# Probability.

Ma 162 Spring 2010

Ma 162 Spring 2010

April 14, 2010

# Sample Spaces.

- Sample space: The scientific investigation of a phenomenon consists of the following steps. Make some hypothesis about a possible theory and perform a suitable experiment (or a survey) to collect appropriate data.
- The various possible outcomes of an experiment (or responses to the survey) are defined as sample points and the set of sample points is called the sample space.
- An event by definition consists of a set of sample points. The whole sample space is then the universal set and it is denoted by S (to remind us of the word sample).

# Sample Spaces.

- Sample space: The scientific investigation of a phenomenon consists of the following steps. Make some hypothesis about a possible theory and perform a suitable experiment (or a survey) to collect appropriate data.
- The various possible outcomes of an experiment (or responses to the survey) are defined as sample points and the set of sample points is called the sample space.
- An event by definition consists of a set of sample points. The whole sample space is then the universal set and it is denoted by S (to remind us of the word sample).

# Sample Spaces.

- Sample space: The scientific investigation of a phenomenon consists of the following steps. Make some hypothesis about a possible theory and perform a suitable experiment (or a survey) to collect appropriate data.
- The various possible outcomes of an experiment (or responses to the survey) are defined as sample points and the set of sample points is called the sample space.
- An event by definition consists of a set of sample points. The whole sample space is then the universal set and it is denoted by S (to remind us of the word sample).

# Sample Spaces.

- Sample space: The scientific investigation of a phenomenon consists of the following steps. Make some hypothesis about a possible theory and perform a suitable experiment (or a survey) to collect appropriate data.
- The various possible outcomes of an experiment (or responses to the survey) are defined as sample points and the set of sample points is called the sample space.
- An event by definition consists of a set of sample points. The whole sample space is then the universal set and it is denoted by S (to remind us of the word sample).

# Sample Spaces.

- Sample space: The scientific investigation of a phenomenon consists of the following steps. Make some hypothesis about a possible theory and perform a suitable experiment (or a survey) to collect appropriate data.
- The various possible outcomes of an experiment (or responses to the survey) are defined as sample points and the set of sample points is called the sample space.
- An event by definition consists of a set of sample points. The whole sample space is then the universal set and it is denoted by S (to remind us of the word sample).

# Examples.

- Here are some typical examples of Sample Spaces.
- A chemist carries out an experiment with some chemical mixture and observes the resulting temperature.
- The sample space then consists of all possible positive numbers, if he uses the absolute scale. This is an infinite sample space. Of course, many of these are unrealistic (being too large to be a valid temperature) and in reality, the records might be decimals with at most two digits past the decimal point.
- Thus, the sample space is, in practice, a finite space and this is usually the case in practical situations.

# Examples.

- Here are some typical examples of Sample Spaces.
- A chemist carries out an experiment with some chemical mixture and observes the resulting temperature.
- The sample space then consists of all possible positive numbers, if he uses the absolute scale. This is an infinite sample space. Of course, many of these are unrealistic (being too large to be a valid temperature) and in reality, the records might be decimals with at most two digits past the decimal point.
- Thus, the sample space is, in practice, a finite space and this is usually the case in practical situations.

# Examples.

- Here are some typical examples of Sample Spaces.
- A chemist carries out an experiment with some chemical mixture and observes the resulting temperature.
- The sample space then consists of all possible positive numbers, if he uses the absolute scale. This is an infinite sample space. Of course, many of these are unrealistic (being too large to be a valid temperature) and in reality, the records might be decimals with at most two digits past the decimal point.
- Thus, the sample space is, in practice, a finite space and this is usually the case in practical situations.

# Examples.

- Here are some typical examples of Sample Spaces.
- A chemist carries out an experiment with some chemical mixture and observes the resulting temperature.
- The sample space then consists of all possible positive numbers, if he uses the absolute scale. This is an infinite sample space. Of course, many of these are unrealistic (being too large to be a valid temperature) and in reality, the records might be decimals with at most two digits past the decimal point.
- Thus, the sample space is, in practice, a finite space and this is usually the case in practical situations.

# Examples.

- Here are some typical examples of Sample Spaces.
- A chemist carries out an experiment with some chemical mixture and observes the resulting temperature.
- The sample space then consists of all possible positive numbers, if he uses the absolute scale. This is an infinite sample space. Of course, many of these are unrealistic (being too large to be a valid temperature) and in reality, the records might be decimals with at most two digits past the decimal point.
- Thus, the sample space is, in practice, a finite space and this is usually the case in practical situations.

# Examples.

- Here are some typical examples of Sample Spaces.
- A chemist carries out an experiment with some chemical mixture and observes the resulting temperature.
- The sample space then consists of all possible positive numbers, if he uses the absolute scale. This is an infinite sample space. Of course, many of these are unrealistic (being too large to be a valid temperature) and in reality, the records might be decimals with at most two digits past the decimal point.
- Thus, the sample space is, in practice, a finite space and this is usually the case in practical situations.

# More Examples.

- A simpler example is to throw a die and observe the number of dots on the top face. These range from 1 to 6 and this is a sample space with six sample points.
- Another experiment might be the value of a stock as observed at the opening of the stock market every day. The sample space is again a set of non negative decimal numbers, with up to two decimal places.
  This is again a finite space in practice.
- We may survey students in a course and ask their opinion of the course content on a scale of 0 to 10. This is a sample space of 11 sample points.
- A person enters a lottery and either wins or does not. This is a sample space with two sample points.

# More Examples.

- A simpler example is to throw a die and observe the number of dots on the top face. These range from 1 to 6 and this is a sample space with six sample points.
- Another experiment might be the value of a stock as observed at the opening of the stock market every day. The sample space is again a set of non negative decimal numbers, with up to two decimal places.
  This is again a finite space in practice.
- We may survey students in a course and ask their opinion of the course content on a scale of 0 to 10. This is a sample space of 11 sample points.
- A person enters a lottery and either wins or does not. This is a sample space with two sample points.

# More Examples.

- A simpler example is to throw a die and observe the number of dots on the top face. These range from 1 to 6 and this is a sample space with six sample points.
- Another experiment might be the value of a stock as observed at the opening of the stock market every day. The sample space is again a set of non negative decimal numbers, with up to two decimal places.
  This is again a finite space in practice.
- We may survey students in a course and ask their opinion of the course content on a scale of 0 to 10. This is a sample space of 11 sample points.
- A person enters a lottery and either wins or does not. This is a sample space with two sample points.

# More Examples.

- A simpler example is to throw a die and observe the number of dots on the top face. These range from 1 to 6 and this is a sample space with six sample points.
- Another experiment might be the value of a stock as observed at the opening of the stock market every day. The sample space is again a set of non negative decimal numbers, with up to two decimal places.
  This is again a finite space in practice.
- We may survey students in a course and ask their opinion of the course content on a scale of 0 to 10. This is a sample space of 11 sample points.
- A person enters a lottery and either wins or does not. This is a sample space with two sample points.

# More Examples.

- A simpler example is to throw a die and observe the number of dots on the top face. These range from 1 to 6 and this is a sample space with six sample points.
- Another experiment might be the value of a stock as observed at the opening of the stock market every day. The sample space is again a set of non negative decimal numbers, with up to two decimal places.
  This is again a finite space in practice.
- We may survey students in a course and ask their opinion of the course content on a scale of 0 to 10. This is a sample space of 11 sample points.
- A person enters a lottery and either wins or does not. This is a sample space with two sample points.

# More Examples.

- A simpler example is to throw a die and observe the number of dots on the top face. These range from 1 to 6 and this is a sample space with six sample points.
- Another experiment might be the value of a stock as observed at the opening of the stock market every day. The sample space is again a set of non negative decimal numbers, with up to two decimal places.
  This is again a finite space in practice.
- We may survey students in a course and ask their opinion of the course content on a scale of 0 to 10. This is a sample space of 11 sample points.
- A person enters a lottery and either wins or does not. This is a sample space with two sample points.

# Events.

- Another example: A medicine is prescribed and it reduces the symptoms of a disease in a certain number of days. We keep track of how many days it takes for various patients. Thus the sample space is the number of days needed for the reduction to be observed. Again a theoretically infinite, but practically finite space, since any number larger than a life span has no meaning for this experiment.
- Events: Let us take the temperature example above. An event might be that the resulting temperature is between 95 and 124 degrees centigrade. The corresponding set of temperatures in this interval is the mathematical description of the event.
- In the experiment of casting a die, the event might be that of getting the top face to have an even number of dots, so the event is $\{2, 4, 6\}$.

# Events.

- Another example: A medicine is prescribed and it reduces the symptoms of a disease in a certain number of days. We keep track of how many days it takes for various patients. Thus the sample space is the number of days needed for the reduction to be observed. Again a theoretically infinite, but practically finite space, since any number larger than a life span has no meaning for this experiment.
- Events: Let us take the temperature example above. An event might be that the resulting temperature is between 95 and 124 degrees centigrade. The corresponding set of temperatures in this interval is the mathematical description of the event.
- In the experiment of casting a die, the event might be that of getting the top face to have an even number of dots, so the event is $\{2, 4, 6\}$.

# Events.

- Another example: A medicine is prescribed and it reduces the symptoms of a disease in a certain number of days. We keep track of how many days it takes for various patients. Thus the sample space is the number of days needed for the reduction to be observed. Again a theoretically infinite, but practically finite space, since any number larger than a life span has no meaning for this experiment.
- Events: Let us take the temperature example above. An event might be that the resulting temperature is between 95 and 124 degrees centigrade. The corresponding set of temperatures in this interval is the mathematical description of the event.
- In the experiment of casting a die, the event might be that of getting the top face to have an even number of dots, so the event is $\{2, 4, 6\}$.

# Events.

- Another example: A medicine is prescribed and it reduces the symptoms of a disease in a certain number of days. We keep track of how many days it takes for various patients. Thus the sample space is the number of days needed for the reduction to be observed. Again a theoretically infinite, but practically finite space, since any number larger than a life span has no meaning for this experiment.
- Events: Let us take the temperature example above. An event might be that the resulting temperature is between 95 and 124 degrees centigrade. The corresponding set of temperatures in this interval is the mathematical description of the event.
- In the experiment of casting a die, the event might be that of getting the top face to have an even number of dots, so the event is $\{2, 4, 6\}$.

# Events.

- Another example: A medicine is prescribed and it reduces the symptoms of a disease in a certain number of days. We keep track of how many days it takes for various patients. Thus the sample space is the number of days needed for the reduction to be observed. Again a theoretically infinite, but practically finite space, since any number larger than a life span has no meaning for this experiment.
- Events: Let us take the temperature example above. An event might be that the resulting temperature is between 95 and 124 degrees centigrade. The corresponding set of temperatures in this interval is the mathematical description of the event.
- In the experiment of casting a die, the event might be that of getting the top face to have an even number of dots, so the event is $\{2, 4, 6\}$.

# Making New Sample Spaces.

- Sometimes, one can generate more complicated sample spaces from simpler spaces.
- For example, we can throw two dice and add up the number of dots on the top faces. The sample space now changes to numbers between 2 and 12.
- Or we can take the student survey, but also ask questions about the course, the teacher and the textbook. Then one event might be that the average score of all three is at least 7 out of 10 - indicating an overall satisfaction.
- The sum of the scores on the three questions is a number between 0 and 30, but the average is a fraction between 0 and 10 with denominator 3. Thus the sample space has 30 sample points.

# Making New Sample Spaces.

- Sometimes, one can generate more complicated sample spaces from simpler spaces.
- For example, we can throw two dice and add up the number of dots on the top faces. The sample space now changes to numbers between 2 and 12.
- Or we can take the student survey, but also ask questions about the course, the teacher and the textbook. Then one event might be that the average score of all three is at least 7 out of 10 - indicating an overall satisfaction.
- The sum of the scores on the three questions is a number between 0 and 30, but the average is a fraction between 0 and 10 with denominator 3. Thus the sample space has 30 sample points.

# Making New Sample Spaces.

- Sometimes, one can generate more complicated sample spaces from simpler spaces.
- For example, we can throw two dice and add up the number of dots on the top faces. The sample space now changes to numbers between 2 and 12.
- Or we can take the student survey, but also ask questions about the course, the teacher and the textbook. Then one event might be that the average score of all three is at least 7 out of 10 - indicating an overall satisfaction.
- The sum of the scores on the three questions is a number between 0 and 30, but the average is a fraction between 0 and 10 with denominator 3. Thus the sample space has 30 sample points.

# Making New Sample Spaces.

- Sometimes, one can generate more complicated sample spaces from simpler spaces.
- For example, we can throw two dice and add up the number of dots on the top faces. The sample space now changes to numbers between 2 and 12.
- Or we can take the student survey, but also ask questions about the course, the teacher and the textbook. Then one event might be that the average score of all three is at least 7 out of 10 - indicating an overall satisfaction.
- The sum of the scores on the three questions is a number between 0 and 30, but the average is a fraction between 0 and 10 with denominator 3. Thus the sample space has 30 sample points.

# Making New Sample Spaces.

- Sometimes, one can generate more complicated sample spaces from simpler spaces.
- For example, we can throw two dice and add up the number of dots on the top faces. The sample space now changes to numbers between 2 and 12.
- Or we can take the student survey, but also ask questions about the course, the teacher and the textbook. Then one event might be that the average score of all three is at least 7 out of 10 - indicating an overall satisfaction.
- The sum of the scores on the three questions is a number between 0 and 30, but the average is a fraction between 0 and 10 with denominator 3. Thus the sample space has 30 sample points.

# Making New Sample Spaces.

- Sometimes, one can generate more complicated sample spaces from simpler spaces.
- For example, we can throw two dice and add up the number of dots on the top faces. The sample space now changes to numbers between 2 and 12.
- Or we can take the student survey, but also ask questions about the course, the teacher and the textbook. Then one event might be that the average score of all three is at least 7 out of 10 - indicating an overall satisfaction.
- The sum of the scores on the three questions is a number between 0 and 30, but the average is a fraction between 0 and 10 with denominator 3. Thus the sample space has 30 sample points.

# Counting Events.

- How many sample points does the described event have? Since the events are sets, we can use set theory to do standard set operations on events.
- Thus we talk of the union, the intersection and the complement of events.
- In natural language, an event occurs if we get a sample point that belongs to the event (set) during the experiment/observation.
- If two events cannot occur at the same time, then they have an empty intersection and are said to be mutually exclusive.
- An event which can never occur corresponds to an empty set.
- An event that always happens corresponds to the universal set, which is now called $S$ (and not $U$).

# Counting Events.

- How many sample points does the described event have? Since the events are sets, we can use set theory to do standard set operations on events.
- Thus we talk of the union, the intersection and the complement of events.
- In natural language, an event occurs if we get a sample point that belongs to the event (set) during the experiment/observation.
- If two events cannot occur at the same time, then they have an empty intersection and are said to be mutually exclusive.
- An event which can never occur corresponds to an empty set.
- An event that always happens corresponds to the universal set, which is now called $S$ (and not $U$).

# Counting Events.

- How many sample points does the described event have? Since the events are sets, we can use set theory to do standard set operations on events.
- Thus we talk of the union, the intersection and the complement of events.
- In natural language, an event occurs if we get a sample point that belongs to the event (set) during the experiment/observation.
- If two events cannot occur at the same time, then they have an empty intersection and are said to be mutually exclusive.
- An event which can never occur corresponds to an empty set.
- An event that always happens corresponds to the universal set, which is now called $S$ (and not $U$).

# Counting Events.

- How many sample points does the described event have? Since the events are sets, we can use set theory to do standard set operations on events.
- Thus we talk of the union, the intersection and the complement of events.
- In natural language, an event occurs if we get a sample point that belongs to the event (set) during the experiment/observation.
- If two events cannot occur at the same time, then they have an empty intersection and are said to be mutually exclusive.
- An event which can never occur corresponds to an empty set.
- An event that always happens corresponds to the universal set, which is now called $S$ (and not $U$).

# Counting Events.

- How many sample points does the described event have? Since the events are sets, we can use set theory to do standard set operations on events.
- Thus we talk of the union, the intersection and the complement of events.
- In natural language, an event occurs if we get a sample point that belongs to the event (set) during the experiment/observation.
- If two events cannot occur at the same time, then they have an empty intersection and are said to be mutually exclusive.
- An event which can never occur corresponds to an empty set.
- An event that always happens corresponds to the universal set, which is now called $S$ (and not $U$).

# Counting Events.

- How many sample points does the described event have? Since the events are sets, we can use set theory to do standard set operations on events.
- Thus we talk of the union, the intersection and the complement of events.
- In natural language, an event occurs if we get a sample point that belongs to the event (set) during the experiment/observation.
- If two events cannot occur at the same time, then they have an empty intersection and are said to be mutually exclusive.
- An event which can never occur corresponds to an empty set.
- An event that always happens corresponds to the universal set, which is now called $S$ (and not $U$).

# Counting Events.

- How many sample points does the described event have? Since the events are sets, we can use set theory to do standard set operations on events.
- Thus we talk of the union, the intersection and the complement of events.
- In natural language, an event occurs if we get a sample point that belongs to the event (set) during the experiment/observation.
- If two events cannot occur at the same time, then they have an empty intersection and are said to be mutually exclusive.
- An event which can never occur corresponds to an empty set.
- An event that always happens corresponds to the universal set, which is now called $S$ (and not $U$).

# Counting Events.

- How many sample points does the described event have? Since the events are sets, we can use set theory to do standard set operations on events.
- Thus we talk of the union, the intersection and the complement of events.
- In natural language, an event occurs if we get a sample point that belongs to the event (set) during the experiment/observation.
- If two events cannot occur at the same time, then they have an empty intersection and are said to be mutually exclusive.
- An event which can never occur corresponds to an empty set.
- An event that always happens corresponds to the universal set, which is now called $S$ (and not $U$).

# Mathematical Probability.

- In ordinary conversation, the word probability or chance has a rather loose meaning. It represents a personal observation/feeling about an event being likely or unlikely to happen.
- In mathematical treatment the probability of an event is a well defined number between 0 and 1,
- Assume that we know the size $n(S)$ of the sample space and know the size $n(E)$ of the event $E$ under consideration. Then we define:

$$P(E) = \frac{n(E)}{n(S)}$$

and call this the probability of the event $E$.

# Mathematical Probability.

- In ordinary conversation, the word probability or chance has a rather loose meaning. It represents a personal observation/feeling about an event being likely or unlikely to happen.
- In mathematical treatment the probability of an event is a well defined number between 0 and 1,
- Assume that we know the size $n(S)$ of the sample space and know the size $n(E)$ of the event $E$ under consideration. Then we define:

$$P(E) = \frac{n(E)}{n(S)}$$

and call this the probability of the event $E$.

# Mathematical Probability.

- In ordinary conversation, the word probability or chance has a rather loose meaning. It represents a personal observation/feeling about an event being likely or unlikely to happen.
- In mathematical treatment the probability of an event is a well defined number between 0 and 1,
- Assume that we know the size $n(S)$ of the sample space and know the size $n(E)$ of the event $E$ under consideration. Then we define:

$$P(E) = \frac{n(E)}{n(S)}$$

and call this the probability of the event $E$.

# Mathematical Probability.

- In ordinary conversation, the word probability or chance has a rather loose meaning. It represents a personal observation/feeling about an event being likely or unlikely to happen.
- In mathematical treatment the probability of an event is a well defined number between 0 and 1,
- Assume that we know the size $n(S)$ of the sample space and know the size $n(E)$ of the event $E$ under consideration. Then we define:

$$P(E) = \frac{n(E)}{n(S)}$$

and call this the probability of the event $E$.

# Mathematical Probability.

- In ordinary conversation, the word probability or chance has a rather loose meaning. It represents a personal observation/feeling about an event being likely or unlikely to happen.
- In mathematical treatment the probability of an event is a well defined number between 0 and 1,
- Assume that we know the size $n(S)$ of the sample space and know the size $n(E)$ of the event $E$ under consideration. Then we define:

$$P(E) = \frac{n(E)}{n(S)}$$

and call this the probability of the event $E$.

# Probability Continued.

- Now, if we our space $S$ is infinite, then suitable calculus has to be used to define and calculate the analog of $\frac{n(E)}{n(S)}$. Typically, this becomes an integration problem.
- Note that $P(E) = 0$ exactly when $E$ is empty or when the event never happens! Similarly, $P(E) = 1$ exactly when the set $E = S$, or the event always happens.
- For these two cases, the calculation makes sense even for infinite sets.

# Probability Continued.

- Now, if we our space $S$ is infinite, then suitable calculus has to be used to define and calculate the analog of $\frac{n(E)}{n(S)}$. Typically, this becomes an integration problem.
- Note that $P(E) = 0$ exactly when $E$ is empty or when the event never happens! Similarly, $P(E) = 1$ exactly when the set $E = S$, or the event always happens.
- For these two cases, the calculation makes sense even for infinite sets.

# Probability Continued.

- Now, if we our space $S$ is infinite, then suitable calculus has to be used to define and calculate the analog of $\frac{n(E)}{n(S)}$. Typically, this becomes an integration problem.
- Note that $P(E) = 0$ exactly when $E$ is empty or when the event never happens! Similarly, $P(E) = 1$ exactly when the set $E = S$, or the event always happens.
- For these two cases, the calculation makes sense even for infinite sets.

# Probability Continued.

- Now, if we our space $S$ is infinite, then suitable calculus has to be used to define and calculate the analog of $\frac{n(E)}{n(S)}$. Typically, this becomes an integration problem.
- Note that $P(E) = 0$ exactly when $E$ is empty or when the event never happens! Similarly, $P(E) = 1$ exactly when the set $E = S$, or the event always happens.
- For these two cases, the calculation makes sense even for infinite sets.

# Probability Continued.

- Now, if we our space $S$ is infinite, then suitable calculus has to be used to define and calculate the analog of $\frac{n(E)}{n(S)}$. Typically, this becomes an integration problem.
- Note that $P(E) = 0$ exactly when $E$ is empty or when the event never happens! Similarly, $P(E) = 1$ exactly when the set $E = S$, or the event always happens.
- For these two cases, the calculation makes sense even for infinite sets.

# Some probability calculations.

- Now we illustrate some simple problems.
  As before, we toss a die and observe the number on top.
  Thus $S = \{1, 2, 3, 4, 5, 6\}$.
- Consider the event $E$ that the top number is even or
  $E = \{2, 4, 6\}$.
  So $P(E) = \frac{3}{6} = 0.5$.
- Consider the event $F$ that the top number is prime. Then
  $F = \{2, 3, 5\}$ and $P(F)$ is also 0.5.
  Thus very different events can have identical probabilities.

# Some probability calculations.

- Now we illustrate some simple problems.
  As before, we toss a die and observe the number on top.
  Thus $S = \{1, 2, 3, 4, 5, 6\}$.
- Consider the event $E$ that the top number is even or
  $E = \{2, 4, 6\}$.
  So $P(E) = \frac{3}{6} = 0.5$.
- Consider the event $F$ that the top number is prime. Then
  $F = \{2, 3, 5\}$ and $P(F)$ is also 0.5.
  Thus very different events can have identical probabilities.

# Some probability calculations.

- Now we illustrate some simple problems.
  As before, we toss a die and observe the number on top.
  Thus $S = \{1, 2, 3, 4, 5, 6\}$.
- Consider the event $E$ that the top number is even or
  $E = \{2, 4, 6\}$.
  So $P(E) = \frac{3}{6} = 0.5$.
- Consider the event $F$ that the top number is prime. Then
  $F = \{2, 3, 5\}$ and $P(F)$ is also 0.5.
  Thus very different events can have identical probabilities.

# Some probability calculations.

- Now we illustrate some simple problems.
  As before, we toss a die and observe the number on top.
  Thus $S = \{1, 2, 3, 4, 5, 6\}$.
- Consider the event $E$ that the top number is even or $E = \{2, 4, 6\}$.
  So $P(E) = \frac{3}{6} = 0.5$.
- Consider the event $F$ that the top number is prime. Then $F = \{2, 3, 5\}$ and $P(F)$ is also 0.5.
  Thus very different events can have identical probabilities.

# Some probability calculations.

- Now we illustrate some simple problems.
  As before, we toss a die and observe the number on top.
  Thus $S = \{1, 2, 3, 4, 5, 6\}$.
- Consider the event $E$ that the top number is even or $E = \{2, 4, 6\}$.
  So $P(E) = \frac{3}{6} = 0.5$.
- Consider the event $F$ that the top number is prime. Then $F = \{2, 3, 5\}$ and $P(F)$ is also 0.5.
  Thus very different events can have identical probabilities.

# More Calculations.

- Now consider the experiment of tossing two dice and observing both numbers on top in sequence.
- Thus the sample space consists of 36 pairs
$$(1,1),(1,2),\cdots,(1,6),(2,1),\cdots,(2,6),\cdots,(6,1),(6,2),\cdots,(6,$$
.
It is customary to use the set builder notation:
$$S = \{(a,b) \mid 1 \le a, b \le 6\}.$$
- Consider the event
$$E = \{(a,b) \mid (a,b) \in S \text{ and } a+b \text{ is even. }\}$$
- We can enumerate $E$ thus:
If $a+b$ is even, then either both $(a,b)$ are odd or both are even. Clearly each of these sub events have $3 \cdot 3 = 9$ sample points and thus $n(E) = 18$.

# More Calculations.

- Now consider the experiment of tossing two dice and observing both numbers on top in sequence.
- Thus the sample space consists of 36 pairs

$$(1,1),(1,2),\cdots,(1,6),(2,1),\cdots,(2,6),\cdots,(6,1),(6,2),\cdots,(6,$$

.

It is customary to use the set builder notation:

$$S = \{(a,b) \mid 1 \leq a, b \leq 6\}.$$

- Consider the event

$$E = \{(a,b) \mid (a,b) \in S \text{ and } a+b \text{ is even. } \}$$

- We can enumerate $E$ thus:
If $a+b$ is even, then either both $(a,b)$ are odd or both are even. Clearly each of these sub events have $3 \cdot 3 = 9$ sample points and thus $n(E) = 18$.

# More Calculations.

- Now consider the experiment of tossing two dice and observing both numbers on top in sequence.
- Thus the sample space consists of 36 pairs

  $(1, 1), (1, 2), \cdots, (1, 6), (2, 1), \cdots, (2, 6), \cdots, (6, 1), (6, 2), \cdots, (6,$

  .

  It is customary to use the set builder notation:

  $$S = \{(a, b) \mid 1 \leq a, b \leq 6\}.$$

- Consider the event

  $$E = \{(a, b) \mid (a, b) \in S \text{ and } a + b \text{ is even. } \}$$

- We can enumerate $E$ thus:
  If $a + b$ is even, then either both $(a, b)$ are odd or both are even. Clearly each of these sub events have $3 \cdot 3 = 9$ sample points and thus $n(E) = 18$.

# More Calculations.

- Now consider the experiment of tossing two dice and observing both numbers on top in sequence.
- Thus the sample space consists of 36 pairs

$$(1,1),(1,2),\cdots,(1,6),(2,1),\cdots,(2,6),\cdots,(6,1),(6,2),\cdots,(6,$$

.

It is customary to use the set builder notation:

$$S = \{(a,b) \mid 1 \leq a, b \leq 6\}.$$

- Consider the event

$$E = \{(a,b) \mid (a,b) \in S \text{ and } a+b \text{ is even. } \}$$

- We can enumerate $E$ thus:
  If $a+b$ is even, then either both $(a,b)$ are odd or both are even. Clearly each of these sub events have $3 \cdot 3 = 9$ sample points and thus $n(E) = 18$.

# More Calculations.

- Now consider the experiment of tossing two dice and observing both numbers on top in sequence.
- Thus the sample space consists of 36 pairs

  $(1,1),(1,2),\cdots,(1,6),(2,1),\cdots,(2,6),\cdots,(6,1),(6,2),\cdots,(6,$

  .

  It is customary to use the set builder notation:

  $$S = \{(a,b) \mid 1 \leq a, b \leq 6\}.$$

- Consider the event

  $$E = \{(a,b) \mid (a,b) \in S \text{ and } a+b \text{ is even. } \}$$

- We can enumerate $E$ thus:
  If $a+b$ is even, then either both $(a,b)$ are odd or both are even. Clearly each of these sub events have $3 \cdot 3 = 9$ sample points and thus $n(E) = 18$.

# More Calculations.

- Now consider the experiment of tossing two dice and observing both numbers on top in sequence.
- Thus the sample space consists of 36 pairs

  $$(1,1),(1,2),\cdots,(1,6),(2,1),\cdots,(2,6),\cdots,(6,1),(6,2),\cdots,(6,$$

  .

  It is customary to use the set builder notation:

  $$S = \{(a,b) \mid 1 \le a, b \le 6\}.$$

- Consider the event

  $$E = \{(a,b) \mid (a,b) \in S \text{ and } a+b \text{ is even. } \}$$

- We can enumerate $E$ thus:
  If $a+b$ is even, then either both $(a,b)$ are odd or both are even. Clearly each of these sub events have $3 \cdot 3 = 9$ sample points and thus $n(E) = 18$.

# A paradox?

- So, $P(E) = \frac{18}{36} = 0.5$ again. Since the event of getting an odd sum is the complement of $E$, it must have 18 sample points as well, so probability 0.5 also.
- Now imagine that someone has put in pairs of dice in boxes so that different boxes have different pairs of dots on top. The difference is that the dice are no longer ordered and thus we have only one box for the choice $a = 3, b = 5$ and not the two $(3, 5), (5, 3)$ as before.
- We shall have 21 such boxes now. Why 21? There will be six boxes of the type $(a, a)$ and the boxes with two different numbers will be $C(6, 2) = 15$.
- Thus our sample space is now: [1] $S = \{\{a, b\} \mid 1 \leq a, b \leq 6\}$.

[1] Technically this is slightly wrong since this replaces a box with two copies of $a$ by the singleton set $\{a\}$. But it represents the same idea.

# A paradox?

- So, $P(E) = \frac{18}{36} = 0.5$ again. Since the event of getting an odd sum is the complement of $E$, it must have 18 sample points as well, so probability 0.5 also.
- Now imagine that someone has put in pairs of dice in boxes so that different boxes have different pairs of dots on top. The difference is that the dice are no longer ordered and thus we have only one box for the choice $a = 3, b = 5$ and not the two $(3, 5), (5, 3)$ as before.
- We shall have 21 such boxes now. Why 21? There will be six boxes of the type $(a, a)$ and the boxes with two different numbers will be $C(6, 2) = 15$.
- Thus our sample space is now: [1] $S = \{\{a, b\} \mid 1 \leq a, b \leq 6\}$.

[1] Technically this is slightly wrong since this replaces a box with two copies of $a$ by the singleton set $\{a\}$. But it represents the same idea.

# A paradox?

- So, $P(E) = \frac{18}{36} = 0.5$ again. Since the event of getting an odd sum is the complement of $E$, it must have 18 sample points as well, so probability 0.5 also.
- Now imagine that someone has put in pairs of dice in boxes so that different boxes have different pairs of dots on top. The difference is that the dice are no longer ordered and thus we have only one box for the choice $a = 3, b = 5$ and not the two $(3, 5), (5, 3)$ as before.
- We shall have 21 such boxes now. Why 21? There will be six boxes of the type $(a, a)$ and the boxes with two different numbers will be $C(6, 2) = 15$.
- Thus our sample space is now: [1] $S = \{\{a, b\} \mid 1 \leq a, b \leq 6\}$.

[1] Technically this is slightly wrong since this replaces a box with two copies of $a$ by the singleton set $\{a\}$. But it represents the same idea.

# A paradox?

- So, $P(E) = \frac{18}{36} = 0.5$ again. Since the event of getting an odd sum is the complement of $E$, it must have 18 sample points as well, so probability 0.5 also.
- Now imagine that someone has put in pairs of dice in boxes so that different boxes have different pairs of dots on top. The difference is that the dice are no longer ordered and thus we have only one box for the choice $a = 3, b = 5$ and not the two $(3, 5), (5, 3)$ as before.
- We shall have 21 such boxes now. Why 21? There will be six boxes of the type $(a, a)$ and the boxes with two different numbers will be $C(6, 2) = 15$.
- Thus our sample space is now: [1] $S = \{\{a, b\} \mid 1 \le a, b \le 6\}$.

[1] Technically this is slightly wrong since this replaces a box with two copies of $a$ by the singleton set $\{a\}$. But it represents the same idea.

# A paradox?

- So, $P(E) = \frac{18}{36} = 0.5$ again. Since the event of getting an odd sum is the complement of $E$, it must have 18 sample points as well, so probability 0.5 also.
- Now imagine that someone has put in pairs of dice in boxes so that different boxes have different pairs of dots on top. The difference is that the dice are no longer ordered and thus we have only one box for the choice $a = 3, b = 5$ and not the two $(3, 5), (5, 3)$ as before.
- We shall have 21 such boxes now. Why 21? There will be six boxes of the type $(a, a)$ and the boxes with two different numbers will be $C(6, 2) = 15$.
- Thus our sample space is now: [1] $S = \{\{a, b\} \mid 1 \leq a, b \leq 6\}$.

[1] Technically this is slightly wrong since this replaces a box with two copies of $a$ by the singleton set $\{a\}$. But it represents the same idea.

# A paradox?

- So, $P(E) = \frac{18}{36} = 0.5$ again. Since the event of getting an odd sum is the complement of $E$, it must have 18 sample points as well, so probability 0.5 also.
- Now imagine that someone has put in pairs of dice in boxes so that different boxes have different pairs of dots on top. The difference is that the dice are no longer ordered and thus we have only one box for the choice $a = 3, b = 5$ and not the two $(3, 5), (5, 3)$ as before.
- We shall have 21 such boxes now. Why 21? There will be six boxes of the type $(a, a)$ and the boxes with two different numbers will be $C(6, 2) = 15$.
- Thus our sample space is now: [1] $S = \{\{a, b\} \mid 1 \leq a, b \leq 6\}$.

[1] Technically this is slightly wrong since this replaces a box with two copies of $a$ by the singleton set $\{a\}$. But it represents the same idea.

# Paradox continued.

- Now we shall determining the probability of getting an "even box" i.e. a box with with a pair giving an even sum.
- Such even boxes can be counted thus.
  Boxes with both numbers identical: 6.
  Boxes with both numbers odd: 3:
  $$\{1,3\},\{1,5\},\{3,5\}$$
  Boxes with both numbers even: also 3. Total 12.
- Thus the probability of getting an even sum is now $\frac{12}{21} = 0.5714$, clearly bigger than 0.5.
- Thus, a careful analysis of the sample space is essential for determination of probability.
- Which calculation is correct? People can and do argue about it. Ultimately, it is a question of interpreting or observing what the true sample population is!

# Paradox continued.

- Now we shall determining the probability of getting an "even box" i.e. a box with with a pair giving an even sum.
- Such even boxes can be counted thus.
  Boxes with both numbers identical: 6.
  Boxes with both numbers odd: 3:

  $$\{1,3\},\{1,5\},\{3,5\}$$

  Boxes with both numbers even: also 3. Total 12.
- Thus the probability of getting an even sum is now $\frac{12}{21} = 0.5714$, clearly bigger than 0.5.
- Thus, a careful analysis of the sample space is essential for determination of probability.
- Which calculation is correct? People can and do argue about it. Ultimately, it is a question of interpreting or observing what the true sample population is!

# Paradox continued.

- Now we shall determining the probability of getting an "even box" i.e. a box with with a pair giving an even sum.
- Such even boxes can be counted thus.
  Boxes with both numbers identical: 6.
  Boxes with both numbers odd: 3:

$$\{1,3\},\{1,5\},\{3,5\}$$

  Boxes with both numbers even: also 3. Total 12.
- Thus the probability of getting an even sum is now $\frac{12}{21} = 0.5714$, clearly bigger than 0.5.
- Thus, a careful analysis of the sample space is essential for determination of probability.
- Which calculation is correct? People can and do argue about it. Ultimately, it is a question of interpreting or observing what the true sample population is!

# Paradox continued.

- Now we shall determining the probability of getting an "even box" i.e. a box with with a pair giving an even sum.
- Such even boxes can be counted thus.
  Boxes with both numbers identical: 6.
  Boxes with both numbers odd: 3:
  $$\{1,3\},\{1,5\},\{3,5\}$$
  Boxes with both numbers even: also 3. Total 12.
- Thus the probability of getting an even sum is now $\frac{12}{21} = 0.5714$, clearly bigger than 0.5.
- Thus, a careful analysis of the sample space is essential for determination of probability.
- Which calculation is correct? People can and do argue about it. Ultimately, it is a question of interpreting or observing what the true sample population is!

# Paradox continued.

- Now we shall determining the probability of getting an "even box" i.e. a box with with a pair giving an even sum.
- Such even boxes can be counted thus.
  Boxes with both numbers identical: 6.
  Boxes with both numbers odd: 3:

$$\{1,3\},\{1,5\},\{3,5\}$$

  Boxes with both numbers even: also 3. Total 12.
- Thus the probability of getting an even sum is now $\frac{12}{21} = 0.5714$, clearly bigger than 0.5.
- Thus, a careful analysis of the sample space is essential for determination of probability.
- Which calculation is correct? People can and do argue about it. Ultimately, it is a question of interpreting or observing what the true sample population is!

# Paradox continued.

- Now we shall determining the probability of getting an "even box" i.e. a box with with a pair giving an even sum.
- Such even boxes can be counted thus.
  Boxes with both numbers identical: 6.
  Boxes with both numbers odd: 3:

  $$\{1,3\},\{1,5\},\{3,5\}$$

  Boxes with both numbers even: also 3. Total 12.
- Thus the probability of getting an even sum is now $\frac{12}{21} = 0.5714$, clearly bigger than 0.5.
- Thus, a careful analysis of the sample space is essential for determination of probability.
- Which calculation is correct? People can and do argue about it. Ultimately, it is a question of interpreting or observing what the true sample population is!

# Paradox continued.

- Now we shall determining the probability of getting an "even box" i.e. a box with with a pair giving an even sum.
- Such even boxes can be counted thus.
  Boxes with both numbers identical: 6.
  Boxes with both numbers odd: 3:
  $$\{1,3\},\{1,5\},\{3,5\}$$
  Boxes with both numbers even: also 3. Total 12.
- Thus the probability of getting an even sum is now $\frac{12}{21} = 0.5714$, clearly bigger than 0.5.
- Thus, a careful analysis of the sample space is essential for determination of probability.
- Which calculation is correct? People can and do argue about it. Ultimately, it is a question of interpreting or observing what the true sample population is!

# An Infinite Sample Space.

- Consider the experiment of tossing a coin repeatedly, until you get a head up. Thus, for each $n = 1, 2, \cdots$ you can represent a sample point as $T \cdots TH$ where the first $(n-1)$ letters are $T$ (tail) and the last is $H$.
  Let $E_n$ denote the simple event represented by it.
- What is $P(E_n)$?
- Also, consider the following game: You get a dollar for every tail showing up. You get nothing when a head shows up and it also stops the game. Also, you get noting more.
  How much would you expect to win in this game? (Thus, how much would you be willing to pay as a fair fee to play this game?)
- We analyze both these questions, assuming a fair coin is being tossed, so that probability of getting a head or tail is the same, namely 0.5.

# An Infinite Sample Space.

- Consider the experiment of tossing a coin repeatedly, until you get a head up. Thus, for each $n = 1, 2, \cdots$ you can represent a sample point as $T \cdots TH$ where the first $(n-1)$ letters are $T$ (tail) and the last is $H$.
  Let $E_n$ denote the simple event represented by it.
- What is $P(E_n)$?
- Also, consider the following game: You get a dollar for every tail showing up. You get nothing when a head shows up and it also stops the game. Also, you get noting more.
  How much would you expect to win in this game? (Thus, how much would you be willing to pay as a fair fee to play this game?)
- We analyze both these questions, assuming a fair coin is being tossed, so that probability of getting a head or tail is the same, namely 0.5.

# An Infinite Sample Space.

- Consider the experiment of tossing a coin repeatedly, until you get a head up. Thus, for each $n = 1, 2, \cdots$ you can represent a sample point as $T \cdots TH$ where the first $(n-1)$ letters are $T$ (tail) and the last is $H$.
  Let $E_n$ denote the simple event represented by it.
- What is $P(E_n)$?
- Also, consider the following game: You get a dollar for every tail showing up. You get nothing when a head shows up and it also stops the game. Also, you get noting more.
  How much would you expect to win in this game? (Thus, how much would you be willing to pay as a fair fee to play this game?)
- We analyze both these questions, assuming a fair coin is being tossed, so that probability of getting a head or tail is the same, namely 0.5.

# An Infinite Sample Space.

- Consider the experiment of tossing a coin repeatedly, until you get a head up. Thus, for each $n = 1, 2, \cdots$ you can represent a sample point as $T \cdots TH$ where the first $(n-1)$ letters are $T$ (tail) and the last is $H$.
  Let $E_n$ denote the simple event represented by it.
- What is $P(E_n)$?
- Also, consider the following game: You get a dollar for every tail showing up. You get nothing when a head shows up and it also stops the game. Also, you get noting more.
  How much would you expect to win in this game? (Thus, how much would you be willing to pay as a fair fee to play this game?)
- We analyze both these questions, assuming a fair coin is being tossed, so that probability of getting a head or tail is the same, namely 0.5.

# An Infinite Sample Space.

- Consider the experiment of tossing a coin repeatedly, until you get a head up. Thus, for each $n = 1, 2, \cdots$ you can represent a sample point as $T \cdots TH$ where the first $(n-1)$ letters are $T$ (tail) and the last is $H$.
  Let $E_n$ denote the simple event represented by it.
- What is $P(E_n)$?
- Also, consider the following game: You get a dollar for every tail showing up. You get nothing when a head shows up and it also stops the game. Also, you get noting more.
  How much would you expect to win in this game? (Thus, how much would you be willing to pay as a fair fee to play this game?)
- We analyze both these questions, assuming a fair coin is being tossed, so that probability of getting a head or tail is the same, namely 0.5.

# An Infinite Sample Space.

- Consider the experiment of tossing a coin repeatedly, until you get a head up. Thus, for each $n = 1, 2, \cdots$ you can represent a sample point as $T \cdots TH$ where the first $(n-1)$ letters are $T$ (tail) and the last is $H$.
  Let $E_n$ denote the simple event represented by it.
- What is $P(E_n)$?
- Also, consider the following game: You get a dollar for every tail showing up. You get nothing when a head shows up and it also stops the game. Also, you get noting more.
  How much would you expect to win in this game? (Thus, how much would you be willing to pay as a fair fee to play this game?)
- We analyze both these questions, assuming a fair coin is being tossed, so that probability of getting a head or tail is the same, namely 0.5.

# Coin Example Continued.

- A little thought shows that $P(E_n) = \frac{1}{2^n}$. Now we know that $P(S)$ has to be equal to 1, since the sample space has to contain all possible sample points.

$$P(S) = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1$$

  as desired. (We used the G.P. series formula.)

- Now we answer the second question. The payoff of $E_n$ is exactly $(n-1)$ dollars. From what we showed above, the probability of winning $n-1$ dollars is exactly $\frac{1}{2^n}$.
- The expected payoff of the game is then defined as

$$\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots = \sum_{1}^{\infty} \frac{n-1}{2^n}.$$

  It can be shown that it evaluates to 1. Thus a fair fee to play this game is \$1.

# Coin Example Continued.

- A little thought shows that $P(E_n) = \frac{1}{2^n}$. Now we know that $P(S)$ has to be equal to 1, since the sample space has to contain all possible sample points.

$$P(S) = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1$$

  as desired. (We used the G.P. series formula.)

- Now we answer the second question. The payoff of $E_n$ is exactly $(n-1)$ dollars. From what we showed above, the probability of winning $n-1$ dollars is exactly $\frac{1}{2^n}$.
- The expected payoff of the game is then defined as

$$\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots = \sum_1^\infty \frac{n-1}{2^n}.$$

  It can be shown that it evaluates to 1. Thus a fair fee to play this game is $1.

# Coin Example Continued.

- A little thought shows that $P(E_n) = \frac{1}{2^n}$. Now we know that $P(S)$ has to be equal to 1, since the sample space has to contain all possible sample points.

$$P(S) = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1$$

  as desired. (We used the G.P. series formula.)
- Now we answer the second question. The payoff of $E_n$ is exactly $(n-1)$ dollars. From what we showed above, the probability of winning $n-1$ dollars is exactly $\frac{1}{2^n}$.
- The expected payoff of the game is then defined as

$$\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots = \sum_{1}^{\infty} \frac{n-1}{2^n}.$$

  It can be shown that it evaluates to 1. Thus a fair fee to play this game is \$1.

# More Expected Value.

- How did we calculate this infinite sum? You need to review your Calculus and theory of infinite series.
- Here is a more complicated game which can also be analyzed with Calculus.
- If you get \$1, for the first tail, \$2 for the second tail and so on, then what is your expected payoff?
  Thus, if you get 9 tails following by head, you expect to win $1+2+3+4+5+6+7+8+9=45$ dollars!
- The answer comes out to be the infinite sum
  $$\sum_1^\infty \frac{(n-1)n}{2} \cdot \frac{1}{2^n}.$$
  Here we use the sum of A.P. formula to write the terms.
- The answer can be shown to be \$2. And so, even though the game looks a lot more attractive, its fair fee is only \$2.

# More Expected Value.

- How did we calculate this infinite sum? You need to review your Calculus and theory of infinite series.
- Here is a more complicated game which can also be analyzed with Calculus.
- If you get \$1, for the first tail, \$2 for the second tail and so on, then what is your expected payoff?
  Thus, if you get 9 tails following by head, you expect to win $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ dollars!
- The answer comes out to be the infinite sum

$$\sum_{1}^{\infty} \frac{(n-1)n}{2} \cdot \frac{1}{2^n}.$$

  Here we use the sum of A.P. formula to write the terms.
- The answer can be shown to be \$2. And so, even though the game looks a lot more attractive, its fair fee is only \$2.

# More Expected Value.

- How did we calculate this infinite sum? You need to review your Calculus and theory of infinite series.
- Here is a more complicated game which can also be analyzed with Calculus.
- If you get \$1, for the first tail, \$2 for the second tail and so on, then what is your expected payoff?
  Thus, if you get 9 tails following by head, you expect to win $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ dollars!
- The answer comes out to be the infinite sum

$$\sum_{1}^{\infty} \frac{(n-1)n}{2} \cdot \frac{1}{2^n}.$$

  Here we use the sum of A.P. formula to write the terms.
- The answer can be shown to be \$2. And so, even though the game looks a lot more attractive, its fair fee is only \$2.

# More Expected Value.

- How did we calculate this infinite sum? You need to review your Calculus and theory of infinite series.
- Here is a more complicated game which can also be analyzed with Calculus.
- If you get \$1, for the first tail, \$2 for the second tail and so on, then what is your expected payoff?
  Thus, if you get 9 tails following by head, you expect to win $1+2+3+4+5+6+7+8+9=45$ dollars!
- The answer comes out to be the infinite sum

$$\sum_{1}^{\infty} \frac{(n-1)n}{2} \cdot \frac{1}{2^n}.$$

  Here we use the sum of A.P. formula to write the terms.
- The answer can be shown to be \$2. And so, even though the game looks a lot more attractive, its fair fee is only \$2.

# More Expected Value.

- How did we calculate this infinite sum? You need to review your Calculus and theory of infinite series.
- Here is a more complicated game which can also be analyzed with Calculus.
- If you get \$1, for the first tail, \$2 for the second tail and so on, then what is your expected payoff?
  Thus, if you get 9 tails following by head, you expect to win $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ dollars!
- The answer comes out to be the infinite sum

$$\sum_{1}^{\infty} \frac{(n-1)n}{2} \cdot \frac{1}{2^n}.$$

  Here we use the sum of A.P. formula to write the terms.
- The answer can be shown to be \$2. And so, even though the game looks a lot more attractive, its fair fee is only \$2.