# Lecture on sections 1.3,1.4

Ma 162 Spring 2010

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# Review: Basic Definitions.

- A linear function of one variable $x$ is a function $f(x) = mx + c$ where $m, c$ are constants. Its graph is a straight line, hence it is called linear.
  **Example:** $f(x) = 3x + 4$.
- A linear function of two variables $x, y$ is of the form $f(x, y) = ax + by + c$. Its graph is a plane in three space.
  **Example:** $f(x, y) = 3x + 4y + 5$.
- A natural generalization is a linear function of $n$ variables $f(x_1, x_2, \cdots, x_n) = a_1 x_1 + a_2 x_2 + \cdots a_n x_n + b$ where $a_1, a_2, \cdots, a_n, b$ are constants.
  **Example:** $f(x, y, z) = 3x + 4y - 5z + 7$.
- These functions are useful in many applications.
- What are examples of functions which are **not** linear?
  **Example:**
  $f(x) = x^2 + 3x, g(x, y) = x^3 - y^3, h(x, y, z) = xy + yz + zx$.

# Some interesting Linear Functions

Here are some examples of real life functions which behave like linear functions.

- **Tax Calculations.** Typically, tax calculation on an income of $x$ dollars is a linear function when $x$ lies in a specific tax bracket. The function formula, however, changes with tax brackets. This is a good example of a step function which is defined by different formulas in different ranges of $x$ values. A typical formula looks like

$$t(x) = f + r(x - b)$$

where $x$ is assumed to be at least $b$, $f$ is the fixed tax for income $b$ and $r$ is the tax on each dollar earned above $b$.

## Tax Calculations continued.

**Example:** What is the tax on $49,000$, if tax is charged at 6% on all income above $15,000$?

**Answer:** Tax is $0.06(49000 - 15000) = 2040$ dollars.

**Example continued:** What is the tax if amounts above $50,000$ are charged at the rate of 7% and the income is $60,000$?

**Answer:**

Tax on $50,000$ by the first formula is $f = 0.06(50000 - 15000) = 2100$. Tax for the excess of $(60000 - 50000)$ or $10,000$ is $0.07(10000) = 700$.

Thus, the net tax is $2100 + 700 = 2800$ dollars.

# Depreciation.

- **Depreciation.** If an initial value $b$ is to be depreciated to value zero in $d$ years, then the depreciated value for any $t$ between 0 and $d$ is given by the formula:

$$v(t) = b - \frac{b}{d}t.$$

**Example:** If a car worth $45,000$ is to be depreciated to zero in 6 years, what is its value after 4 years.

**Answer:** Here $b = 45000$, $d = 6$ and $t = 4$.

So the formula gives $45000 - \frac{45000}{6}4 = 15000$.

# Financial Functions

- **Review of Business Functions.** If $x$ is the number of units sold or manufactured, then we have three natural functions associated with it.
- The cost function is $C(x) = cx + f$ where $c$ is the production cost per unit and $f$ is the fixed cost.
- The revenue function is $R(x) = px$ where $p$ is the selling price of a unit.
- The profit function is then given by
  $P(x) = R(x) - C(x) = (p - c)x - f$.

# Break even Analysis

We now describe how to determine the minimum production level which guarantees no net loss.

- **Break even Production.** A production level $x$ is said to be break even if the net profit is zero, i.e.

$$P(x) = R(x) - C(x) = (p - c)x - f = 0.$$

- **Another viewpoint.** The break even production can be interpreted alternatively as follows.
  Consider the graphs of the revenue function $R(x)$ and the cost function $C(x)$ plotted on the same axes. Both graphs are lines.
  Then the break even production is simply the $x$-coordinate of the common point.

## Example of Break even Analysis.

**Example.** Suppose that a certain toy can be manufactured for
$3.5 each and the cost of maintaining the workshop is $21,000 per
month.

If the toy is sold for $5, determine the monthly break even
production.

**Answer:** Let $x$ denote the monthly production. Then the revenue
function is $R(x) = 5x$. The cost function is $C(x) = 3.5x + 21000$.
We find the common point of the graphs of $y = R(x)$ and
$y = C(x)$, thus we solve:

$$5x = 3.5x + 21000 \text{ or } 1.5x = 21000.$$

The break even production is $x = \dfrac{21000}{1.5} = 14000.$

# Intersecting lines

The above example motivates a review of techniques to intersect two lines.

We consider two linear equations $ax + by = c$ and $px + qy = r$ and discuss their common solutions.

- **Substitution method.** The most familiar method is to solve one of the equations for $y$ and substitute the solution in the other to find the $x$-coordinate of the common point. Then use it and one of the equations to find $y$. **Example:** Solve

$$E1: \quad 3x - y = 5 \text{ and } E2: \quad 2x + 3y = 7.$$

**Answer:** Solving E1 for $y$, we get $y = 3x - 5$. Substitution in E2 gives: $2x + 3(3x - 5) = 7$ or $11x = 22$. This gives $x = 2$ and finally $y = 3x - 5 = 3(2) - 5 = 1$.

# Continued Intersections of Lines.

- Thus, the intersection of the lines

$$E1: \quad 3x - y = 5 \text{ and } E2: \quad 2x + 3y = 7$$

  is given by $x = 2, y = 1$.

- **Comment.** Though easy to understand, the above method is not always the most efficient and it is helpful to learn other techniques as well.

- **Cramer's Rule.** This technique lets us write down the answer to a system of two linear equations in two variables by a formula.
  It can also generalize to several equations in several variables.

# Determinants.

- **Determinant.** Given a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we define its determinant:

$$\det(A) = ad - bc.$$

Sometimes this is also written as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

**Example:**

$$\begin{vmatrix} 5 & -3 \\ 2 & 4 \end{vmatrix} = (5)(4) - (-3)(2) = 26.$$

# Cramer's Rule.

- Now we describe the Cramer's rule for solving two equations in two variables:

$$E1 : ax + by = c, \quad E2 : px + qy = r.$$

- Calculate the determinants:

$$\Delta = \left| \begin{array}{cc} a & b \\ p & q \end{array} \right|, \ \Delta_x = \left| \begin{array}{cc} c & b \\ r & q \end{array} \right|, \ \Delta_y = \left| \begin{array}{cc} a & c \\ p & r \end{array} \right|.$$

- Then the answers are:

$$x = \frac{\Delta_x}{\Delta} \text{ and } y = \frac{\Delta_y}{\Delta}.$$

# Cramer's Rule: continued.

- Recall the answers:

$$x = \frac{\Delta_x}{\Delta} \text{ and } y = \frac{\Delta_y}{\Delta}.$$

- These are undefined when $\Delta = 0$ and in that case we have:
  - If one of $\Delta_x, \Delta_y$ is non zero, then the system has no solution.
  - If all three determinants are zero then one equation is a multiple of another and we could have infinitely many solutions **unless** we happen to have an equation where all coefficients of variables are zero and some right hand side is non zero!

# Example of Cramer's Rule.

- We redo the old example.
  **Example:** Solve

  $$E1: \quad 3x - y = 5 \text{ and } E2: \quad 2x + 3y = 7.$$

  **Answer:** We see that

  $$\Delta = \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} 5 & -1 \\ 7 & 3 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} 3 & 5 \\ 2 & 7 \end{vmatrix}.$$

- Evaluation gives:

  $$\begin{array}{rclcl} \Delta &=& (3)(3) - (-1)(2) &=& 11 \\ \Delta_x &=& (5)(3) - (-1)(7) &=& 22 \\ \Delta_y &=& (3)(7) - (5)(2) &=& 11 \end{array}$$

- So the answers are:

  $$x = \frac{22}{11} = 2, \quad y = \frac{11}{11} = 1$$

  as already known!