Lectures on Sets.

Ma 162 Spring 2010

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Set Notations.

- Sets are collections of objects. For convenience and to avoid some logical problems, we always agree to a universal set U and understand that all sets under discussion are contained in it.
- An empty set is denoted by the symbol $\emptyset$. A set can be described in essentially one of two ways:
  - An explicit list (roster notation) as in $A = \{1, 2, 3\}$
  - or by a defining property (set builder notation)

$$A = \{x \mid x \text{ is one of the first three natural numbers.}\}.$$

- We can build new sets from given sets, say A, B by repeated application of one of these operations:

Union $A \bigcup B$ Intersection $A \bigcap B$ Subtraction $A \setminus B$.

Union

- Let $U = \{1,2,3,4,5,6,7,8\}$ and $A = \{1,2,3\}, B = \{2,3,4,5,6\}$. We define and illustrate the various set operations.
- $A \bigcup B$ is the set obtained by putting elements of A as well as B together. By convention, repeated elements are written only once.
  So $A \bigcup B = \{1,2,3,4,5,6\}$.
- It is worth noting that $X \bigcup U = U$ and $X \bigcup \emptyset = X$ for all sets X.
- We can also take the union of several sets and the final result does not depend on the order of taking the unions. In fancy words, this says that the union is an associative and commutative binary operation.
  Thus, if $C = \{2,4,8\}$, then

$$A \bigcup B \bigcup C = \{1,2,3,4,5,6,8\}.$$

Intersection and Subtraction.

- Let $U = \{1,2,3,4,5,6,7,8\}$ and $A = \{1,2,3\}$, $B = \{2,3,4,5,6\}$. We define and illustrate the various set operations.
- $A \bigcap B$ is defined as the set of common elements of A and B. Thus, for our example, it gives $A \bigcap B = \{2,3\}$.
- $A \setminus B$ is defined by listing elements of A except for elements of B. This may also be written as $A - B$.
  Thus, for our example, $A \setminus B = \{1\}$. Note that by the same definition $B \setminus A = \{4,5,6\}$.
- The set $U \setminus A$ is also called the complement of A and has a special notation A^C. Thus, for our example

$$A^C = \{1,2,3,4,5,6,7,8\} \setminus \{1,2,3\} = \{4,5,6,7,8\}.$$

Calculations in Set Builder notation.

- For finite sets, the lister notation is easy to work with and set operations are easily handled. For large sets or infinite sets, one has to use the set builder notation and the calculation of set operations also has to be understood in the same way. Here is an illustration.
- Let U be the set of all integers $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.
  Let

  $$A = \{x \mid x = 2t \text{ for some } t \in U\}, B = \{x \mid x = 5t \text{ for some } t \in U\}.$$

- We claim the following:

  $$A \bigcup B = \{x \mid x = 2t \text{ or } x = 5t \text{ for some } t \in U\}.$$

  A little thought shows that the set consists of all even numbers as well as all elements of $\{\pm 5, \pm 15, \pm 25, \pm 35, \cdots\}$.

Continued Set Builder Notation.

- With the above definition of U, A, B we see that

  $$A \bigcap B = \{x \mid x = 2t \text{ and } x = 5t \text{ for some } t \in U\}.$$

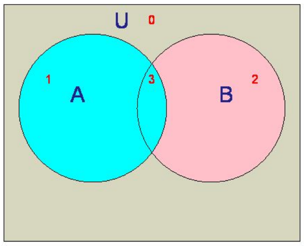
  This set is easily seen to be all multiples of 10.
- The complement A^C is easily seen as the set of all odd integers and can be described as $\{2t + 1 \mid t \in U\}$.
- Also:

  $$A \setminus B = \{x \mid x = 2t \text{ and } x \neq 5t \text{ for any } t \in U\}.$$

  These are all even integers which don't end in a zero!

Venn Diagrams.

- Often a graphical technique helps us understand the set operations and helps in deducing properties of calculated sets.
- If one is working with two sets A, B, then it is easy to see that any combination of operations between them results in a set which is a union of some of the following sets in a Venn diagram:



2 set Venn Diagram

- We shall find it convenient to number them so that the set 1 is $A \setminus B$, set 2 is $B \setminus A$, set 3 is $A \bigcap B$ and set 0 is $(A \bigcup B)^C$.
- This is useful to prove properties of two set combinations.

Using the Two-set Venn Diagram.

- We can use the set numbering to “prove theorems” about two sets. Thus, we show how to prove:

$$(A \bigcup B)^C = A^C \bigcap B^C.$$

- Note from the Venn diagram that $A \bigcup B$ is composed of the pieces $1, 2, 3$ so its complement is the piece 0. This says that the set on the left hand side is the 0 piece.
- From the Venn diagram, we also see that A^C and B^C are composed of sets “2, 0” and “1, 0” respectively. Thus, there intersection is also the piece “0”.
- The same technique can be used to prove or disprove any statements about two sets.

Counting Formula 1.

- For any finite set X, let $n(X)$ denote the number of elements in X. For two sets A, B we denote by d_1, d_2, d_3, d_0 the respective number of elements in the Venn diagram pieces $1, 2, 3, 0$.
- Then from the Venn diagram, we note the following:
- $$n(A) = d_1 + d_3, \quad n(B) = d_2 + d_3.$$
  $$n(A \bigcup B) = d_1 + d_2 + d_3, \quad n(A \bigcap B) = d_3$$
- It follows that

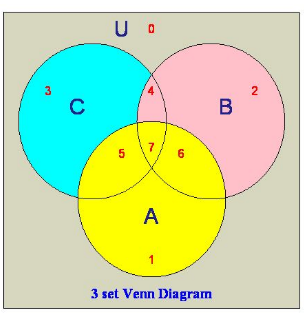
  $$n(A)+n(B)-n(A \bigcap B) = (d_1+d_3)+(d_2+d_3)-(d_3) = d_1+d_2+d_3.$$

  Hence:

  $$n(A \bigcup B) = n(A) + n(B) - n(A \bigcap B).$$
- This is one of the important counting formula that we shall use.

Venn Diagram of three sets.

- Here is similar idea to handle three sets A, B, C.



Note the numbering of the pieces from 0 to 7. Here is a sample theorem:

$$A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C).$$

- The second part on LHS has pieces $4, 7$ and the first part has pieces $1, 5, 6, 7$, so the union is composed of $1, 4, 5, 6, 7$.
- On the RHS, the two parts are respectively composed of $1, 5, 6, 7, 2, 4$ and $1, 5, 6, 7, 3, 4$. So the intersection gives, $1, 5, 6, 7, 4$ which is the same as the LHS.
- This proves the theorem!

Counting Formula 3.

- We can make a three set counting formula using the Venn diagram as before.
- The formula is: $n(A \bigcup B \bigcup C) = n(A) + n(B) + n(C) - n(A \bigcap B) - n(B \bigcap C) - n(C \bigcap A) + n(A \bigcap B \bigcap C)$.
- Using a notation similar to the previous calculation, the LHS is clearly the sum of $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7$.
- The first three terms on the RHS give the sum $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + (d_4 + d_5 + d_6 + 2d_7)$.
- The next three terms on the RHS subtract off $d_6 + d_7 + d_4 + d_7 + d_5 + d_7 = d_4 + d_5 + d_6 + 3d_7$, so we are now left with $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + (-d_7)$.
- The last term on RHS is exactly d_7 and adding it matches the LHS. This finishes the proof!

More on Venn Diagrams.

- It is tempting to hope that we can have Venn diagrams to handle 4 or more sets. However, this does not work. If you have four sets, then you need 16 pieces and these cannot be made by intersecting four circles!
- So, for four or more sets, one has to learn how to use use the set builder notation carefully and resolve set operations.
- As a result, some people prefer to do the two and three set calculations also using formulas and set builder notation.
- The two basic counting formulas that one should remember are:

$$n(A \bigcup B) = n(A) + n(B) - n(A \bigcap B)$$

and

$$\begin{aligned} n(A \bigcup B \bigcup C) &= n(A) + n(B) + n(C) \\ &\quad -(n(A \bigcap B) + n(b \bigcap C) + n(C \bigcap A)) \\ &\quad +n(A \bigcap B \bigcap C) \end{aligned}.$$

Using the Counting Formulas.

- **Question.** Given that A, B and C are sets with $95, 69$ and 85 members respectively,
  If $B \bigcap A$ has 44 members, then $B \bigcup A$ has ??? members.
- **Answer:** Use the two set formula
  $n(B \bigcup A) = n(B) + n(A) - n(B \bigcap A) = 69 + 95 - 44 = 120.$
- ii) If it is further known that $C \bigcap A$ has 57 members, then $C \bigcup A$ has ??? members.
- **Answer:** Using the two set formula for C and A gives:
  $n(C \bigcup A) = n(C) + n(A) - n(c \bigcap A) = 85 + 95 - 57 = 123.$
- iii) If, in addition $B - C$ has 39 members, then $C \bigcap B$ has ??? members.
- Note that $n(B) = n(B - C) + n(C \bigcap B)$, so
  $69 = 39 + n(C \bigcap B)$. Hence the answer is 30.

Question continued.

- iv) Finally, if we are given that the intersection of all three sets A, B and C has 17 members, then the union of all three sets has ??? members.
- **Answer:** Apply the three set formula and numbers known from above to get:
  $n(A \bigcup B \bigcup C) = 95 + 69 + 85 - (44 + 30 + 57) + 17 = 135.$