Lecture continuation of 2.6

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Matrices for solving equations.

Now we describe the main use of matrices for solving systems of linear equations. In this lecture, we would mainly consider systems where the number of equations equals the number of variables.

- A linear system of n equations in n variables can be descibed by a single matrix equation of the form $AX = B$.

Examples.

- For example:
  The equations $2x - 3y = 1, x - 2y = 5$ can be written as

$$\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

- The solution, in turn can also be described as $IX = C$ which reduces to $X = C$.
- Thus, the solution to the above system is $x = -13, y = -9$ and can be written as:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -13 \\ -9 \end{bmatrix}.$$

Examples continued.

- The equations $x - y - z = 5, x + y + 2z = 0, 2x + y + 3z = 1$ can be written as

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

- The solution, as before can be written as $IX = C$ which reduces to $X = C$. Thus, the solution to the above system is $x = 4, y = 2, z = -3$ and can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}.$$

Inverse Philosophy.

- Both the solutions above can be described by the following simple philosophy.
  Let the original equations be $AX = B$ where we assume that A is a square $n \times n$ matrix, X is the column of n variables and B denotes the right hand sides.
  We **find** an $n \times n$ matrix M such that $AM = MA = I_n$.
- Multiplying both sides of the equation $AX = B$ by M on the left, we get $MAX = MB$ which becomes

$$IX = MB \text{ and yields the solution } X = MB.$$

- Thus, it would be good to have a mechanism for finding such a matrix M when possible.

Inverse Defined.

- We define the inverse of a **square matrix** A to be a square matrix M such that $MA = AM = I$.
- The matrix M can be shown to be uniquely defined by A, when it exists and is called the **inverse of A.**
- The matrix A is said to be invertible (or non singular) if its inverse exists and it is said to be non invertible or singular otherwise.

Finding the Inverse $\mathbf{2} \times \mathbf{2}$ case.

- If
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
then we have a very simple answer.
Let $\Delta = \det(A) = ad - bc$.
- Then A is invertible if and only if $\Delta \neq 0$.
- Moreover, if $\Delta \neq 0$ then the inverse of A is the matrix
$$\frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
- Thus, for our first example above $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$. Then $\Delta = 2 \cdot (-2) - 1 \cdot (-3) = -1$ and hence the inverse is
$$M = \frac{1}{-1} \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

Inverse Calculation continued.

- It is easy to check that $M\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}-13\\-9\end{bmatrix}$ is the old solution. As a side note, we observe that in this case the inverse M is the same as A, or $AA = A^2 = I$. Such matrices are said to be **unipotent**.
- **Important notation.** When the inverse of A exists, it is denoted by the convenient notation A^{-1}.
  Do not ever write $\frac{1}{A}$ in place of A^{-1}; it is both illegal and meaningless.

An observation.

- We solved the equation $AX = B$ above as $X = MB = A^{-1}B$ for a specific 2×2 matrix A. Note that B did not enter the calculation until the product MB.
- Thus we observe that if A is invertible, then the equation $AX = B$ has a unique solution $X = A^{-1}B$.
- This should be compared with the statement: If a,b are numbers and if $a \neq 0$ then the equation $ax = b$ has a unique solution $x = \frac{b}{a}$.
- What happens if $\Delta = 0$. Let $P = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and consider the equations $PX = Q$ where $Q = \begin{bmatrix} u \\ v \end{bmatrix}$.
- We invite you to check that when $v = 2u$ this system has infinitely many solution, but it has no solution when $v \neq 2u$.

General Inverses.

- Now we discuss the general inverse. the formula is not as convenient as in the 2×2 case. So we give a procedure.
- Suppose we are trying to find the inverse of a matrix $A = A_{n \times n}$.
  Start with the augmented matrix $[A|I]$ and row reduce it, i.e. find its RREF.
- The matrix A is invertible if and only if the RREF becomes $[I|M]$ for some $n \times n$ matrix M. Moreover, this M is the desired A^{-1}.
- If one of the pivots is on the right hand side of the separator bar, then the matrix A is non invertible or singular.

Example of inverse.

- We now illustrate the procedure on our 3×3 matrix in the second example.
- Start with:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right].$$

- When we do row transformations $R_2 - R_1$, $R_3 - 2R_1$ and $R_3 - \frac{3}{2}R_2$, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1/2 & -1/2 & -3/2 & 1 \end{array}\right].$$

This is REF.

Inverse continued.

- Now we go on to make RREF.
- The operations

$$2R_3, R_2 - 3R_3, R_1 + R_3, \frac{1}{2}R_2, R_1 + R_2$$

produce the RREF:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 5 & -3 \\ 0 & 0 & 1 & -1 & -3 & 2 \end{array}\right]$$

- Thus the desired inverse is:

$$A^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 5 & -3 \\ -1 & -3 & 2 \end{bmatrix}$$

.

Summary.

- Verify that our answer is correct.
  Thus:
  $$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 5 & -3 \\ -1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}.$$
- We note that when we finish the work of converting $[A|I]$ we can either find the inverse or determine that the inverse does not exist.
- If A has an inverse, then the equations $AX = B$ always have a unique solution $X = A^{-1}B$.
- The main drawback of this method is the calculation of the RREF which can be lengthy.

Testing Invertibility.

- Thus, it would be useful to know if we are likely to find an inverse before doing the full work.
- Luckily, we already have such a tool. Let A be an $n \times n$ matrix.
  We can convert $[A|I]$ to REF. It is easy to see that we have exactly n pivots.
- The inverse exists if and only if all the pivots are on the left hand side of the separator bar.
  It is instructive to observe the following:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \quad R_2 - 2R_1 \Rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

Since the second row has pivot on RHS, we have no inverse!

Transpose.

- There is one important but easy operation on matrices called the transpose. Given a matrix $A = A_{m\times n}$ we flip it or turn its rows into columns and vice verse to get a new matrix of type $n \times m$ denoted by A^T.
- For example, we have:

$$\text{For } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ we get } A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

- It is not too hard to show that this satisfies the product rule $(AB)^T = B^T A^T$. It is recommended that you test this out by examples.
- A similar result holds for inverses. Namely, for square matrices A, B we have $(AB)^{-1} = B^{-1}A^{-1}$.