), \sin (0))=(1,0)
$$

from the known position of $P(0)$.
Using similar expressions for points $P(s), P(t)$ and $P(t-s)$ in $(*)$, we get:

$$
(\cos (t)-\cos (s))^{2}+(\sin (t)-\sin (s))^{2}=(\cos (t-s)-1)^{2}+(\sin (t-s)-0)^{2} .
$$

Expansion and rearrangement of the left hand side (LHS) gives:

$$
L H S=\cos ^{2}(t)+\sin ^{2}(t)+\cos ^{2}(s)+\sin ^{2}(s)-2(\cos (t) \cos (s)+\sin (t) \sin (s))
$$

Using the fundamental identity, this simplifies to:

$$
L H S=1+1-2(\cos (t) \cos (s)+\sin (t) \sin (s)) .
$$

Similarly, the right hand side (RHS) becomes:

$$
\left.R H S=\cos ^{2}(t-s)+\sin ^{2}(t-s)\right)+(1+0)-2(\cos (t-s)(1)+\sin (t-s)(0))
$$

After simplification, it gives:

$$
R H S=1+1-2(\cos (t-s)) .
$$

Comparing the two sides, we have our addition formula: ${ }^{14}$
Fundamental Identity 4. $\quad \cos (t-s)=\cos (t) \cos (s)+\sin (t) \sin (s)$

### 8.11 Identities Galore.

Optional section. The fourth fundamental identity together with known values leads to a whole set of new identities. Deriving them is a fruitful, fun exercise. We have already learned most of these, but we have not seen the proofs. This section is designed to guide you through the proofs.

[^52]1. Take $t=0$ in the fourth fundamental identity and write:

$$
\cos (-s)=\cos (0) \cos (s)+\sin (0) \sin (s)
$$

and using the known values $\cos (0)=1, \sin (0)=0$ we get that
Evenness of the cosine function. $\cos (-s)=\cos (s)$
Note: Recall that function $y=f(x)$ is said to be even if $f(-x)=f(x)$ for all $x$ in its domain. It is said to be an odd function, if, on the other hand $f(-x)=-f(x)$ for all $x$ in the domain. Geometrically, the evenness is recognized by the symmetry of the graph of the function about the $y$-axis, while the oddness is recognized by its symmetry about the origin.
2. We know that

$$
\cos (\pi / 2)=0 \text { and } \sin (\pi / 2)=1
$$

Take $t=\pi / 2$ in the fourth fundamental identity and write:

$$
\cos (\pi / 2-s)=(0) \cos (s)+(1) \sin (s)=\sin (s)
$$

Replace $s$ by $\pi / 2-s$ in this identity and get: ${ }^{15}$

$$
\cos (\pi / 2-(\pi / 2-s))=\sin (\pi / 2-s)
$$

This simplifies to

$$
\cos (s)=\sin (\pi / 2-s)
$$

Definition: Complementary Angles. The angles $s$ and $\pi / 2-s$ are said to be complementary angles. Indeed, in a right angle triangle, the two angles different from the right angle are complementary. But our definition is more general.
Thus we have established
Complementary Angle Identity.

$$
\cos (\pi / 2-s)=\sin (s), \quad \sin (\pi / 2-s)=\cos (s)
$$

3. Definition: Supplementary Angles. The angles $s$ and $\pi-s$ are said to be supplementary angles. We continue as above, except we now note that $P(\pi)=(-1,0)$ and take $t=\pi$ in the fourth fundamental identity.
We get:

$$
\cos (\pi-s)=(-1) \cos (s)+(0) \sin (s)=-\cos (s)
$$

[^53]This is the first part of the Supplementary Angle Identity. Now we want to work out the formula for $\sin (\pi-s)$. This is rather tricky, since we have only a limited number of formulas proved so far. But here is how it goes:

$$
\begin{array}{rl|l}
\sin (\pi-s) & =\sin (\pi / 2+(\pi / 2-s)) & \text { Algebra } \\
& =\sin (\pi / 2-(s-\pi / 2)) & \text { Algebra } \\
& =\cos (s-\pi / 2) & \text { Complementary Angle Identity } \\
& =\cos (\pi / 2-s) \\
& =\sin (s) & \text { Evenness of the cosine function }
\end{array}
$$

Thus we have established
Supplementary Angle Identity. $\cos (\pi-s)=-\cos (s), \sin (\pi-s)=\sin (s)$
4. Now we replace $t$ by $\pi / 2-t$ in the fourth fundamental identity to get:

$$
\begin{array}{ll}
\cos ((\pi / 2-t)-s)) & =\cos (\pi / 2-t) \cos (s)+\sin (\pi / 2-t) \sin (s) \\
\cos (\pi / 2-(t+s)) & =\sin (t) \cos (s)+\cos (t) \sin (s) \\
\sin (t+s) & =\sin (t) \cos (s)+\cos (t) \sin (s)
\end{array}
$$

The second and third line simplifications are done using the complementary angle identities. We record the final conclusion as the fifth fundamental identity:

Fundamental Identity 5. $\quad \sin (t+s)=\sin (t) \cos (s)+\cos (t) \sin (s)$
5. Now take $s=-t$ in the fifth fundamental identity to write:

$$
\sin (t-t)=\sin (t) \cos (-t)+\cos (t) \sin (-t)
$$

Notice that the left hand side is $\sin (0)=0$. Since we know that $\cos (-t)=\cos (t)$ from the evenness of the cosine function, the equation becomes:

$$
0=\cos (t)(\sin (t)+\sin (-t)) .
$$

We deduce that we must have ${ }^{16}$

## Oddness of the sine function.

$$
\sin (-t)=-\sin (t)
$$

[^54]6. It is worth recording the identity:

Fundamental Identity 4-variant. $\cos (t+s)=\cos (t) \cos (s)-\sin (t) \sin (s)$

To do this, simply replace $s$ by $-s$ in the fundamental identity 4 and use the oddness of the sine function.

Armed with the fundamental identities 4 and 5, we can give a whole set of new and useful formulas. We sketch partial argument for some and leave the rest to the reader. Indeed, it is recommended that the reader practices developing more by trying out special cases.

Here is a useful set:

1. Double angle formulas Taking $s=t$ in the fundamental identities 4 and 5 we get two important identities:

Fundamental Identity 6. $\quad \cos (2 t)=\cos ^{2}(t)-\sin ^{2}(t)$

Fundamental Identity 7. $\sin (2 t)=2 \sin (t) \cos (t)$
The fundamental identity 6 has two variations and they both are very useful, so we record them individually.

Fundamental Identity 6-variant 1. $\quad \cos (2 t)=2 \cos ^{2}(t)-1$

## Fundamental Identity 6-variant 2. <br> $$
\cos (2 t)=1-2 \sin ^{2}(t)
$$

These are simply obtained using the fundamental identity 1 to replace $\sin ^{2}(t)$ by $1-\cos ^{2}(t)$ or $\cos ^{2}(t)$ by $1-\sin ^{2}(t)$.
2. Theoretical use of the double angle formula. Remember the complicated proof we had to make to find the sine and cosine of $30^{\circ}$ ?
The double and half angle formulas will make a short work of finding the sine and cosine for $\frac{\pi}{6}=30^{\circ}$ as well as $\frac{\pi}{3}=60^{\circ}$.
Here is the work.
For convenience of writing, let us:

$$
\text { use the notation } s=\frac{\pi}{6} \text { and } t=\frac{\pi}{3} \text {. }
$$

Notice that

$$
s+t=\frac{\pi}{2} \text { and } t=2 s
$$

So by complementary angle identity, we get

$$
\sin (t)=\cos (s) \text { and } \cos (t)=\sin (s)
$$

Since $t=2 s$, from the fundamental identity 7 , we get $\sin (t)=2 \sin (s) \cos (s)$ and from the above we have $\sin (t)=\cos (s)$. Thus we have:

$$
\cos (s)=2 \sin (s) \cos (s)
$$

Since $\cos (s)$ clearly is non zero, we cancel it to deduce

$$
1=2 \sin (s) \text { which means } \sin (s)=\frac{1}{2}
$$

As before, from the fundamental identity 1 we deduce that

$$
\cos (s)= \pm \sqrt{1-\left(\frac{1}{2}\right)^{2}}= \pm \sqrt{\frac{3}{4}}
$$

And in our case we get

$$
\cos (s)=\sqrt{\frac{3}{4}}
$$

due to the position of the angle in the first quadrant. We have thus established:

$$
\cos \left(60^{\circ}\right)=\sin \left(30^{\circ}\right)=\frac{1}{2} \text { and } \cos \left(30^{\circ}\right)=\sin \left(60^{\circ}\right)=\frac{\sqrt{3}}{2}
$$

Due to their frequent use, these formulas are worth memorizing!
3. Half Angle formulas. These are easy but powerful consequences of the above variants of identity 6 .
Replace $t$ by $\frac{t}{2}$ in variant 1 of the fundamental identity 6 to get

$$
\cos (t)=2 \cos ^{2}\left(\frac{t}{2}\right)-1
$$

and solve for $\cos \left(\frac{t}{2}\right)$ :
Fundamental Identity 8. $\quad \cos \left(\frac{t}{2}\right)= \pm \sqrt{\frac{1+\cos (t)}{2}}$

Similarly, replace $t$ by $\frac{t}{2}$ in variant 2 of the fundamental identity 6 to get

$$
\cos (t)=1-2 \sin ^{2}\left(\frac{t}{2}\right)
$$

and solve for $\sin \left(\frac{t}{2}\right)$ :
Fundamental Identity 9. $\quad \sin \left(\frac{t}{2}\right)= \pm \sqrt{\frac{1-\cos (t)}{2}}$
The $\pm$ sign comes due to the square root and has to be determined by the location of the angle $\frac{t}{2}$.
The second one is similarly obtained from variant 2 of the identity 6 .
Clearly, there is no end to the kind of formulas that we can develop. There is no point in "learning" them all. You should learn the technique of deriving them, namely convenient substitution and algebraic manipulation.

## Chapter 9

## Looking closely at a function

We have visited many functions $y=f(x)$ where $f(x)$ is given by a formula based on a rational function. We have also seen plane algebraic curves given by $F(x, y)=0$ where it is not easy to explicitly solve for $y$ in terms of $x$ or even to express both $x$ and $y$ as nice functions of a suitable parameter.

It is, however, often possible to approximate a curve near a point by nice polynomial parameterization. We can even do this with a desired degree of accuracy. This helps us analyze the behavior of a function which may lack a good formula.

### 9.1 Introductory examples.

We shall begin by discussing two simple examples.

## - Analyzing a parabola.

Consider a parabola given by $y=4 x^{2}-2$. We shall analyze it near a point $P(1,2)$. Like a good investigator, we shall first go near the point by changing our origin to $P$.
As we know, this is easily done by the
Substitution: $x=1+u, y=2+v$ with $u, v$ as the new coordinates.
We shall then treat $\mathbf{u}, \mathbf{v}$ as our local coordinates and

$$
\text { The original equation } \quad y=4 x^{2}-2
$$

becomes:

$$
(2+v)=4(1+u)^{2}-2 \text { or } 2+v=4\left(1+2 u+u^{2}\right)-2
$$

and this simplifies to:
The local equation.

$$
v=4 u^{2}+8 u
$$

Since we are studying the equation near the point $u=0, v=0$ we rewrite the equation in increasing degrees in $u, v$ as

$$
(v-8 u)-4 u^{2}=0
$$

It is clear that when $u, v$ are sufficiently small, the linear part $v-8 u$ is going to be much bigger relative to the quadratic part $-4 u^{2}$. We say that the expression

$$
v-8 u=y-2-8(x-1)=y-8 x+6
$$

is a Linear Approximation near the point $P(1,2)$ to the expression

$$
y-4 x^{2}+2
$$

which describes the parabola when set equal to zero.
We can express this alternatively by saying that the equation $y-8 x+6=0$ is a linear approximation to the equation of the parabola near $P(1,2)$.
The graph below illustrates the closeness between the parabola and its linear approximation near the point $P(1,2)$.


The linear approximation to the equation of the parabola, namely $y-8 x+6=0$ defines a line and is called the tangent line to the parabola at $P(1,2)$.

## Alternative organization of calculations.

There is a different way of doing the above calculation which avoids introducing new variables $u, v$ but requires you to be more disciplined in your work.
For the reader's convenience, we present this alternative technique, which can be used in all the following calculations.
We shall illustrate this by analyzing $y=4 x^{2}-2$ at the same point $(1,2)$.

We agree to rewrite:

$$
x \text { as } 1+(x-1) \text { and } y \text { as } 2+(y-2) .
$$

Then our parabola becomes:

$$
2+(y-2)=4\left(1+2(x-1)+(x-1)^{2}\right)-2
$$

which rearranges as

$$
(y-2)=(-2+4-2)+(4)(2)(x-1)+4(x-1)^{2}=8(x-1)+4(x-1)^{2} .
$$

Thus we again get

$$
(y-2)=8(x-1)+\text { higher degree terms in }(x-1) .
$$

and the linear approximation is again:

$$
(y-2)-8(x-1) \text { or } y-8 x-6 .
$$

The reader may follow this alternate procedure if desired.
Now, let us find the tangent to the same curve at another point while listing the various steps.

- Choose point. Choose the point with $x$-coordinate -1 . Its $y$-coordinate must be $4(-1)^{2}-2=2$. Thus the point is $Q(-1,2)$.
- Shift the origin. We move the origin to the point $Q(-1,2)$ by

$$
x=-1+u, y=2+v .
$$

The equation transforms to: ${ }^{1}$
$(2+v)=4(-1+u)^{2}-2$ or $v=\left(4\left(1-2 u+u^{2}\right)-2\right)-2$ i.e. $v=4 u^{2}-8 u$.

- Get the linear approximation and the tangent line. This is rewritten in increasing degree terms as

$$
v+8 u-4 u^{2}=0
$$

The linear approximation is:

$$
v+8 u \text { or }(y-2)+8(x+1) \text { simplified to } y+8 x+6 \text {. }
$$

Thus the tangent line at this point must be

$$
v+8 u=0 \text { or } y+8 x+6=0 \text { or } y=-8 x-6 \text {. }
$$

[^55]Now let us do such calculations once and for all for a general point $(a, b)$.
First make the origin shift: $x=a+u, y=b+v$. Then rearrange the equation:

$$
y=4 x^{2}-2 \text { becomes } b+v=4(a+u)^{2}-2
$$

After a little rearrangement, this becomes simplified local equation:

$$
\left(b-4 a^{2}+2\right)+(v-8 a u)-4 u^{2}=0
$$

Note that this final form tells us many things:

1. The point $(a, b)$ is on our parabola if and only if the constant term vanishes. This means that $b-4 a^{2}+2=0$ or $b=4 a^{2}-2$.
2. If the point is on the parabola, then the linear approximation is

$$
v-8 a u=(y-b)-8 a(x-u) \text { or } y-8 a x+8 a u-b .
$$

3. The tangent line is

$$
v-8 a u=0 \text { or } v=8 a u
$$

In original coordinates, it becomes ${ }^{2}$

$$
y-b=8 a(x-a) \text { and is rearranged to } y=8 a x+\left(b-8 a^{2}\right)
$$

4. The slope of the tangent line at the point $(a, b)$ is $8 a$ and we could simply write down its equation using the point slope formula.
Thus, we really only need to know the slope of the tangent 8a and we could write down the equation of the tangent line.

## - Analyzing a circle.

Now consider a circle $x^{2}+y^{2}=25$ of radius 5 centered at the origin. Consider the point $P(3,4)$ on it. We shall repeat the above procedure and analyze the equation near $P$. So, make the shift $x=3+u, y=4+v$. The equation becomes:

$$
(3+u)^{2}+(4+v)^{2}=25 \text { or } u^{2}+6 u+9+v^{2}+8 v+16=25 .
$$

When simplified and arranged by increasing degrees we get:

[^56]$$
(6 u+8 v)+\left(u^{2}+v^{2}\right)=0 .
$$
${ }^{3}$ Thus we know that the tangent line must be
$$
6 u+8 v=0 \text { or } 6(x-3)+8(y-4)=0 \text { or } y=-\frac{6}{8} x+\frac{50}{8}
$$

Thus, the slope of the tangent line is $-\frac{3}{4}$ and the $y$-intercept is $\frac{25}{4}$.


As before, the linear approximation to the circle near the point $P(3,4)$ is given by

$$
6 u+8 v=6(x-3)+8(y-4)=6 x+8 y-50 .
$$

Now we try to do this for a general point $(a, b)$ of the circle. As before, make the shift $x=a+u, y=b+v$. Substitute and rearrange the equation of the circle:

$$
x^{2}+y^{2}-25=0 \text { becomes }\left(a^{2}+b^{2}-25\right)+2(a u+b v)+u^{2}+v^{2}=0 .
$$

As before, we note:

1. The point $(a, b)$ is on our circle if and only if $a^{2}+b^{2}=25$.
2. If the point is on the circle, then the tangent line is $2 a u+2 b v=0$ or $a u+b v=0$ after dropping a factor of 2 .
In original coordinates, it is $a(x-a)+b(y-b)=0$ and rearranged to $y=-\frac{a}{b} x+\frac{a^{2}+b^{2}}{b}$. This simplifies to: $y=-\frac{a}{b} x+\frac{25}{b}$.
3. The slope of the tangent line for the point $(a, b)$ is $-\frac{a}{b}$ and we could simply write down the equation using the point slope formula.

[^57]4. Indeed, we note that the slope of the line joining the center $(0,0)$ to the point $(a, b)$ is $\frac{b}{a}$ and hence this line is perpendicular to our tangent with slope $-\frac{a}{b}$. Thus, we have essentially proved the theorem that the radius of a circle is perpendicular to its tangent. This, in fact, is well known for well over 2000 years, since the time of the Greek geometry at least.

### 9.2 Analyzing a general curve $y=f(x)$ near a point ( $a, f(a)$ ).

Using the experience of the parabola, let us formally proceed to analyze a general function. We follow the same steps without worrying about the geometry.

We formally state the procedure and then illustrate it by a few examples.

1. Consider the point $P(a, b)$ on the curve $y=f(x)$ where, for now, we assume that $f(x)$ is a polynomial in $x$. Since $P(a, b)$ is on the curve, we have $b=f(a)$.
2. Substitute $x=a+u, y=b+v$ in $y-f(x)$ to get $b+v-f(a+u)$. Expand and rearrange the terms as: ${ }^{4}$

$$
(b-f(a))+(v-m u)+(\text { higher degree terms in } u, v) .
$$

3. The constant term $b-f(a)$ is zero by assumption and we declare the equation of the tangent line to $y=f(x)$ at $x=a$ to be the linear part set to zero:

$$
v-m u=0 \text { or } y-b=m(x-a) \text { or, finally } y=m x+(b-m a) .
$$

We may also call it the tangent line at the point $(a, f(a))$ and the expression $v-m u$ or $y-b-m(x-a)$ is said to be a linear approximation at this point to the curve $y-f(x)$.
Definition: Linear approximation to a function. We now define the linear approximation of a function $f(x)$ near $x=a$ to be simply

$$
f(a)+m(x-a) .
$$

Let us note that then

$$
f(x)=\text { linear approximation }+ \text { terms involving higher powers of }(x-a)
$$

and we could simply take this as a direct definition of a linear approximation of a function without discussing the curve $y=f(x)$.

[^58]4. Thus we note that we only need to find the number $m$ which can be defined as follows:

Definition: The derivative of a polynomial function $f(x)$ at a point a. A number $m$ is the derivative of $f(x)$ at $x=a$ if $f(a+u)-f(a)-m u$ is divisible by $u^{2}$.

Let us name the desired number $\mathbf{m}$ as something related to $f$ and $a$ and the customary notation is $\mathbf{f}^{\prime}(\mathbf{a})$, a quantity to be derived from $f$ and $a$, or the so-called derivative of $f$ at $x=a$.
This gives us a better way of remembering and using our definition of the derivative:

The derivative $f^{\prime}(a)$ is defined as that real number for which

$$
f(a+u)=f(a)+f^{\prime}(a) u+\text { terms divisible by } u^{2} .
$$

Here are some examples of the above procedure.

1. Let $f(x)=p$, where $p$ is a constant. Then we have:

$$
f(a+u)=p=f(a)+0
$$

so our $m$ must be 0 .
2. Let $f(x)=p x$, where $p$ is a constant. Then we have:

$$
f(a+u)=p(a+u)=p a+p u=f(a)+p u
$$

so our $m$ is simply $p$.
3. $f(x)=p x^{2}$, where $p$ is a constant. Then we have:

$$
f(a+u)=p(a+u)^{2}=p a^{2}+(2 p a) u+(p) u^{2}=f(a)+2 p a u+u^{2}(p)
$$

so our $m$ is simply $2 p a$.
4. $f(x)=p x^{3}$, where $p$ is a constant. Then we have:
$f(a+u)=p(a+u)^{3}=p a^{3}+\left(3 p a^{2}\right) u+(3 p a) u^{2}+(p) u^{3}=f(a)+3 p a^{2} u+u^{2}(3 p a+p u)$ so our $m$ is simply $3 p a^{2}$.

Clearly we can go on doing this, but it involves more and more calculations. We need to do it more efficiently. The true power of algebra consists of making a good definition and recognizing a pattern!

### 9.3 The slope of the tangent, calculation of the derivative.

As we know, given a function $f(x)$ we wish to substitute $a+u$ for $x$ and rearrange the expanded terms $f(a+u)$ as $f(a)+m u+$ higher degree terms .

A more convenient notation for the derivative. We note that the parameter $a$ used in the above analysis is just a convenient place holder and $f^{\prime}(a)$ is really a function of $a$. Hence, there is no harm in using the same letter $x$ as a variable name again.

Thus, for a function $f(x)$ we will record its derivative $f^{\prime}(a)$ as simply $f^{\prime}(x)$. As an example, note that our calculation for the parabola $y=f(x)=4 x^{2}-2$ with $f^{\prime}(a)=8 a$ can thus be recorded as $f^{\prime}(x)=8 x$.

Finally, we introduce two shortcuts to our notation.
We will find it convenient not to introduce the function name $f$ and would like to denote the derivative directly in terms of the symbol $y$. We would, thus prefer to write $y^{\prime}=8 x$. If the mark ' gets confusing we shall use the notation: $D_{x}(y)=8 x$, where the symbol $D$ denotes the word derivative and the subscript $x$ reminds us of the main variable that $y$ is a function of.

Thus, our earlier results for the derivatives of $p x^{n}$ for $n=1,2,3$ can be restated as

$$
D_{x}(p x)=p, D_{x}\left(p x^{2}\right)=2 p x, D_{x}\left(p x^{3}\right)=3 p x^{2}
$$

We formally declare that we shall use the notation: $D_{x}(y)$ to denote the derivative $f^{\prime}(x)$ of the function $y=f(x)$ even though we have no explicit name $f$ declared.

We note that there is yet another popular notation $\frac{d}{d x}(y)$ or $\frac{d y}{d x}$ for writing the derivative, but we will mostly use the $D_{x}$ notation. ${ }^{5}$

We begin by making some very simple observations:

## Formula 1 The constant multiplier.

If $c$ is a constant and $g(x)=c f(x)$, then $g^{\prime}(a)=c f^{\prime}(a)$. In better notation:

$$
D_{x}(c y)=c D_{x}(y)
$$

Proof: Write $f(a+u)=f(a)+f^{\prime}(a) u+u^{2}$ ( more terms ). Multiplying by $c$ we see that $c f(a+u)=c f(a)+c f^{\prime}(a) u+u^{2}$ ( more terms ) which says that $g(a+u)=g(a)+c f^{\prime}(a) u+u^{2}($ more terms $)$ and hence we get $g^{\prime}(a)=c f^{\prime}(a)$.

[^59]Formula 2 The sum rule. Suppose that for two functions $f(x), g(x)$ we have the derivatives calculated.
Then

$$
(f+g)^{\prime}(x)=f^{\prime}(x)+g^{\prime}(x)
$$

## In better notation:

$$
D_{x}(y+z)=D_{x}(y)+D_{x}(z) .
$$

Proof. We have:

$$
f(a+u)=f(a)+f^{\prime}(a) u+u^{2}(\cdots) \text { and } g(a+u)=g(a)+g^{\prime}(a) u+u^{2}(\cdots)
$$

Then we clearly have:

$$
(f+g)(a+u)=f(a+u)+g(a+u)=f(a)+g(a)+\left(f^{\prime}(a)+g^{\prime}(a)\right) u+u^{2}(\cdots) .
$$

This proves that
$(f+g)^{\prime}(a)=f^{\prime}(a)+g^{\prime}(a)$ or, in compact notation $D_{x}(y+z)=D_{x}(y)+D_{x}(z)$.
Formula 3 The product rule. As above, we assume that the derivatives $f^{\prime}(a), g^{\prime}(a)$ are known. We now consider the product function $h(x)=f(x) g(x)$.

$$
h^{\prime}(x)=f^{\prime}(x) g(x)+f(x) g^{\prime}(x)
$$

## In better notation:

$$
D_{x}(y z)=y D_{x}(z)+D_{x}(y) z .
$$

Proof. We have:
$h(a+u)=f(a+u) g(a+u)=\left[f(a)+f^{\prime}(a) u+u^{2}(\cdots)\right]\left[g(a)+g^{\prime}(a) u+u^{2}(\cdots)\right]$.
The right hand side simplifies to:

$$
f(a) g(a)+\left[f(a) g^{\prime}(a)+f^{\prime}(a) g(a)\right] u+u^{2}(\cdots)
$$

Thus we get the product rule:

$$
h^{\prime}(a)=(f g)^{\prime}(a)=f(a) g^{\prime}(a)+f^{\prime}(a) g(a)
$$

or, in compact notation:

$$
D_{x}(y z)=y D_{x}(z)+D_{x}(y) z .
$$

We illustrate the use of this formula by calculating the derivative of $h(x)=x^{4}$. Write $h(x)=g(x) g(x)$ where $g(x)=x^{2}$. Note that we have already determined that $g^{\prime}(a)=2 a$. Then by the product formula we get $h^{\prime}(a)=g(a) g^{\prime}(a)+$ $g^{\prime}(a) g(a)=a^{2}(2 a)+(2 a) a^{2}=4 a^{3}$. Remember, we record this conveniently as

$$
h^{\prime}(x)=D_{x}\left(x^{4}\right)=4 x^{3}
$$

We generalize this in the next formula.
Formula 4 The general power of $x$. If $f(x)=x^{n}$ where $n=1,2, \cdots$, then $f^{\prime}(x)=n x^{n-1}$.

## In better notation:

$$
D_{x}\left(x^{n}\right)=n x^{n-1} \text { for } n=1,2, \cdots .
$$

Proof. We already know this for $n=1,2,3,4$ from the previous section. We now assume that we have handled all the values of $n$ up to some $k$ and show how to do the next value of $k+1$. ${ }^{6}$
Thus we assume that $f(x)=x^{k+1}=(x)\left(x^{k}\right)=g(x) h(x)$ say. By what we have assumed, we know that $g^{\prime}(x)=1$ and $h^{\prime}(x)=k x^{k-1}$. Then the product formula gives

$$
f^{\prime}(x)=g(x) h^{\prime}(x)+g^{\prime}(x) h(x)=x\left(k x^{k-1}\right)+(1) x^{k}=(k+1) x^{k} .
$$

Thus we have proved our formula for the next value of $k+1$.
We are done with the induction!
Formula 5 The combined formula. Consider a polynomial in $x$ with constant coefficients $p_{0}, p_{1}, \cdots, p_{n}$ :

$$
f(x)=p_{0} x^{n}+p_{1} x^{n-1}+\cdots+p_{n-1} x+p_{n}
$$

Then its derivative is given by

$$
f^{\prime}(x)=n p_{0} x^{n-1}+(n-1) p_{1} x^{n-2}+\cdots+p_{n-1}
$$

## In compact notation:

$$
D_{x}\left(p_{0} x^{n}+p_{1} x^{n-1}+\cdots+p_{n-1} x+p_{n}\right)=n p_{0} x^{n-1}+(n-1) p_{1} x^{n-2}+\cdots+p_{n-1} .
$$

[^60]This is easily seen by combining the formulas $1,2,4$.
Thus if

$$
f(x)=3 x^{4}-5 x^{2}+x-5
$$

then

$$
f^{\prime}(x)=(4)(3) x^{3}-(2)(5) x+1=12 x^{3}-10 x+1
$$

### 9.4 Derivatives of more complicated functions.

Optional section. The proofs in this section can be omitted in a first reading. Many of the formulas can serve as shortcuts, but can be avoided, if you are willing to work a bit longer!

Even though we defined the derivatives only for a polynomial function so far, we have really worked them out for even more complicated functions, without realizing it!

Just consider our tangent line calculation for the circle where the explicit function would have involved a square root of a quadratic function! $\left(y=\sqrt{25-x^{2}}\right)$.

Inspired by our calculation with the circle, we shall attempt to calculate the derivative of $\frac{1}{x}$.

We begin by setting $y=\frac{1}{x}$ and simplifying it into a polynomial equation $x y=1$. Following earlier practice, let us substitute $x=a+u, y=b+v$ and simplify the resulting expression:

$$
x y-1=(a+u)(b+v)-1=(a b-1)+(a v+b u)+(u v) .
$$

Thus the point $P(a, b)$ is on the curve if and only if $a b-1=0$, i.e. $b=\frac{1}{a}$ and we get:

$$
\text { The tangent line } a v+b u=a(y-b)+b(x-a)=a y+b x-2 a b=0
$$

Thus the tangent line at $x=a, y=\frac{1}{a}$ is given by

$$
a y+\frac{x}{a}=2 \text { or } y=\frac{2}{a}-\frac{1}{a^{2}} x .
$$

Thus the slope of the tangent is $-\frac{1}{a^{2}}$ and it makes sense to declare that

$$
D_{x}\left(\frac{1}{x}\right)=D_{x}\left(x^{-1}\right)=-\frac{1}{x^{2}}=-x^{(-2)} .
$$

The reader should note that formally, it follows the same pattern as our earlier power formula.

Would this follow the original definition of the derivative in some new sense? Yes indeed! Let us set $f(x)=\frac{1}{x}$. Consider (and verify) these calculations:

$$
f(a+u)=\frac{1}{a+u}=\frac{1}{a}+\frac{-u}{a^{2}+a u}=-\frac{1}{a^{2}} \cdot u \cdot \frac{1}{a+u / a} .
$$

Thus, we have, after further manipulation:

$$
f(a+u)=f(a)+\left(-\frac{1}{a^{2}}\right) u+\frac{u^{2}}{(a+u) a^{2}} .
$$

The last term, is $u^{2}$ times a rational function (instead of a polynomial) and it does not have $u$ as a factor of the denominator. Thus we simply have to modify our definition of the derivative to allow for such complications:

Enhanced definition of the derivative of a rational function $\mathbf{f}(x)$ at a point a. A number $m$ is the derivative of $f(x)$ at $x=a$ if $f(a+u)-f(a)-m(u)$ is equal to $u^{2}$ times a rational function of $u$ which does not have $u$ as a factor of its denominator.

Of course, this definition is only needed as a formality; our calculation with the polynomial equation $x y=1$ had given us the answer neatly anyway!

So, let us make a
Final definition of the derivative of an algebraic function. Suppose we have a polynomial relation $f(x, y)=0$ between variables $x, y$. Suppose that $P(a, b)$ is a point of the resulting curve so that $f(a, b)=0$. Further, suppose that the substitution $x=a+u, y=b+v$ results in a simplified form

$$
f(a+u, b+v)=r u+s v+\text { terms of degree } 2 \text { or more in }(u, v)
$$

where $r, s$ are constants such that at least one of $r, s$ is non zero. Then

$$
r u+s v=r(x-a)+s(y-b)
$$

is said to be a linear approximation of $f(x, y)$ at $x=a, y=b$. The line $r(x-a)+$ $s(y-b)=0$, is defined to be the tangent line to the curve $f(x, y)=0$ at $P(a, b)$.

Finally, if $s \neq 0$ then $y$ can be described as a well defined algebraic function of $x$ near the point $P(a, b)$ and we can define $D_{x}(y)=-\frac{r}{s}$ or the slope of the tangent line. In case $s=0$, sometimes we may be still able to define an algebraic function, but it will not be well defined.

The above paragraph leaves many details out and the interested reader in encouraged to consult higher books on algebraic geometry to learn them. The aim of the calculations and the definition is to illustrate how mathematics develops. We do some simple calculations which lead to possibly new ideas for functions which we did not have before. In this manner, many new functions are born.

### 9.5 General power and chain rules..

We give proofs of two powerful rules. The student should learn to use these before worrying about the proof.

Armed with these new ideas, let us push our list of formulas further.
Formula 6 Enhanced power formula. Can we handle a function like $y=x^{\frac{2}{3}}$ ? Yes, indeed.
We shall show: $D_{x}\left(x^{n}\right)=n x^{n-1}$ when $n$ is any rational number. ${ }^{7}$
Proof: ${ }^{8}$ First suppose $n=\frac{p}{q}$ where $p, q$ are positive integers. Let $y=x^{n}$. Consider a polynomial $f(X, Y)=Y^{q}-X^{p}$. Substituting $x$ for $X$ and $y$ for $Y$, note that:

$$
f(x, y)=0
$$

We know that $D_{x}\left(x^{p}\right)=p x^{p-1}$, so

$$
(a+u)^{p}=a^{p}+p a^{p-1} u+u^{2}(\text { higher degree terms in } u)
$$

Similarly, we must have $D_{y}\left(y^{q}\right)=q y^{q-1}$ and hence:

$$
(b+v)^{q}=b^{q}+q b^{q-1} v+v^{2}(\text { higher degree terms } v)
$$

It follows that

$$
\begin{aligned}
f(a+u, b+v)=(b+v)^{q}-(a+u)^{p}= & \left(b^{q}-a^{p}\right)+q b^{q-1}(v)-p a^{p-1}(u) \\
& +(\text { terms of degree } 2 \text { or more in }(u, v)) .
\end{aligned}
$$

Thus $(a, b)$ is a point on the curve if and only if $a^{p}=b^{q}$ and the tangent line is given by

$$
q b^{q-1}(v)-p a^{p-1}(u)=0 \text { or } q b^{q-1}(y-b)-p a^{p-1}(x-a)=0 .
$$

Note that since $a^{p}=b^{q}$.

$$
\frac{p a^{p-1}}{q b^{q-1}}=\frac{p}{q} \frac{a^{-1}}{b^{-1}}=n \frac{b}{a} .
$$

Finally, since $b=a^{n}=a^{\frac{p}{q}}$, we get

$$
\frac{b}{a}=a^{\frac{p}{q}-1}=a^{n-1} .
$$

Thus the slope of the tangent is: $n a^{n-1}$.

[^61]In particular, we get that $D_{x}(y)=n x^{n-1}$ as required! Our claim is thus proved for all positive fractional powers.
We now think of negative powers, i.e. we assume that $y=x^{-n}$ where $n$ is a positive rational number. Set $z=x^{n}$ and note that $y z=1$.
Our product rule (Formula 3) says that $D_{x}(1)=y D_{x}(z)+D_{x}(y) z$ or from known facts

$$
0=y n x^{n-1}+D_{x}(y) x^{n}
$$

When solved for $D_{x}(y)$, we get ${ }^{9}$

$$
D_{x}(y)=\frac{-y n x^{n-1}}{x^{n}}=-\frac{x^{-n} n x^{n}}{x^{n}}=-\frac{n}{x^{n}} .
$$

Thus, we have

$$
D_{x}\left(x^{-n}\right)=-n x^{-n}
$$

We thus have the: General power rule.

$$
D_{x}\left(x^{n}\right)=n x^{n-1} \text { where } n \text { is any rational number. }
$$

We should point out that the function $x^{n}$ has definition problems when $x \leq 0$ and our derivatives usually work only for positive $x$ unless $n$ is special.

Formula 7 The chain rule. We now know how to find the derivative of $g(x)=x^{2}+x+1$ and also we know how to find the derivative of $f(x)=x^{5}$. Can we find the derivative of $f(g(x))=\left(x^{2}+x+1\right)^{5}$ ?
Of course, we could expand it out and use our combined formula, but that is not so pleasant!

It would be nice to have a simple mechanism to write the derivative of a composite function. Let us work it out.
Write

$$
g(a+u)=g(a)+m u+u^{2} w
$$

where we note that $m=g^{\prime}(a)$ and $w$ contains remaining terms. For convenience, set

$$
h=m u+u^{2}(w),
$$

so we get:

$$
f(g(a+u))=f(g(a)+h)
$$

[^62]and then by definition of the derivative of $f(x)$ we can write
$$
f(g(a+u))=f(g(a))+m^{*} h+h^{2} r
$$
where $m^{*}=f^{\prime}(g(a))$ and $r$ contains the remaining terms. Note that $h$ is divisible by $u$ and hence the part $h^{2} r$ is divisible by $u^{2}$. Thus we can write $h^{2} r=u^{2} s$ for some $s$.
Combining terms, we write:
$f(g(a+u))=f(g(a))+m^{*}\left(m u+u^{2} w\right)+u^{2} s=f(g(a))+m^{*} m u+u^{2}\left(m^{*} w+s\right)$
and hence we deduce that the derivative of $f(g(x))$ at $x=a$ is
$$
m^{*} m=f^{\prime}(g(a)) g^{\prime}(a)
$$

Thus we write

$$
D_{x}\left(f(g(x))=f^{\prime}(g(x)) g^{\prime}(x)\right.
$$

For example, with our $f(x)=x^{5}$ and $g(x)=x^{2}+x+1$ as stated above we get

$$
D_{x}\left(\left(x^{2}+x+1\right)^{5}\right)=f^{\prime}\left(x^{2}+x+1\right) g^{\prime}(x)=5\left(x^{2}+x+1\right)^{4}(2 x+1) .
$$

For instance, at $x=1$ we have the derivative of $\left(x^{2}+x+1\right)^{5}$ equal to

$$
5(1+1+1)^{4}(2+1)=(5)(81)(3)=1215
$$

and hence the tangent line to $y=\left(x^{2}+x+1\right)^{5}$ at $x=1$ would have the equation

$$
y-(1+1+1)^{5}=(1215)(x-1)
$$

simplified to $y=1215 x+243-1215$ or further simplified to $y=1215 x-972$. As before, if our functions get more complicated, the chain rule is rather subtle to prove and we need to be careful in providing its proof.

Let us, however, note the following rule for the derivative of a power of a function that can be deduced from the above example:

$$
D_{x}\left(f(x)^{n}\right)=n f(x)^{(n-1)} D_{x}(f(x))
$$

When the theory is completely developed, this formula will work for any $n$ for which $f(x)^{n}$ can make sense.

### 9.6 Using the derivatives for approximation.

As we discussed in the introductory section on functions, in practice, the calculation of $f(a)$ for a given function $f(x)$ can be difficult and even impossible for real life functions given only by a few data points. If for some reason, we know the function to be linear, then obviously it is easy to calculate its values quickly.

If a function is behaving nicely near a point, we can act as if it is a linear function near the point and hence can take advantage of the ease of calculation (at the cost of some accuracy, of course). In the days before calculators, many important functions were often used as tables of values at equally spaced values of $x$ and for values in between then, one used an "interpolation", i.e. guess based on the assumption that the function was linear in between.

Even when modern calculators or computers give a value of the function to a desired accuracy, they are really doing higher order versions of linear approximations and doing them very fast; thus giving the appearance of a complete knowledge of the function. We will illustrate some of the higher approximations below.

## Examples of linear approximations.

## 1. Find an approximate value of $\sqrt{3.95}$ using linear approximation.

Since we are discussing a square root, consider the function

$$
f(x)=\sqrt{x}=x^{1 / 2} .
$$

We know $f(4)=2$ precisely. So we find a linear approximation to $f(x)$ near $x=4$.

We have already figured out that

$$
f^{\prime}(x)=D_{x}\left(x^{\frac{1}{2}}\right)=\frac{1}{2} x^{-\frac{1}{2}}=\frac{1}{2 \sqrt{x}},
$$

using the enhanced power formula.
Thus the linear approximation

$$
\begin{gathered}
L(x)=f(a)+f^{\prime}(a)(x-a) \text { becomes } \\
L(x)=\sqrt{4}+\left(\frac{1}{2 \sqrt{4}}(x-4)\right) \text { or } L(x)=2+\frac{x-4}{4} .
\end{gathered}
$$

Thus the estimated value of $\sqrt{3.95}$ using the linear estimate is

$$
L(3.95)=2+\frac{3.95-4}{4}=1.9875
$$

Note that any reasonable calculator will spit out the answer 1.987460691 which is certainly more accurate, but not as easy to get to! Considering the ease of getting a value correct to 4 decimal places, our method is well justified!

Technically, we can use the linear approximation to calculate any desired value, but clearly it only makes sense near a known good value of $x$.
2. Improving a known approximation. Suppose we start with the above approximation 1.9875 which is not quite right and its square is actually 3.95015625 . Suppose we use $a=3.95015625$ and then $f(a)=\sqrt{a}=1.9875$. Then we get a new linear approximation

$$
L_{1}(x)=1.9875+\frac{x-3.95015625}{3.9750}
$$

If we now set $x=3.95$ and calculate the linear approximation, we get

$$
1.9875+\frac{3.95-3.95015625}{3.9750}=1.987460692
$$

an even better value!
Thus with a repeated application of the linear approximation technique, one can get fairly accurate values. ${ }^{10}$
Indeed, for the square root function, there is a classical algorithm known for over 2000 years to find successive decimal digits to any desired degree of accuracy. This algorithm used to be routinely taught in high schools before the days of the calculators.

A similar but more complicated algorithm exists for cube roots as well.
3. Estimating polynomials: Introduction to higher approximations. Suppose that we are trying to analyze a polynomial function $f(x)$ near a point $x=a$. In principle, a polynomial is "easy" to calculate precisely, but it may not be practical for a large degree polynomial. So, we might consider approximation for it.

Let us discuss a concrete example.
Consider $f(x)=4 x^{4}-5 x^{3}+3 x^{2}-3 x+1$. Let us take $a=1$ and note that $f(a)=f(1)=4-5+3-3+1=0$. What is $f(a+u)$ ?
The reader should calculate:

$$
f(x)=f(a+u)=f(1+u)=4 u^{4}+11 u^{3}+12 u^{2}+4 u .
$$

Thus for value of $x$ near $a=1$, we have $u=x-1$ is small and hence the function $f(x)$ can be approximated by various degree polynomials.

[^63] of the final conclusions. This topic is studied in the branch of mathematics called numerical analysis.

The Linear approximation will be $L(x)=4 u=4(x-1)$. The quadratic approximation will be

$$
Q(x)=12 u^{2}+4 u=12(x-1)^{2}+4(x-1)=8-20 x+12 x^{2} .
$$

The cubic approximation shall be
$C(x)=11 u^{3}+12 u^{2}+4 u=11(x-1)^{3}+12(x-1)^{2}+4(x-1)=-3+13 x-21 x^{2}+11 x^{3}$.
The quartic (degree four) approximation will become the function itself!
Can we calculate the necessary approximations efficiently, without the whole substitution process?
The answer is yes for polynomial functions, since we have already studied the Binomial Theorem.
4. Let $f(x)=x^{12}$. What is a linear approximation near $x=a$ ? We set $x=a+u$ and we know that:

$$
f(x)=f(a+u)=a^{12}+12 C_{1} a^{11} u+\text { terms with higher powers of } u .
$$

By our known formula for the binomial coefficient, we get: $12 C_{1}=\frac{12}{1!}=12$ and so the linear approximation is

$$
L(x)=a^{12}+12 a^{11} u \text { or alternatively } L(x)=a^{12}+12(x-a)
$$

As above, for a quadratic approximation, we can keep the next term and use:
$Q(x)=a^{12}+12 a^{11}(x-a)+\frac{12 \cdot 11}{2!} a^{10}(x-a)^{2}=a^{12}+12 a^{11}(x-a)+66 a^{10}(x-a)^{2}$.
Evidently, we can continue this, if needed.

## Chapter 10

## Root finding

### 10.1 Newton's Method

We now find a new use for the idea of linear approximation, namely given a polynomial $f(x)$ we try to find its root, which means a value $x=a$ such that $f(a)=0$.

For a linear polynomial $f(x)=p x+q$ this is a triviality, namely the answer is $x=-\frac{q}{p}$.

Note that for higher degree polynomials, clearly there could be many answers, as the simple polynomial $x^{2}-1=(x-1)(x+1)$ illustrates. Indeed, if we can factor the polynomial $f(x)$ into linear factors, then it is easy to find the roots; they are simply the roots of the various factors. In general, it is hard to find exact roots of polynomials with real coefficients, unless we are lucky. Moreover, polynomials like $x^{2}+1$ show that there may not even be any roots!

What we are about to describe is a simple idea to find the roots approximately. The idea goes like this:

- Choose a starting point $x=a$ suspected to be close to a root.
- Take the linear approximation $L(x)$ at this $x=a$ for the given function $f(x)$.
- Find the root of $L(x)$, call it $a_{1}$.
- Now take $x=a_{1}$ as a new convenient starting point point and repeat!
- Under reasonable conditions, you expect to land close to a root, after a few iterations of these steps.

Let us try an example:
Let $f(x)=x^{2}-2$. By algebra, we know that $x=\sqrt{2}$ is a root, but to write $\sqrt{2}$ as a decimal number is not a finite process. Its decimal expansion continues forever, without any repeating pattern. So, we are going to try and find a good decimal approximation, something like what a calculator will spit out, or may be even better!

We know that $\sqrt{2}$ is between 1 and 2. So, let us start with $a=1$. Note that $f(x)=x^{2}-2$ and hence $f^{\prime}(x)=2 x$.

Then near $x=1$ we have:

$$
L(x)=f^{\prime}(1)(x-1)+f(1)=(2(1))(x-1)+\left(1^{2}-2\right)=2(x-1)-1=2 x-3 .
$$

The root of this $L(x)$ is clearly $x=3 / 2=1.5$ So now we take $a_{1}=1.5$ and repeat the process! Here is a graph of the function, its linear approximation and the new root.


Thus the linear approximation near $x=a_{1}=1.5$ is

$$
L(x)=f^{\prime}(1.5)(x-1.5)+f(1.5)=(2(1.5))(x-1.5)+\left((1.5)^{2}-2\right)=3 x-4.25 .
$$

The root of this $L(x)$ is now $\frac{4.25}{3}=\frac{17}{12}=1.4167$. This is now our $a_{2} .{ }^{1}$ The linear approximation near $a_{2}$ is

$$
L(x)=f^{\prime}(1.4167)(x-1.4167)+f(1.4167)=2.8334 x-4.007 .
$$

Thus $a_{3}$ will be 1.4142 .
If you repeat the procedure and keep only 5 digit accuracy, then you get the same answer back!

Thus, we have found the desired answer, at least for the chosen accuracy!
Let us summarize the technique and make a formula.

1. Given a function $f(x)$ and a starting value $a$, the linear approximation is:

$$
L(x)=f^{\prime}(a)(x-a)+f(a) \text { and hence it has the root } x=a-f(a) / f^{\prime}(a)
$$

2. Hence, if $f^{\prime}(a)$ can be computed, replace $a$ by $a-\frac{f(a)}{f^{\prime}(a)}$.
3. Repeat this procedure until $f(a)$ or better yet $\frac{f(a)}{f^{\prime}(a)}$ becomes smaller than the desired accuracy.

[^64]4. When this happens, declare $a$ to be a root of $f(x)$ to within the desired accuracy.

Let us redo our calculations above, using this procedure. Note that for our $f(x)=$ $x^{2}-2$ we have $f^{\prime}(x)=2 x$ and hence $\frac{f(a)}{f^{\prime}(a)}$ evaluates to $\frac{a^{2}-2}{2 a}$. Thus it makes sense to calculate a nice table as follows: ${ }^{2}$

$$
\left[\begin{array}{ccccc}
a & y=a^{2}-2 & y^{\prime}=2 a & \frac{y}{y^{\prime}} & a-\frac{y}{y^{\prime}} \\
1.0 & -1.0 & 2.0 & -0.50000 & 1.5000 \\
1.5000 & 0.25000 & 3.0 & 0.083333 & 1.4167 \\
1.4167 & 0.0070389 & 2.8334 & 0.0024843 & 1.4142 \\
1.4142 & -0.00003836 & 2.8284 & -0.000013562 & 1.4142
\end{array}\right]
$$

### 10.2 Limitations of the Newton's Method

Even though it is easy to understand and nice to implement, the above method can fail for several reasons. We give these below to help the reader appreciate why one should not accept the results blindly.

In spite of all the listed limitations, the method works pretty well for most reasonable functions.

1. Case of no real roots!. If we start with a function like $f(x)=x^{2}+2$, then we know that it has no real roots. What would happen to our method?
The method will keep you wandering about the number line, leading to no value! You should try this!
2. Reaching the wrong root. Usually you expect that if you start near a potential root, you should land into it. It depends on the direction in which the tangent line carries you.

[^65]3. Not reaching a root even when you have them! Consider $f(x)=x^{3}-5 x$. Then $f^{\prime}(x)=3 x^{2}-5$. Then the starting value of $a=1$ takes you to
$$
1-\frac{f(1)}{f^{\prime}(1)}=1-\frac{-4}{-2}=-1
$$

Now the value $a=-1$ leads to

$$
-1-\frac{f(-1)}{f^{\prime}(-1)}=-1-\frac{4}{-2}=1
$$

Thus you will be in a perpetual cycle between -1 and 1 , never finding any of the roots $0, \sqrt{5},-\sqrt{5}$. The reader should observe that taking other values of $a$ does the trick. For example $a=2$ takes you thru:

$$
2 \rightarrow 2.2857 \rightarrow 2.2376 \rightarrow 2.2361
$$

This is the $\sqrt{5}$ to 5 decimal places.
Other starting values will take you to other roots.
4. Horizontal tangent. If at any stage of the process, the derivative becomes zero, then the value of the new "a" is undefined and we get stuck. For more complicated functions, the derivative may not even be defined! Often changing the starting value takes care of this.
5. Accuracy problems. In addition to the above, there are possible problems with roundoff errors of calculations both in the calculation of the derivative and the function. There are also problems of wild jumps in values if the derivative becomes rather small and so $\frac{y}{y^{\prime}}$ gets very large!
6. Conclusion. We have only sketched the method. An interested reader can pursue it further to study the branch of mathematics called Numerical Analysis, which studies practical calculation techniques to ensure desired accuracy.

## Chapter 11

## Summation of series.

We now discuss a simple application of elementary polynomials which can be used to build elegant useful formulas.

Given any function $f(x)$ defined for all positive integers, by a series we mean a sum:

$$
f(1)+f(2)+\cdots+f(n) \text { where } n \text { is some positive integer. }
$$

In usual books on this subject, a compact notation is presented and it goes like this:
$\sum_{i=1}^{n} f(i)$ is a short hand notation for the sum of all values obtained by setting $i=1, i=2, \cdots, i=n$ in succession.

Thus, if $f(x)=x^{2}$, then we get:

$$
\sum_{i=1}^{5} f(i)=\sum_{i=1}^{5} i^{2}=1^{2}+2^{2}+3^{2}+4^{2}+5^{2}=55
$$

The symbol $\sum$ stands for the Greek letter " S " to suggest a sum. The reader should note that the particular "index" $i$ in the expression does not affect the final answer, i.e.

$$
\sum_{i=1}^{5} f(i)=\sum_{j=1}^{5} f(j)
$$

(Just expand each and see!)
It is possible to create manipulation techniques for this summation symbol, but we refrain from getting into it. Our aim, in this chapter, is to build formulas for such summations for some convenient functions. We have actually avoided the summation notation, choosing to write out the long form instead. The reader is advised to try and convert our statements to the summation notation for practice and efficiency.

### 11.1 Application of polynomials to summation formulas.

Consider any polynomial expression $g(x)$ and define a new polynomial

$$
f(x)=g(x)-g(x-1)
$$

Consider the following sequence of substitutions.

$$
\begin{array}{|llll|}
\hline x & f(x) & = & g(x)-g(x-1) \\
\hline 1 & f(1) & = & g(1)-g(0) \\
2 & f(2) & = & g(2)-g(1) \\
\cdots & \cdots & \cdots & \cdots \\
i & f(i) & = & g(i)-g(i-1) \\
\cdots & \cdots & \cdots & \cdots \\
n & f(n) & = & g(n)-g(n-1) \\
\hline
\end{array}
$$

If we add up the $f$-column and equate it to the sum of the last column, we see that most values of $g$ cancel leaving us with a very simple formula

$$
f(1)+f(2)+\cdots+f(i)+\cdots+f(n)=g(n)-g(0)
$$

This gives us very efficient and clever ways for adding up a series of terms.
For example, let $g(x)=x$. Then $f(x)=1$ and we get the trivial result

$$
f(1)+\cdots+f(n)=1+\cdots+1=g(n)-g(0)=n-0=n .
$$

If we take $g(x)=x^{2}+x$, then $f(x)=2 x$ and we get the identity

$$
f(1)+\cdots+f(n)=2+4+\cdots+2 n=g(n)-g(0)=\left(n^{2}+n\right)-(0)=n(n+1)
$$

Division of both sides by 2 gives the well known

$$
\text { Arithmetic series formula: } 1+2+\cdots n=\frac{n(n+1)}{2}
$$

Thus to add up values of a gives $f(x)$ all we have to do is to come up with a cleverly constructed $g(x)$ such that $f(x)=g(x)-g(x-1)$. The best strategy is to try lots of simple $g(x)$ and learn to combine.

Here is a table from simple calculations.

$$
\begin{array}{ll}
g(x) & g(x)-g(x-1) \\
x & 1 \\
x^{2} & 2 x-1 \\
x^{3} & 3 x^{2}-3 x+1 \\
x^{4} & 4 x^{3}-6 x^{2}+4 x-1 \\
x^{5} & 5 x^{4}-10 x^{3}+10 x-5 x+1
\end{array}
$$

Indeed, these expressions should be deducible by using our old formulas for $(x+t)^{n}$ by taking $t=-1$.

Now suppose we wish to find the sum

$$
1^{2}+2^{2}+\cdots+n^{2}
$$

Here $f(x)=x^{2}$ and we can manipulate the first three polynomials above and write

$$
x^{2}=(1 / 3)\left(3 x^{2}-3 x+1\right)+(1 / 2)(2 x-1)+(1 / 6)(1)
$$

and this tells us that $g(x)=x^{3} / 3+x^{2} / 2+x / 6$ will do the trick! We have done a lot of manipulation, be sure to check the claim and think how we might have found the expression!

Thus the desired answer is:

$$
1^{2}+2^{2}+\cdots+n^{2}=g(n)-g(0)=n^{3} / 3+n^{2} / 2+n / 6=\frac{(n)(n+1)(2 n+1)}{6} .
$$

Clearly, with little work, you can make your own formulas. ${ }^{1}$

### 11.2 Examples of summation of series.

- Example 1: Using the idea for the Arithmetic series formula determine the formula for the sum of $n$ terms $2+5+8+\cdots$.

Answer: This looks like adding up terms of $f(x)=-1+3 x$. (Note that this gives $f(1)=2, f(2)=5$ etc. Then $f(n)=3 n-1$ and we want the formula for $2+5+\cdots+(3 n-1)$.

By looking up our table, we see that $f(x)=(3 / 2)(2 x-1)+(1 / 2)$, so we use $g(x)=(3 / 2)\left(x^{2}\right)+(1 / 2)(x)$. Indeed, check that

$$
g(x)-g(x-1)=\left(3 x^{2}+x\right) / 2-\left(\left(3 x^{2}-5 x\right) / 2-1\right)=3 x-1
$$

as desired. ${ }^{2}$

[^66]- Example 2: A general arithmetic series (A.P.) ${ }^{3}$ with beginning term $a$ and common difference $d$ is a sum of $n$ terms $a, a+d, \cdots, a+(n-1) d$. Find the formula for its sum.
Answer: We can, of course work this out just as above. But to illustrate the power of independent thinking, we shall give two other methods.

1. Guess the function method. Let us guess that there is a quadratic expression:

$$
F(x)=p x^{2}+q x+r
$$

such that the sum of the $n$ terms $a, a+d, \cdots, a+(n-1) d$ is $F(n)$.
We just try to find suitable $p, q, r$ so the answer is correct for first few values of $n$. Here is the work.
If we take $n=0$, i.e. no terms at all, then the sum must be 0 , so

$$
F(0)=p(0)^{2}+q(0)+r
$$

This says that $r=0$, so $F(x)=p x^{2}+q x$.
For $n=1$ the sum is $a$ and hence we want
Equation1: $\quad F(1)=p+q=a$.
For $n=2$ the sum is $a+(a+d)=2 a+d$, so we want
Equation2:. $\quad F(2)=p(2)^{2}+q(2)+r=4 p+2 q=2 a+d$
To solve equations 1 and 2 , we subtract twice the equation 1 from equation 2 and get:

$$
(4 p+2 q)-2(p+q)=(2 a+d)-2(a) \text { i.e. } 2 p=d \text { or } p=d / 2 .
$$

Using equation 1 , we deduce that $q=a-p=a-d / 2$.
Thus, we have

$$
F(x)=d x^{2} / 2+(a-d / 2) x=x\left(a+\frac{(x-1) d}{2}\right)=x\left(\frac{(a)+(a+(x-1) d)}{2}\right) .
$$

Hence our final answer is going to be $F(n)$. This gives us:
A.P. formula An A.P. with first term $a$ and common difference $d$ has $n$-th term as $a+(n-1) d$. The sum of the first $n$ terms is given by the expression "number of terms multiplied by the average of the first and the last terms", i.e.

$$
F(n)=n \frac{(a)+(a+(n-1) d)}{2}
$$

[^67]2. A trick! Our second method is much cuter and easy to see, however, it does not generalize very well.
Let us write the terms of the A.P. in a better notation:
Let the terms be named thus:

| term | description | value |
| :--- | :--- | :--- |
| $t_{1}$ | the first term | $a$ |
| $t_{2}$ | the second term | $a+d$ |
| $t_{3}$ | the third term | $a+2 d$ |
| $\cdots$ |  |  |
| $t_{n}$ | the $n$-th term | $a+(n-1) d$ |

Let $F(n)$ denote the sum $t_{1}+t_{2}+\cdots+t_{n}$.
Consider the following table: ${ }^{4}$

$$
\begin{array}{llllll}
F(n) & = & t_{1}+ & t_{2}+ & \cdots & +t_{n} \\
F(n) & = & t_{n}+ & t_{n-1}+ & \cdots & +t_{1}
\end{array}
$$

And this last line gives the desired formula:

$$
F(n)=(1 / 2) n(2 a+(n-1) d) .
$$

For example, consider the A.P. formed by $5,8, \cdots$. Here $a=5, d=3$ and the $n$-th term is $5+3(n-1)=2+3 n$. The average of the first and the $n$-th terms is simply $(1 / 2)(5+2+3 n)=(7+3 n) / 2$. Thus $f(n)=n(7+3 n) / 2$. The reader should verify the first 10 values of this function:
$5,13,24,38,55,75,98,124,153,185$.

[^68]
## Index

Āryabhaṭa algorithm, 39
A.P., 172
absolute
extremum, 88
maximum, 88
minimum, 88
angle
degree, 123
locator point, 122
radian, 122, 123
arithmetic series, 172
axes, 57
axis, 57
binomial, 6
Binomial coefficient
formulas for, 15
Binomial Theorem, 13
Binomial theorem
Newton's generalization, 16
center, 104
Chinese Remainder Problem, 39
Chinese Remainder Theorem, 42
circle, 103
angles in a semi circle, 111
center and radius of, 103
Constant angle in a sector, 135
diameter form of the equation, 111
intersection of two circles, 112
line of intersection of two circles, 113
rational parametric form, 106
tangent to a line, 114
through three points, 113
fails for collinear points, 114
unit, 121
with given center and tangent line, 115
complementary angles, 144
completing the square, 18
conic, 108
coordinate system, 57
cos, 124
cot, 127
Cramer's Rule, 30
exceptions to, 31
csc, 127
curve
plane algebraic, 95
rational, 95
derivative, 155
definition for a rational function, 160
definition for an algebraic function, 160
notation $f^{\prime}(a), 155$
better notation: $f^{\prime}(x)=D_{x}(f(x))$, 156
popular notation: $D_{x}(y)=\frac{d}{d x}(y), 156$
rules of
chain of functions, 162
constant multiplier, 156
enhanced power, 161
general power, 158
polynomial expression, 158
power of a function, 163
product, 157
sum, 157
use for approximation, 164
determinant of 2 by 2 matrix, 30
direction numbers, 60
distance, 58
between a point and a line, 116
dividend, 48
divisibility of integers, 37
divisibility of polynomials, 44
division algorithm, 44, 45
quotient, 45
remainder, 45
divisor, 48
duck principle, 77, 113
Efficient Euclidean algorithm, 39
ellipse, 109
enhanced remainder theorem, 55
equation
consistent, 23
equations
solution of a system, 23
system of, 23
Euclidean algorithm, 39
exponential, 127
field, 1
finite, 2
Fourier analysis, 127
function, 97
domain of, 97
range of, 97
G.P., 173

GCD, 49
geometric series, 53, 173
graphing advice, 99
greatest common divisor, 49
Halāyudha triangle, 15
hyperbola, 109
indeterminate, 4
interval notation, 87
isolating the variable, 24
Kuttaka, 39, 42
LCM, 50
least common multiple, 50
line
direction numbers of, 70
distance formula, 72
division formula, 72
half plane defined by, 118
intercept form, 78
intercepts, 75
locus of points equidistant from two points, 81
meaning of the parameter, 71
midpoint formula, 72
parametric form, 67
perpendicular bisector, 81
slope, 75
slope intercept form, 76
two point equational form, 75
variation, 77
linear approximation, 150
linear equation, 21
identity, 25
inconsistent, 25
unique solution, 24
linear equations
back substitution, 28
eliminating variables from a system, 28
Principle of $\mathbf{0}, \mathbf{1}, \infty$ number of solutions, 24
substitution method, 25
linear fit, 99
linear polynomial
behavior of, 85
linear polynomials, 85
logarithmic, 127
modulo, 36
monomial, 4
coefficient of, 4
degree of, 4
zero monomial, 4
negative angles, 127

Newton's Method, 167
numbers
complex , 1
rational, 1
real, 1
origin, 61
parabola, 109
parallel lines, 78
parameter, 3
Pascal triangle, 15
perpendicular lines, 78
$\pi, 121$
polynomial, 6
applications: fast calculations, 13
coefficient of a monomial in, 7
degree of , 8
leading coefficient of, 9
monic, 50
term of , 6
zero polynomial, 7
polynomial functions, 98
Pythagorean Theorem, 60
Pythagorean triple, 107
quadrant, 60
quadratic fit, 101
quadratic formula, 91
quadratic polynomial
behavior of, 90
extremum value of, 90
factored, 86
general, 90
zeros of, 90
quadratic polynomials, 85
radius, 104
rational function, 7
operations of, 8
rational functions, 98
ray, 59
Remainder theorem, 53
roots of functions

Newton's method, 167
row echelon form, 28
sec, 127
shift, 59
sin, 124
slope, 109
solution to an equation, 22
step functions, 98
floor and ceil, 98
summation of series, 171
supplementary angles, 144
surds, 139
$\tan , 127$
tangent line, 150
transcendental, 121
transformation, 62
triangles
right angle, 81
The cosine law, 138
The sine law, 138
trigonometric functions, 127
addition formulas, 129
additional formulas for the tangent, 134
complementary angles, 130
double angle formulas, 131
half angle formulas, 133
oddness and evenness, 130
supplementary angles, 130
trinomial, 6
unit circle, 122
vectors, 59
vertex theorem, 92


[^0]:    ${ }^{1}$ Partially supported by NSF grant thru AMSP(Appalachian Math Science Partnership)

[^1]:    ${ }^{2}$ Amy has now completed her masters degree in mathematics and joined Sarah on the mathematics faculty of Paul Laurence Dunbar High School in Lexington, Ky.

[^2]:    ${ }^{3}$ Here is a thought for the philosophical readers. Think what you mean when you say that $\sqrt{2}$ is a solution to the equation $x^{2}=2$. Do we really have an independent evidence that $\sqrt{2}$ makes sense, other than saying that it is that positive number whose square equals 2? And is it not just an alternate way of saying that it is a solution of the desired equation $\left(x^{2}=2\right)$ ? Now having thought of a symbol or name $\sqrt{2}$ for the answer, we can proceed to compare it with other known numbers, find various decimal digits of its expansion and so on.

    In higher mathematics, we find a way of turning this abstract thought into solutions of equations by declarations and into the art of finding the properties of solutions of equations without ever solving them! You will see more examples of this later.

[^3]:    ${ }^{1}$ Recall that the line over 3 indicates the digit 3 repeated indefinitely.
    ${ }^{2}$ The expansion of $\sqrt{2}$ for first 20 places is 1.4142135623730950488 , but the complete decimal never stops!

[^4]:    ${ }^{3}$ We illustrate how you would check the first of these. By cross multiplying, we see that we want to show:

    $$
    2(1)=(1+i)(1-i)
    $$

[^5]:    ${ }^{4}$ In short, the distinction between an indeterminate, a variable and a parameter, is, like beauty, in the eye of the beholder!

[^6]:    ${ }^{5}$ Note that this famous number has a long history associated with it and people have spent enormous energy in determining its decimal expansion to higher and higher accuracy. There is no hope of ever listing all the infinitely many digits, since they cannot have any repeating pattern because the number is not rational. Indeed, it has been proved to be transcendental, meaning it is not the solution to any polynomial equation with integer coefficients!

    It is certainly very interesting to know more about this famous number!
    ${ }^{6}$ For the reader who likes formalism, here is a general definition. A monomial in variables $x_{1}, \cdots, x_{r}$ is an expression of the form $c x^{n_{1}} \cdots x^{n_{r}}$ where $n_{1}, \cdots, n_{r}$ are non negative integers and $c$ is any expression which is free of the variables $x_{1}, \cdots, x_{r}$. The coefficient of the monomial is $a$, the degree is $n_{1}+\cdots+n_{r}$ and the exponents are said to be $n_{1}, \cdots, n_{r}$.

    If convenient, $n_{1}, \cdots, n_{r}$ may be allowed to be more general.

[^7]:    ${ }^{7}$ Can you find or make up names for a polynomial with four or five or six terms?

[^8]:    ${ }^{8}$ The situation is similar to integers. If we wish to divide 7 by 2 then we get a rational number $\frac{7}{2}$ which is not an integer any more. It is fine if you are willing to work with the rational numbers. You might have also seen the idea of division with remainder which says divide 7 by 2 and you get a quotient of 3 and a remainder of 1, i.e. $7=(2)(3)+1$.

    As promised, we shall take this up later.

[^9]:    ${ }^{9}$ An alert reader may see similarities with the process of multiplying two integers by using one digit of the multiplier at a time and then adding up the resulting rows of integers. Indeed, it is the result of thinking of the numbers as polynomials in 10 , but we have to worry about carries. Here is

    a sample multiplication for your understanding: |  | 4 | 3 | 1 |
    | :---: | :---: | :---: | :---: |
    | $\times$ | 1 | 2 |  |
    |  | 8 | 6 | 2 |
    | 4 | 3 | 1 |  |
    |  | 1 | 7 | 2 |

[^10]:    ${ }^{10}$ Why the word "binomial"? These are the coefficients coming from powers of a "two term expression" which is called a binomial, where "bi" means two. Indeed, "poly" means many, so our polynomial means a many term expression.
    ${ }^{11}$ The main point to realize what we are doing. Here is a nice organization method.

    1. Take the known expansion of $(x+t)^{2}$, say.
    2. Multiply the expansion by $x$ and $t$ separately.
    3. Add up the two multiplication results by writing the terms below each other while lining up like degree terms vertically.
    4. add up the terms and report the answer.
    5. Now repeat this idea by replacing 2 by $3,4,5, \cdots, n$. That is the inductive process.
[^11]:    ${ }^{12}$ It might be tempting to memorize the whole final answer. We recommend only memorizing the substitution $u-\frac{b}{2 a}$. It is best to carry out the rest of the simplification as needed.

[^12]:    ${ }^{1}$ This may seem unnecessarily complicated. It is tempting to define the equation to be linear by requiring that the maximum degree of its non zero terms is 1 .

    However, the problem is that this maximum could be zero, since the supposed variables might be absent!

    Why would one intentionally write an equation in $x$ which has no $x$ in it?
    This can happen if we are moving some of the parameters of the equation around and $x$ may accidentally vanish! For example the equation $y=m x+c$ is clearly linear in $x$, but if $m=0$ it has no $x$ in it!

[^13]:    ${ }^{2}$ Even though we have stated it, we are far from proving this principle; its proof forms the basic investigation in the course on Linear Algebra and it is a very important result, useful in many branches of mathematics.

[^14]:    ${ }^{3}$ The three solutions which give solutions for each of $x, y, z$ in terms of later variables can be considered an almost final answer. Indeed, in a higher course on linear algebra, this is exactly what is done!

    Given a set of equations, we order our variables somehow and try to get a sequence of solutions giving each variable in terms of the later ones. The resulting set of solution equations is often called the row echelon form.

[^15]:    ${ }^{4}$ You should note that these familiar operations are special cases of our permissible operation. Thus, to add a quantity " $p$ " to both sides can be described as adding the true equation $p=p$ to the given equation. Similarly, multiplying by a non zero number " $a$ " can be explained as writing the same equation twice and adding $(a-1)$-times the second to the first. Why do we want $a$ to be non zero?. If we take $a=0$, then we get a true equation, but its solutions have nothing to do with the given equation!

[^16]:    ${ }^{5}$ Thus $\Delta$ lets us "determine" the nature of the solution. This is the reason for the term "determinant".

[^17]:    ${ }^{6}$ Technically, what we wrote is illegal! A true algebraist would insist on writing the equations as

    $$
    (\Delta) x=\Delta_{x} \text { or }(0) x=10-k
    $$

    and

    $$
    (\Delta) y=\Delta_{y} \text { or }(0) y=2 k-20 .
    $$

    Our work is valid as long as we interpret the answers carefully and double check them.
    ${ }^{7}$ We shall see such answers when we study the parametric forms of equations of lines later.

[^18]:    ${ }^{1}$ You should verify why $q=4$ does not work.

[^19]:    ${ }^{2}$ As in the polynomial case, any integer dividing the top two integers in our "Integers" column

[^20]:    ${ }^{4}$ It is well worth checking all such answers again: Using the calculator verify: 611797/623 $=982.02$ and $611797-(982)(623)=11$. Similarly, $611797 / 7553=81.0005$ and $611797-(7553)(81)=4$.

[^21]:    ${ }^{5}$ Note that $\operatorname{deg}_{x}(u(x))=\operatorname{deg}_{x}(w(x))+\operatorname{deg}_{x}(v(x))$ or $3=\operatorname{deg}_{x}(w(x))+2$. Hence the conclusion.

[^22]:    ${ }^{6}$ Hint for the proof: Suppose that $u-q_{1} v=r_{1}$ and $u-q_{2} v=r_{2}$ where $r_{1}, r_{2}$ are either 0 or have degrees less than $m$.
    Subtract the second equation from the first and consider the two sides $\left(q_{2}-q_{1}\right) v$ and $\left(r_{1}-r_{2}\right)$.
    If both are 0 , then you have the proof that $q_{2}=q_{1}$ and $r_{2}=r_{1}$. If one of them is zero, then the other must be zero too, giving the proof.

    Finally, if both are non zero, then the left hand side has degree bigger than or equal to $m$ while the right hand side has degree less than $m$; a clear contradiction!

[^23]:    ${ }^{7}$ Note that we use the already done calculation of $u-x v$ and don't waste time redoing the steps. We only figure out the contribution of the new terms.

[^24]:    ${ }^{8}$ If you are confused about how 2 becomes 1 , remember that the only monic polynomial of degree 0 is 1 .

[^25]:    ${ }^{9}$ The informal proof above can me made precise in the following way. Some of you might find it easier to understand this version.

    We have proved that for each $n=0,1, \cdots$ we have polynomials

    $$
    q_{n}(x)=x^{n-1}+a x^{n-2}+\cdots+a^{n-2} x+a^{n-1}
    $$

[^26]:    ${ }^{1}$ What we are doing here is informally introducing the concept of vectors. One idea about vectors is that they are line segments with directions marked on them. This is a useful concept leading to the theory known as Linear Algebra. You may have occasion to study it later in great detail.

[^27]:    ${ }^{2}$ The well known Pythagorean Theorem states that the square of the length of the hypotenuse in a right angle triangle is equal to the sum of the squares of the lengths of the other two sides.

    This theorem is arguably discovered and proved by mathematicians in different parts of the world both before and after Pythagoras!

[^28]:    ${ }^{3}$ An alert reader can verify this thus: The distance between two points $P_{1}\left(a_{1}\right), P_{2}\left(a_{2}\right)$ is given by $\left|a_{2}-a_{1}\right|$. In new coordinates it becomes $\left|\left(a_{2}-3\right)-\left(a_{1}-3\right)\right|=\left|a_{2}-a_{1}\right|$, i.e. it is unchanged. Indeed, the new coordinates also preserve all shifts.

[^29]:    ${ }^{4}$ It would be algebraically better to write $(p / u, q / v)$ since we obtained the answers by solving $u x-p=0$ and $v y-q=0$ for $x, y$ respectively. However, since $u, v$ are $\pm 1$ it does not matter if we divide by them or multiply by them! These are the only two numbers which have this property!

[^30]:    ${ }^{5}$ Here is a sketch of the details, if you want to know it now.
    We know how to translate and for convenience, we assume that there is no translation involved.
    Distance preserving transformations can be achieved exactly when there is an angle $\theta$ and a number $u= \pm 1$ such that

    $$
    x^{\prime}=\cos (\theta) x+\sin (\theta) y, y^{\prime}=-u \sin (\theta) x+u \cos (\theta) y
    $$

    The transformation can be described as a rotation by $\theta$ if $u=1$ and it is a rotation by $\theta$ followed by a flip of the $y^{\prime}$ axis if $u=-1$. The trigonometric functions $\cos (\theta), \sin (\theta)$ have been studied from ancient times and here is a quick definition when $0<\theta<90$. Make a triangle $A B C$ where the angle at $B$ is the desired $\theta$ and the one at $C$ is 90 degrees. Then

    $$
    \cos (\theta)=|\overrightarrow{B C}| /|\overrightarrow{B A}| \text { and } \sin (\theta)=|\overrightarrow{C A}| /|\overrightarrow{B A}| .
    $$

    For other values of angles, these are defined by suitable adjustments. We will give a more general modern definition during the discussion of trigonometry.

[^31]:    ${ }^{1}$ Also note that we did not rush to "evaluate" $\sqrt{65}$ ! A decimal answer like 8.062257748 spit out by a calculator is never accurate and tells us very little about the true nature of the number. The decimal approximation should only be used if the precise number is not needed or so complicated that it impedes understanding. Actually, when the precise number gets so complicated, mathematicians often resort to giving the number a convenient name and using it, rather than an inaccurate value! The most common example of this is $\pi$ - the ratio of the circumference of a circle and its diameter. It can never be written completely precisely by a decimal, so we just use the symbol!

[^32]:    ${ }^{2}$ We may call this the "duck principle" named after the amusing saying: "If it walks like a duck and talks like a duck, then it is a duck." Thus, if an equation looks like the equation of a line and satisfies the required conditions, then it must be the right equation!

[^33]:    ${ }^{3}$ An alert reader might worry about getting different looking expressions, depending on the starting equation. Thus, for a starting equation of $2 x+3 y=1$, we can have $3 x-2 y=k$, but $6 x-4 y=k$ seems like a valid answer as well, since our original equation could have been $4 x+6 y=2$.

    This is not an error. The point is that the equation of a line is never unique, since we can easily multiply it by any non zero number to get an equivalent equation.

[^34]:    ${ }^{4}$ The alert reader has to observe our convention that when we say that the product of slopes of perpendicular lines is -1 , we have to make a special exception when one slope is 0 (horizontal line) and the other is $\infty$ (vertical line). For our problem, clearly neither slope has a chance of becoming infinite, so it is correct to equate the product of slopes to -1 and finish the solution.

    In general, it is possible to set up such an equation and solve it, as long as we are sensitive to cases where some denominator or numerator becomes zero.

[^35]:    ${ }^{1}$ In general this cannot be done with $p$ and $q$ real. However solving hard mathematical problems almost always turns on finding the right special cases to study for insight. It is a maxim among mathematicians that "If there is a hard problem you can't solve then there is an easy problem you can't solve".

[^36]:    ${ }^{2}$ The $a$ and $b$ used in this definition have nothing to do with our coefficients of the quadratic! They are temporary variable names.

[^37]:    ${ }^{3}$ The reader should note that we have already done this case when we assumed a factored form $a(x-p)(x-q)$ for $Q(x)$. We are redoing it just to match the new notation and get the famous quadratic formula established!
    ${ }^{4}$ Indeed, when one learns about complex numbers to find solutions of $Q(x)=0$ even for negative $M$, then the roots can be found from the same quadratic formula and $-b /(2 a)$ shall still be their average. It may no longer have any meaning as maximum or minimum value, but it is still a critical value!

[^38]:    ${ }^{1}$ The proof that a given curve is not rational is not elementary and requires new ideas, found in advanced books on Algebraic Geometry. Pursuing this example is a good entry point into this important branch of mathematics.

[^39]:    ${ }^{2}$ Technically, this domain must be a part of our concept. We get a different function if we use a different domain.

    Thus, the function defined by $y=3 x+4$ where $D=\Re$-the set of all reals and the function defined by $y=3 x+4$ where the domain is $\Re_{+}$-the set of positive real numbers are two different functions. Often, this subtle point is ignored!

[^40]:    ${ }^{1}$ We used the traditional word locus which is a dynamic idea - it signifies the path traced by a point moving according to the given rules; but in practice it is equivalent to the word set instead. Somehow, the locus conveys the ideas about the direction in which the points are traced and may include the idea of retracing portions of the curve.

[^41]:    ${ }^{2}$ Did we forget the $g$ altogether? Not really. If $f=0$ and $h=f-g=0$, then automatically, $g=0$. So, it is enough to solve for $f=0, h=0$.

[^42]:    ${ }^{3}$ The duck principle in action again: It is clearly a line with correct slope and we can choose $k$ to make it pass through any given point!

[^43]:    ${ }^{4}$ There is another possible route to this answer; namely to write the usual equations for $L, L^{\prime}$, solving them for the common point $P$ and then finding the distance $d(A, P)$. The reader should try this approach to appreciate the virtue of using the parametric form for $L^{\prime}$ as suggested!

[^44]:    ${ }^{5}$ The claim that $\frac{q}{q-p}$ is true but needs a careful analysis of cases. The reader should carefully consider the possibilities $p<0, q>0$ and $q<0, p>0$ separately and verify the claim.

[^45]:    ${ }^{6}$ This $\pi$ has a very interesting history and has occupied many talented mathematicians over the ages. Finding more and more digits of the decimal expansion of $\pi$ is a challenging mathematical problem. There are many books and articles on the subject of $\pi$ including a movie with the same title! People have imagined many mysterious properties for the decimal digits of this real number, which is proven to be transcendental. This means that there is no non zero polynomial $p(x)$ with integer coefficients such that $p(\pi)=0$.

[^46]:    ${ }^{7}$ It is not clear why those great old mathematicians wanted "clockwise" to be the negative direction, what did they have against clocks anyway? This convention of positive counter clockwise direction, however, is now firmly entrenched in tradition!

[^47]:    ${ }^{8}$ Here is a hint of the geometric argument. If we draw a circle centered at $H$ and having $O P$ as diameter, then we know from our work with the circles that the point $M$ is on this circle, since the angle between $O M$ and $P M$ is known to be $90^{\circ}$. Therefore, all the distances are equal to the radius of this circle.

[^48]:    ${ }^{9}$ This needs a bit of algebra: First cross multiply to get:

    $$
    \tan (\alpha)\left(x^{2}+y^{2}-x\right)=y
    $$

    and then multiply both sides by $\cot (\alpha)$.
    ${ }^{10} \mathrm{To}$ see this, think of $A, B$ as $O, U$ in our special case. Our choice of special points simplified the equations considerably. If general points $A, B$ are used the equations get more complicated.

    That is why we wrote "essentially proved" and not "proved"!

[^49]:    ${ }^{11}$ There are some missing parts in the argument. We need to argue why our special choice of points $O, U$ still leads to the general result. We also need to argue that the angle is actually constant in each sector, we only proved that it has a constant tangent. All this can be fixed with a careful analysis of the cases.

[^50]:    ${ }^{12}$ Since the sum of the angles $\angle B L M, \angle M L D$ and $\angle D L C$ is $180^{\circ}$ and since $\angle M L D$ is clearly a right angle, we get

    $$
    \angle B L M+\angle D L C=90^{\circ} .
    $$

[^51]:    ${ }^{13}$ You may feel that we are complicating matters too much by naming even the known angles as $s, t$. You will probably find that the reverse is true. Try using the explicit values in the equations and then solve them and compare the work!

    Moreover, this way, you have a wonderful formula at hand. All such height problems can now be solved by changing the various numbers in the final expression!

[^52]:    ${ }^{14}$ It appears like a subtraction formula, but it will easily become converted to the addition formula soon!

[^53]:    ${ }^{15}$ If it bothers you to plug in $\pi / 2-s$ for $s$, you may do it in two steps: First replace $s$ by $\pi / 2-z$ and then replace $z$ by $s$.

[^54]:    ${ }^{16}$ An alert reader might object that our conclusion is not valid since we have not carefully checked the values of $t$ when $\cos (t)=0$. This is a valid objection, but the only values of $t$ with $\cos (t)=0$ are given by $t= \pm \pi / 2$ up to multiples of $2 \pi$. Each of these cases is easily verified by hand.

[^55]:    ${ }^{1}$ Note that the constant term cancelled. If the constant term were to survive, it would indicate that either the chosen point is not on the curve or that there is a mistake in the simplification.

[^56]:    ${ }^{2}$ The line $v=8 a u$ is of interest even when the point is not on our curve. It is sometimes called a polar and has interesting properties for some special curves. Interested reader should look these up in books on algebraic geometry!

[^57]:    ${ }^{3}$ Again, note the vanished constant term as a sign of correctness.

[^58]:    ${ }^{4}$ We are clearly implying that $f(a+u)$ expanded in powers of $u$ looks like $f(a)+m u+$ higher $u$-terms. The fact that the constant term is $f(a)$ is obvious if you notice that the constant term must be obtained by putting $u=0$ in the expansion and putting $u=0$ in $f(a+u)$ clearly makes it $f(a)$.

[^59]:    ${ }^{5}$ The definition looks very much like the definition in Calculus, but our case is much simpler.
    In Calculus one needs to handle more complicated functions and hence needs a much more sophisticated analysis of the linear approximation involving the concept of limits. We shall advance up to the derivatives of rational functions and some special functions and leave the finer details to higher courses.

    Historically, people knew the derivatives for such special functions including the trigonometric functions long before the formal invention of Calculus.

[^60]:    ${ }^{6}$ This is the famous method of Mathematical Induction. It gives concrete minimal argument necessary to give a convincing proof that a statement holds for all values of integers from some point on. There are many logically equivalent ways of organizing the argument. These different forms of the argument often reduce the task of proving a formula to just guessing the right answer and applying the mechanical process of induction. All of our formulas for the summation of series in the last chapter, can be easily proved by this technique and the reader is encouraged to try this.

[^61]:    ${ }^{7}$ In fact, the result is true for any constant exponent, but that proof is well beyond our reach at this point!
    ${ }^{8}$ The proof is easy, but messy in appearance. The student is advised to learn its use before worrying about the details!

[^62]:    ${ }^{9}$ An alert reader should seriously object at this point. We proved the product rule, but the definition of the derivative was only for polynomials at that time! A general product rule can be established, and we are not really going to do it here. We shall leave such finer details for higher courses.

[^63]:    ${ }^{10}$ There are deeper issues of accumulation of roundoff errors during all calculations and reliability

[^64]:    ${ }^{1}$ The reader may try a similar picture. The curve becomes very close to the tangent line and the picture tells you very little, unless it is vastly expanded! It is already a sign that we are very close.

[^65]:    ${ }^{2}$ In this example, the formula which changes $a$ to $a-\frac{a^{2}-2}{2 a}$ simplifies to $\frac{1}{2}\left(a+\frac{2}{a}\right)$. This has a particularly pleasant description. Whichever side of $\sqrt{2}$ we take our positive $a$, we can see that $\frac{2}{a}$ lies on the other side and their average is exactly $\frac{1}{2}\left(a+\frac{2}{a}\right)$. This is easily seen to be closer to $\sqrt{2}$ by observing them as points on the number line. Thus, we know for sure that we are clearly getting closer and closer to our $\sqrt{2}$. We can even estimate the new accuracy as half the distance between $a$ and $\frac{2}{a}$. This particular method was probably well known since ancient times and leads to excellent approximations of any desired accuracy with rational numbers. Techniques for approximating other square roots using rational numbers by special methods are also scattered in ancient mathematical works. Here is a sequence of approximations to $\sqrt{2}$ using this simple formula but keeping the answers as rational numbers, rather than decimals.

    $$
    1,3 / 2, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \frac{886731088897}{627013566048}
    $$

    You will find the second to the sixth commonly used in ancient literature. The fourth is accurate to 5 places. The fifth is accurate to 11 places and the sixth to 23 places. The seventh one, when calculated, will come out correct to 48 places!

[^66]:    ${ }^{1}$ It is interesting to take up the challenge of redoing the geometric series formula with this new idea. Our $f(x)$ is now $r^{(x-1)}$ and the needed $g(x)$ can be $r^{x} /(r-1)$. Verify this.
    ${ }^{2}$ Here is a useful idea for such checking. Instead of straight substitution, first write $g(x)=$ $x(3 x+1) / 2$. Then $g(x-1)=(x-1)(3(x-1)+1) / 2=(x-1)(3 x-2) / 2$. Thus,

    $$
    g(x)-g(x-1)=\frac{x(3 x+1)-(x-1)(3 x-2)}{2}=\frac{\left(3 x^{2}+x\right)-\left(3 x^{2}-5 x+2\right)}{2}=\frac{6 x-2}{2}=3 x-1
    $$

[^67]:    ${ }^{3}$ A.P. is a short form of Arithmetic Progression, which stands for the sequence of terms with common differences. We are using a more technical term "series" which indicates the idea of the sum of such a progression of numbers.

[^68]:    ${ }^{4}$ The main idea is that a typical $i$-th term is $a+(i-1) d$ and so $i$-th and $(n+1-i)$-th terms added together give: $(a+(i-1) d)+(a+(n+1-i-1) d)=2 a+(n-1) d$.

    It is worth thinking about why this gives the right answer!

