Lecture on section 2.4-6

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Definition of a Matrix.

Here we mainly cover 2.5 in great detail. You should study 2.4 from the examples in the book and on WHS.

- A matrix is simply a rectangular array of numbers. If M is a matrix with m rows and n columns, then we say it is of type $m \times n$ and convey the same meaning by saying $M = M_{m\times n}$.
  Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -3 \\ 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 3 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

- These have types $A = A_{4\times 2}, B = B_{1\times 5}, C = C_{5\times 1}$.
- The matrix B above is said to be **a row matrix** and C is said to be **a column matrix** for obvious reasons.

Matrix operations.

- Individual entries of a matrix are conveniently denoted by a subscript notation. Thus for the matrix A above, we have $A_{22} = 4$ and we may find it more convenient to write it as $A(2,2) = 4$.
  Note that
  $$A(2,1) = A(4,1) = 3 = B(1,3) = C(3,1).$$
- **Addition.** Let P, Q be matrices of the same type. We define $P + Q$ by the formula
  $$(P + Q)(i,j) = P(i,j) + Q(i,j).$$
  If the types are mismatched, then **the sum is undefined.**
- **Scalar Multiplication.** Let P be a matrix and c any number (which is also called a scalar. Then we define cP by the formula
  $$(cP)(i,j) = cP(i,j).$$

More definitions.

- **Example.** Let
$$P = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}.$$
Let $R = P + Q$.
Then
$$R = \begin{bmatrix} 2+3 & 5+1 \\ 1-1 & 7+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix}.$$
- Also, if $c = 5$ then
$$cP = \begin{bmatrix} 5 \cdot 2 & 5 \cdot 5 \\ 5 \cdot 1 & 5 \cdot 7 \end{bmatrix} = \begin{bmatrix} 10 & 25 \\ 5 & 35 \end{bmatrix}.$$
- Thus:
$$2P - 3Q = \begin{bmatrix} 2 \cdot 2 - 3 \cdot 3 & 2 \cdot 5 - 3 \cdot 1 \\ 2 \cdot 1 - 3 \cdot (-1) & 2 \cdot 7 - 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ 5 & 8 \end{bmatrix}.$$

Matrix Product.

- Given matrices $A = A_{m \times n}$ and $B = B_{r \times s}$ we define their product **only when** $n = r$. This means that the number of columns of A matches the number of rows of B.
- The definition is:

$$(AB)(i,j) = A(i,1)B(1,j) + A(i,2)B(2,j) + \cdots + A(i,n)C(n,j).$$

- Thus for our earlier examples:

$$B = \begin{bmatrix} 2 & -1 & 3 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

we define

$$BC = \begin{bmatrix} 2 \cdot 2 + (-1) \cdot (1) + 3 \cdot 3 + 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 13 \end{bmatrix}.$$

Understanding the Matrix Product.

- Thus, if we multiply a row and a column of the same length we get a 1×1 matrix and it is often written as a single number without the square brackets.
- This helps us understand the general product thus.
- Consider the old

$$P = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}.$$

- Write $P = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ where R_1, R_2 are its two rows $\begin{bmatrix} 2 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 7 \end{bmatrix}$.
  Similarly, write $Q = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$ where C_1, C_2 are its columns $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Matrix product explained.

- Then we have:

$$PQ = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} R_1 C_1 & R_1 C_2 \\ R_2 C_1 & R_2 C_2 \end{bmatrix}.$$

- Thus, we see

$$PQ = \begin{bmatrix} 2 \cdot 3 + 5 \cdot (-1) & 2 \cdot 1 + 5 \cdot 2 \\ 1 \cdot 3 + 7 \cdot (-1) & 1 \cdot 1 + 7 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 15 \end{bmatrix}.$$

Another view of the product.

- Here is another useful view of a matrix product. Suppose that we have a matrix $A = A_{m \times n}$ and we multiply it by a column $X = X_{n \times 1}$. Then we can interpret AX thus: Write

$$A = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}.$$

Then

$$AX = x_1 C_1 + x_2 C_2 + \cdots + x_n C_n.$$

Calculating the full product.

- Thus we can find the same old PQ by multiplying P by each column of Q and building a matrix from them.
- We already know:

$$\begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

Similarly:

$$\begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \end{bmatrix}.$$

This gives another way to get PQ by as a matrix with these two columns. Both these methods are useful.

Special Matrices.

- A matrix is said to be **square**, if its type is $n \times n$ for some n.
- Note that the product AA is defined if and only if A is square. If A is square, then we use the more natural notation A^2 in place of AA.
  More generally A^m is defined as $AA \cdots A$ where we have exactly m terms of a square matrix A.
- A matrix with all zero entries is called a **zero matrix** and we abuse the notation by simply writing it as 0. It may be written as $0_{m \times n}$ to indicate its type, but this is rarely done. Thus the size of this 0 must be guessed by context.
- Another matrix which has an abused notation is the **identity or unit matrix.** It is denoted by I or I_n if its type is $n \times n$ matrix.
  The identity matrix is defined by the formula:

$$I(i,j) = 0 \text{ if } i \neq j \text{ and } I(i,i) = 1.$$

More Special Matrices.

- Thus, we have

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- These behave like 1 in our usual numbers. Thus $AI = A$ and $IA = A$ for all matrices A with the understanding that the I is chosen to make the product well defined.

Elementary Matrices.

- We performed some elementary operations on matrices to achieve desired forms like REF and RREF. We now show that these operations can be also explained as simply the result of multiplying by suitable special matrices called elementary matrices. We first define them.
- Let $n > 1$ be a chosen integer. Define $E_{rs}(c)$ to be the $n \times n$ matrix defined as follows.
  Start with I_n and write c for its (r, s)-th entry.
  Thus if $n = 3$, then

$$E_{21}(5) = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_{22}(5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using Elementary Matrices.

- Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -5 & -4 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$$

- Calculate $E_{21}(5)A$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -5 & -4 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix}.$$

- Thus we note that **multiplying on the left by** $E_{21}(5)$ has the same effect as our operation $R_2 + 5R_1$.
- It can be proved that when $i \neq j$ then left multiplying by $E_{ij}(c)$ has the same effect as $R_i + cR_j$.

Using Elementary Matrices continued.

- What about $E_{ii}(c)$?
  Consider $E_{22}(5)A$.
- $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -5 & -4 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -25 & -20 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$$
- Thus $E_{22}(5)$ has the same effect as $5R_2$ when we left multiply by it.
- In general $E_{ii}(c)$ has the same effect of cR_i.

The Voodoo Principle.

- How should we remember the effect of elementary matrices? Here is a simple trick.
- Say we wish to do the operation $R_2 + 5R_1$ on some $3 \times n$ matrix.
  Since it has 3 rows, we start with I_3 and perform the desired operation on it.

  $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+5R_1} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

  Thus we get $E_{21}(5)$ by doing the operation on I. We can now left multiply by it to any matrix to get the same row operation.

Voodoo Principle continued.

- Thus we see:

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & -1 \\ -5 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 11 & 1 & -3 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

- Any desired row operation can be thus performed. For instance if we want the sum of the three rows of the above matrix, we can simply multiply it by the sum of the three rows of I_3, i.e. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. Try this out!
- We can perform column operations as well, except we need to multiply by the modified identity matrices on the **right** instead of left.

Permutation matrices.

- The Voodoo principle can also be used to permute rows of a given matrix. Thus, for any matrix $A = A_{3\times n}$ we wish to swap its second and third rows.
- We start with I_3 (an identity matrix with the same number of rows and swap its second and third rows. We get:

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

- Now $P_{23}A$ gives:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -5 & -4 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ -5 & -4 & 0 \end{bmatrix}.$$

More Voodoo Principle.

- In general a permutation matrix is a matrix obtained by permuting rows of some I_n. Left multiplication by such a permutation matrix produces the same row permutation on any chosen matrix (with n rows).
- The same permutation matrix can also be interpreted as a column permutation. Thus our P_{23} above can be thought of as swapping the second and third columns of I_3.
- Then AP_{23} will do the same column permutation of A thus:

$$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -4 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -5 & 0 & -4 \\ 0 & 3 & 1 \end{bmatrix}.$$