Understanding the Simplex algorithm.

Ma 162 Spring 2010

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Standard Optimization Problems.

- A standard maximization problem can be conveniently described in matrix form as follows.
  Maximize $P = CX$ subject to $AX \leq B$ and $X \geq 0$.
  Here, X is a column of the variables used in the problem, C is a row vector, so that $P = CX$ is a linear function of X. It is often similar to a profit function, hence the letter P.
- Moreover, in the current course we assume that $B \geq 0$. This insures that the choice $X = 0$ satisfies all the inequalities, i.e. is a feasible solution.
  Problems can be analyzed without this assumption, but we won't try to solve them not in this course.
- Here is an example (4.1 Example 3): Take

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 14 \\ 26 \\ 28 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}.$$

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Example continued.

- We construct a problem table (or tableau) recording all the coefficients as follows:

$$\left[\begin{array}{ccc|l} x & y & z & Constants \\ \hline 2 & 1 & 2 & 14 \\ 2 & 4 & 1 & 26 \\ 1 & 2 & 3 & 28 \\ \hline 2 & 2 & 1 & * \end{array}\right] \text{ or symbolically } \left[\begin{array}{c|l} X & Constants \\ \hline A & B \\ \hline C & * \end{array}\right].$$

- We are using the title "constants" to match the book notation. When we write the Simplex tableau, it may be changed to RHS, since in the tableau, we have equations, not just inequalities.

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Minimization tableau.

- There is a natural dual problem associated with this table and it has a dual problem table (tableau) as follows:

$$\left[\begin{array}{ccc|l} u & v & w & Constants \\ \hline 2 & 2 & 1 & 2 \\ 1 & 4 & 2 & 2 \\ 2 & 1 & 3 & 1 \\ \hline 14 & 26 & 28 & * \end{array}\right] \text{ or symbolically } \left[\begin{array}{c|l} Y & Constants \\ \hline A' & C' \\ \hline B' & * \end{array}\right]$$

  The matrices A', B', C' are the transposes of A, B, C respectively. Thus the whole table is simply a transpose.

- Our dual problem is described as follows: Minimize $14u + 26v + 28w$ subject to the conditions

$$2u+2v+w \geq 2, u+4v+2w \geq 2, 2u+v+3w \geq 1 \text{ and } u, v, w \geq 0.$$

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Duality theorem.

- The dual problem is described as a minimization problem as follows. We let Y be a row vector of variables, equal in number to the number of rows of A.

  $$\text{Minimize } YB \text{ subject to } YA \geq C \text{ and } Y \geq 0.$$

  Compare this with the inequalities above by taking $Y = \begin{bmatrix} u & v & w \end{bmatrix}$.
- The reason to write Y as a row rather than a column is somewhat technical and would be clarified below.
- The amazing theorem called the "duality theorem" states that any solution of the original maximization problem by the Simplex Algorithm produces a solution to its dual minimization problem by simply reading the final tableau. We describe this next.

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The solution from the algorithm.

- For our maximization problem above, we record the starting and the final tableaux and then show how to interpret them.
- Starting tableau:

$$\left[\begin{array}{ccccccc|c} x & y & z & u & v & w & P & RHS \\ \hline 2 & 1 & 2 & 1 & 0 & 0 & 0 & 14 \\ 2 & 4 & 1 & 0 & 1 & 0 & 0 & 26 \\ 1 & 2 & 3 & 0 & 0 & 1 & 0 & 28 \\ \hline -2 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

- End tableau:

$$\left[\begin{array}{ccccccc|c} x & y & z & u & v & w & P & RHS \\ \hline 1 & 0 & 7/6 & 2/3 & -1/6 & 0 & 0 & 5 \\ 0 & 1 & -1/3 & -1/3 & 1/3 & 0 & 0 & 4 \\ 0 & 0 & 5/2 & 0 & -1/2 & 1 & 0 & 15 \\ \hline 0 & 0 & 2/3 & 2/3 & 1/3 & 0 & 1 & 18 \end{array}\right].$$

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The interpretation of the solution.

- Inspection of the final tableau says that the final basis is x, y, w, P and hence the final basic solution is:

$$(x, y, z, u, v, w, P) = (5, 4, 0, 0, 0, 15, 18).$$

- The Voodoo Principle also tells us the actual row transformations that we performed. This information is read from the 4×4 matrix under the variables u, v, w, P.
- We know that the row transformations can be performed by multiplying the original 4×8 matrix by some 4×4 matrix on the left. We see that this transformation matrix must be

$$\left[\begin{array}{c|c} M & 0 \\ \hline Y & 1 \end{array}\right] = \left[\begin{array}{rrr|r} 2/3 & -1/6 & 0 & 0 \\ -1/3 & 1/3 & 0 & 0 \\ 0 & -1/2 & 1 & 0 \\ \hline 2/3 & 1/3 & 0 & 1 \end{array}\right].$$

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- The first part of the last row $Y = \left[\begin{array}{ccc} 2/3 & 1/3 & 0 \end{array}\right]$ tells us that we must have multiplied the first three rows by $2/3, 1/3, 0$ respectively and added to the last row.
- By looking at the the entries at the foot of the x, y, z columns, we deduce that

$$\left[\begin{array}{ccc} 0 & 0 & 2/3 \end{array}\right] = -C + YA$$

  since the original entries were $-C = \left[\begin{array}{ccc} -2 & -2 & -1 \end{array}\right]$ and we added YA to it. Thus $YA \geq C$.
- By looking at the last entry in the bottom row, we know that it was 0 and we have added YB to it.
- Thus, we have

$$Y \geq 0, YB = (2/3) \cdot (14) + (1/3) \cdot (26) + (0) \cdot (28) = 18.$$

- Thus Y is a feasible solution to the dual problem.

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$$Y \geq 0, YB = (2/3) \cdot (14) + (1/3) \cdot (26) + (0) \cdot (28) = 18.$$

- Thus Y is a feasible solution to the dual problem.

Interpretation continued.

- The first part of the last row $Y = \begin{bmatrix} 2/3 & 1/3 & 0 \end{bmatrix}$ tells us that we must have multiplied the first three rows by $2/3, 1/3, 0$ respectively and added to the last row.
- By looking at the the entries at the foot of the x, y, z columns, we deduce that
$$\begin{bmatrix} 0 & 0 & 2/3 \end{bmatrix} = -C + YA$$
since the original entries were $-C = \begin{bmatrix} -2 & -2 & -1 \end{bmatrix}$ and we added YA to it. Thus $YA \geq C$.
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Why do we have the dual problem solved?

- Recall that the two Linear Programming Problems (LPP) are:

  $$\text{Maximize: } P = CX \text{ s.t. } X \geq 0, AX \leq B$$

  and

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  Here we name the second function Q instead of C since C is used for the coefficients of P.
- Recall that by a feasible solution to either problem we mean a solution which satisfies all the inequalities, but may not give the maximum or minimum.
- If X_0, Y_0 are feasible solutions to the two problems respectively, then we see that $Y_0B \geq Y_0AX_0 \geq CX_0$. Thus the function value $Q_0 = Y_0B$ is always bigger than or equal to the function value $P_0 = CX_0$.

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- Thus, if X_0 and Y_0 are feasible solutions to the maximization and its dual minimization problems respectively and if

$$P_0 = CX_0 = Y_0 B = Q_0$$

then both must be simultaneously the optimum values for the respective problems, hence the solutions of both the problems at once!

- Thus for our dual problems $X_0 = (5, 4, 0)$ and $Y_0 = (2/3, 1/3, 0)$ are the respective solutions of the maximization and the minimization problems with a common function value 18.
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How to handle optimization problems?

- Recall that we always assume all our variables to be non negative in this course. In the following discussion, we are only discussing the remaining inequalities; we shall call them essential inequalities.
- If our essential inequalities are of $\leq$ type with non negative RHS, then we write it as a maximization problem and solve with the Simplex algorithm. (If we happen to be minimizing a function, we can always maximize its negative instead!)
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When do we fail?

- Recall that if we have a negative entry in the last row of a simplex tableau but no suitable pivot above because all such entries are less than or equal to zero, then our maximization problem is unbounded and has no solution.
  For the dual minimization problem, we can claim that there is no feasible solution. This means that its feasible region is empty!
- We can also have a case where the dual minimization problem is unbounded, but then the primal maximization problem shall have no feasible solution! This cannot occur under our standardness assumption.
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