

Lecture on section 2.3

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Terminology for Gauss-Jordan Elimination.

We now explain the general Gauss-Jordan Elimination process. First, some definitions:

- let M be a matrix. In application, we expect it to be the augmented matrix of a system of linear equations, however, this is not relevant to our definitions.

Assume that the **matrix** has m rows and n columns, so it is of **type** $m \times n$.

- A **pivot** in a row is the first non zero entry in it. The **pivot position** is the corresponding column number. The **pivot position sequence (p.p. for short)** is the sequence of such pivot positions in order.

If the row is full of zeros, then we declare that it does not have a pivot and the pivot position is declared to be ∞ .

Examples.

- **Examples.** Consider:

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

What are the pivots and the pivot positions?

Answer

For A , the p.p. is $(1, 2, 2)$ with pivots being 2, 1, 3 respectively.

For B , the p.p. is $(2, 1, 3)$ with pivots being $-1, 1, 2$ respectively.

For C , the p.p. is $(1, 2, \infty)$ with pivots being 2, 5 respectively.

Elementary Operations.

- A matrix is said to be in **Row Echelon Form or REF** if its p.p. is a strictly increasing sequence. **For this definition** we shall consider a sequence of ∞ to be a strictly increasing sequence!
- **Note:** In the above examples, C is in REF, while A, B are not.
- We allow two operations which help us put a matrix in REF.
- The first is a row swap. Thus, swapping the first and second row of B produces

$$B = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \text{ p.p. is } (1, 2, 3)$$

The notation for this is $R_1 \leftrightarrow R_2$.

Elementary Operations continued.

- The second operation consists of adding some multiple of one row to another.

For the matrix A , the p.p. is $(1, 2, 2)$ and we need to make the last row pivot position bigger to get REF.

- It is easy to see that subtracting $3R_2$ from R_3 does the trick. We shall write this operation as $R_3 - 3R_2$ with the convention that the first mentioned row is being replaced!
- The result is:

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

The new p.p. is $(1, 2, 3)$ with pivots $(2, 1, -1)$.

Getting to REF

- Now we outline the REF procedure in general. First find the p.p. of the given matrix.
- Use row swaps as needed to get the pivot positions increasing, but they may not be strictly increasing yet! For example, the matrix A above had p.p. $(1, 2, 2)$.
- If two successive rows have the same pivot position, use the earlier row to push the pivot position of the latter row as described below.
- In the example of matrix A , the pivot in R_2 was 1, while the pivot in R_3 was 3 in the same column. Call this entry 3 to be the target, which needs to become zero!
- If the target is in R_i and pivot in R_j , then the operation can be described as $R_i - \frac{\text{target}}{\text{pivot}} R_j$.
Here it becomes $R_3 - \frac{3}{1} R_2$ or $R_3 - 3R_2$.

RREF or alternate procedure after REF.

- We explained how to make an augmented matrix of a system of linear equations be in REF. We also explained how we can finish the solution process by a “back substitution” method. Now we explain an alternate procedure which is essential for the upcoming Simplex algorithm method. It also has the advantage that the final solution can be simply **read** from its display, without further manipulations.
- This form is called **Reduced Row Echelon Form or RREF**. The book calls it **the row-reduced form**.
- We shall first illustrate how to reach this RREF and then give its formal definition.

RREF example.

- To get RREF, we must first get REF. So, we shall start with the already worked example:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ \textcolor{red}{3} & 2 & 0 & 0 \\ 0 & \textcolor{red}{-5/3} & 2 & 4 \\ 0 & 0 & \textcolor{red}{-3/5} & \frac{49}{5} \end{array} \right]$$

Let us note that the vertical bar separator and the variable names are for understanding only and not part of calculations at this stage.

- We start with the pivot in the last row, namely, $-\frac{3}{5}$. The operation to perform is to make the pivot 1 by multiplying the row by a suitable number and then cleaning up all entries above it to zero.

RREF Example continued.

- The first operation is denoted as $-\frac{5}{3}R_3$ and gives a new matrix:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ \textcolor{red}{3} & 2 & 0 & 0 \\ 0 & \textcolor{red}{-5/3} & 2 & 4 \\ 0 & 0 & \textcolor{red}{-3/5} & \frac{49}{5} \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ \textcolor{red}{3} & \textcolor{blue}{2} & 0 & 0 \\ 0 & \textcolor{red}{-5/3} & \textcolor{blue}{2} & 4 \\ 0 & 0 & \textcolor{red}{1} & -\frac{49}{3} \end{array} \right]$$

- Next we cleanup the entry 2 above the pivot (now made 1) and this is done with $R_2 - 2R_3$.
- It yields:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ \textcolor{red}{3} & \textcolor{blue}{2} & 0 & 0 \\ 0 & \textcolor{red}{-5/3} & \textcolor{blue}{2} & 4 \\ 0 & 0 & \textcolor{red}{1} & -\frac{49}{3} \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ \textcolor{red}{3} & \textcolor{blue}{2} & 0 & 0 \\ 0 & \textcolor{red}{-5/3} & 0 & \frac{110}{3} \\ 0 & 0 & \textcolor{red}{1} & -\frac{49}{3} \end{array} \right]$$

Continued Example.

- Next, we cleanup the entry 2 in row 1 column 2. We use the pivot $-5/3$ so the operation shall be $R_1 - \frac{2}{-5/3}R_2 = R_1 + \frac{6}{5}R_2$.
- Thus we get:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ \textcolor{red}{3} & \textcolor{blue}{2} & 0 & 0 \\ 0 & -5/3 & 0 & \frac{110}{3} \\ 0 & 0 & \textcolor{red}{1} & -\frac{49}{3} \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ \textcolor{red}{3} & 0 & 0 & 44 \\ 0 & -5/3 & 0 & \frac{110}{3} \\ 0 & 0 & \textcolor{red}{1} & -\frac{49}{3} \end{array} \right]$$

- Finally, we make all pivots 1, i.e. we make $\frac{1}{3}R_1$ and $-\frac{3}{5}R_2$. This produces:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ \textcolor{red}{3} & 0 & 0 & 44 \\ 0 & -5/3 & 0 & \frac{110}{3} \\ 0 & 0 & \textcolor{red}{1} & -\frac{49}{3} \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ \textcolor{red}{1} & 0 & 0 & \frac{44}{3} \\ 0 & \textcolor{red}{1} & 0 & -22 \\ 0 & 0 & \textcolor{red}{1} & -\frac{49}{3} \end{array} \right]$$

Reading the Answer.

- Recall that we have the final form:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 1 & 0 & 0 & \frac{44}{3} \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & -\frac{49}{3} \end{array} \right]$$

- Now we see the main advantage of RREF. The final solution is clearly visible on the RHS. To read the value of x_1 , find the pivot under it and read off the value of RHS, namely $\frac{44}{3}$. Similarly, $x_2 = -22$, $x_3 = -\frac{49}{3}$.
- Thus, RREF takes care of the back substitution without writing the equations again.

How to solve equations in RREF.

- We now give an example of a system in RREF with infinitely many solutions.

Consider an augmented matrix:

$$\left[\begin{array}{cccc|c} x & y & z & w & RHS \\ \textcolor{red}{1} & 0 & 2 & 0 & \textcolor{blue}{2} \\ 0 & \textcolor{red}{1} & 3 & 0 & \textcolor{blue}{5} \\ 0 & 0 & 0 & \textcolor{red}{1} & \textcolor{blue}{3} \end{array} \right]$$

- The pivot variables are x, y, w while z is a non pivot variable. Thus we solve the three equations in order for x, y, w in terms of the non pivot (or free) variable z .
- Thus, the answer is: $x = \textcolor{blue}{2} - \textcolor{red}{2}z$, $y = \textcolor{blue}{5} - \textcolor{red}{3}z$, $w = \textcolor{blue}{3}$. Here z is arbitrary.
- All we did was to move all the z terms to the RHS and then read off the solutions as before.

RREF defined.

- An augmented matrix is said to be in RREF if it is in REF and satisfies the following additional conditions:
 - ① The pivot in each row is equal to 1.
 - ② All entries in the pivot column below **or above** the pivot are equal to zero. The book describes this as the pivot column being a unit column, this being a typical column of the identity matrix.
- Of course, it can happen that in RREF, the equations are inconsistent. This happens when the pivot of some row only appears on RHS. Such equations have no solutions.

Some Terminology.

- The number of pivots in RREF only depends on the starting matrix and is called its rank.
- For obvious reasons, rank of a matrix is less than or equal to its number of rows as well as number of columns.
- The system either has no solutions (if an inconsistent equation is present) or a unique solution (if all variables are pivot variables) or infinitely many, if there is at least one non pivot variable.

This is the so-called $0 - 1 - \infty$ principle of linear algebra.

Comments.

- We can make some general observations based on the above.
- Given a system of m equations in n variables, let r be the rank (i.e. the number of pivots) in the RREF of its augmented matrix.
- $r \leq \min\{m, n\}$.
- The system has a unique solution iff $r = n$.
- The system is consistent (i.e. has at least one solution) iff no pivot occurs on the RHS.
- The number of free (arbitrary) variables is $n - r$.
- The general solution expresses the pivot variables as suitable constants plus certain combinations of the non pivot variables.