DEPARTMENT OF MATHEMATICS

Ma162 Samples from old Final Exams

Detailed explanations are supplied only for problems in chapters 6,7.

1. Fred Foy has $\$100,000$ to invest in stocks, bonds and a money market account. The stocks have an expected return rate of 10% per year, the bonds pay 6% per year and the money market earns 2% per year.

   Fred insists that the money in the money market funds should equal the sum of 20% of the amount invested in stocks and 10% of the amount invested in bonds.

   Fred wants to allocate the $\$100,000$ among the three investments in order to provide an expected income of $\$8,000$ per year.

   Use x, y, z to denote the amounts invested in stocks, bonds and money market respectively.

   Answer the following questions.

   Comment: For our answers, we shall assume that x, y, z are amounts in thousands of dollars.

   (a) Write down a linear equation in x, y, z relating to the total investment.
   Answer: $x + y + z = 100$. (Note the comment above!)

   (b) Write down a linear equation relating to the expected annual income.
   Answer:$10x + 6y + 2z = 800$.

   (c) Write down additional linear equation(s) as needed to express the remaining conditions on the investments.
   Answer: $z = 0.2x + 0.1y$ or $2x + y - 10z = 0$.

   (d) Construct an appropriate augmented matrix to solve all the linear equations constructed above. It is not necessary to operate on this matrix any further at this point.
   Answer: Note that the matrix below is practically the answer, except for a few scaling factors.

   (e) Solve the linear equations represented by the following augmented matrix.

$$\left[\begin{array}{ccc|c} x & y & z & RHS \\ 1 & 1 & 1 & 100 \\ 5 & 3 & 1 & 400 \\ 2 & 1 & -10 & 0 \end{array}\right]$$

   Answer: Solution comes out $x = 65, y = 20, z = 15$. This will give the solution to Fred's problem in thousand dollars.

2. Perth mining company operates two mines for the purpose of extracting gold and silver.

   The Saddle mine costs $\$14,000$ per day to operate and it yields 50 ounces of gold and $3,000$ ounces of silver each day. The Horseshoe mine costs $\$16,000$ per day to operate and yields 75 ounces of gold and $1,000$ ounces of silver each day.

   The company management has a target of at least 650 ounces of gold and at least $18,000$ ounces of silver.

Set up a linear programming problem whose solution will determine how many days each mine should be operated to reach the target while minimizing the cost of operation.

You must define all the variables and list all the constraints. It is not necessary to solve the problem.

Answer: Let x, y be respectively the number of days for which the two mines Saddle and Horseshoe operate.

The conditions on x, y are: $x \geq 0, y \geq 0$, Gold target: $50x + 75y \geq 650$, Silver target: $3x + y \geq 18$ in units of 1000 ounces.

Objective function: Minimize $C = 14x + 16y$ in thousand dollars.

3. Consider these inequalities and answer the questions below.

$$x + y \leq 6\ ,\ \ 2x + y \leq 8\ ,\ \ x \geq 0\ ,\ \ y \geq 0.$$

   (a) Graph the feasible set of the above inequalities on the given graph paper.

   (b) List all the corner points of the feasible set. Be sure to mark them on your graph as well.
   Answer: The corner points come out to be: $(0, 0), (4, 0), (2, 4), (0, 6)$.

   (c) Determine the maximum value of $2x + 6y$ on the above feasible set. **Answer:** Evaluation of $2x + 6y$ produces maximum value 36 at $(0, 6)$.

4. In a group of 110 people, it is determined that 52 drink coffee, 41 drink tea and 39 don't drink either. A person is selected at random.

Answer the following:

Answer: First define A, B to be the set of coffee and tea drinkers.

Given:

$$n(A) = 52, n(B) = 41, n(S) = 110, n(S - A \bigcup B) = 39.$$

Deduce from set formulas $n(A \bigcup B) = 110 - 39 = 71$. Then $n(A \bigcap B) = 52 + 41 - 71 = 22$.

   (a) Compute the probability that the selected person drinks at least one of coffee or tea.
   Answer: We want $n(A \bigcup B)/n(S)$. $\frac{71}{110} = 0.6455$ or 64.55%.

   (b) Compute the probability that the selected person drinks both coffee and tea.
   Answer: We want $n(A \bigcap B)/n(S)$. So $\frac{22}{110} = 0.2$ or 20%.

   (c) Compute the probability that the selected person drinks only coffee.
   Answer: We want $n(A - B)/n(S)$. So $\frac{52-22}{110} = 0.2727$ or 27.27%.

   (d) In this experiment of selecting one person at random, are the events "is a coffee drinker" and "is a tea drinker" independent? Be sure to explain your answer.
   Answer: We are asking if $P(A) \cdot P(B) = P(A \bigcap B)$. In other words, is

$$\frac{52 \cdot 41}{110 \cdot 110} = \frac{22}{110}?$$

   The LHS evaluates to 0.1762 while RHS is 0.2 and we decide that they are not independent.

Warning: Note that the numbers would rarely come identically equal! We usually have to agree what is reasonably equal. One rule of thumb is that the percent probability differs by less than 0.5.

Here the percent probabilities are $17.62, 20$.

5. A bin in the hi-fi department of a bargain outlet contains 120 cassette tapes of which 15 are known to be defective. A customer randomly selects 7 of the tapes.

   Answer the following:

   (a) Describe the sample space and determine the number of elements in the sample space.

   Answer: The sample space is $C(120, 7)$ choices.

   (b) Find the probability that exactly 2 of the chosen tapes are defective.

   Answer: For this to happen, the event chooses 2 cassettes from the defective 15 and the remaining 5 from the good 105. The probability is:

$$\frac{C(15,2) \cdot C(105,5)}{C(120,7)} = 0.1704$$

   or 17.04%.

   (c) Find the probability that at least 1 of the chosen tapes is defective.

   Answer: It is easier to find the probability that no tape is defective. It will be $\frac{C(105,7)}{C(120,7)} = 0.3826$ or 38.26%.

   Our answer is then $1 - 0.3826 = 0.6174$ or 61.74%.

   (d) Find the probability that none of the chosen tapes are defective. **Answer:** Already found!!

6. In a high school there are 400 seniors of which 250 are female. 70% of the females and 50% of the males have their driver's licences.

   A student is chosen at random from the senior class.

   Answer the following:

   Note: Let S be the total student set with $n(S) = 400$. Let F and M be the set of females and males with counts $250, 400 - 250 = 150$ respectively.

   If D is the set of drivers, the $n(D) = 0.7 * 250 + 0.5 * 150 = 175 + 75 = 250$.

   (a) What is the probability that the chosen student is a female with a driver's licence?

   Answer: $n(F \bigcap D) = 175$, so the answer is $\frac{175}{400} = 0.4375$ or 43.75%.

   (b) What is the probability that the chosen student is a male with a driver's licence?

   Answer: Similarly $\frac{75}{400} = 0.1875$ or 18.75%.

   (c) What is the probability that the chosen student has a driver's licence?

   Answer: This is $\frac{n(D)}{400} = 0.6250$ or 62.5%.

   (d) Given that the chosen student does not have a driver's licence, what is the probability that the student is a male?

   Answer: The number of males among the non drivers is $150 - 75 = 75$.

   We are asking for $P(M|D^C) = \frac{75}{150} = 0.5$ or 50%.

7. Set this problem up, by stating the chosen variables, the function to be maximized and **all** the inequalities. **Do not solve the problem.**

WidgetSS makes and sells pulleys and sprockets.

Each pulley sells for \$ 18, requires 1.5 hours of finishing and 3.5 hours of machining.

Each sprocket sells for \$ 20, requires 2.5 hours of finishing and 1.5 hours of machining.

The company has 100 hours of finishing time and 90 hours of machining time available.

Set up a LPP whose solution will determine how many pulleys and how many sprockets should be made to maximize the company profit.

i) Define and explain all the variables you use.

Answer: Let x, y be the number of pulleys and sprockets manufactured.

ii) Now describe the LPP explicitly. We have the objective function $P = 18x + 20y$ to be maximized. The conditions are:

$x \geq 0, y \geq 0$.

Finishing constraint: $1.5x + 2.5y \leq 100$.

machining constraint: $3.5x + 1.5y \leq 90$.

iii) The initial Simplex tableau is:

x	y	u	v	P	$Constants$
1.5	2.5	1	0	0	100
3.5	1.5	0	1	0	90
-18	-20	0	0	1	0

.

8. i) Sketch and shade the region described by the inequalities. Compute the coordinates of the corner points and mark them on your graph.

$x + y \leq 5$

$y \geq 3$

$x \geq 0 \ , \ y \geq 0$

Answer: The corners come out to be $(0, 3), (2, 3), (0, 5)$.

ii) Find the maximum value of the function, $P = 3\,x + 5\,y$ on the region.

Answer: From the above corners, the maximum is at $(0, 5)$ with value 25.

9. Here is a final tableau associated with a maximal LPP.

x	y	z	s	t	u	P	constants
-2	0	1	0	-4	-13	0	6
-1	1	0	0	0	1	0	4
1	0	0	1	1	4	0	2
2	0	0	0	1	9	1	7

Using your knowledge of the Simplex algorithm, determine the solution to the maximal LPP.

Answer: The bottom row has no negatives, so it is the final tableau. The variables of the problem must be x, y, z while s, t, u are slack variables. The pivot variables are y, z, s, P with values $4, 6, 2, 7$ respectively.

So, we report the answer: $x = 0, y = 4, z = 6$ with maximum value of P equal to 7.

If we were to read the solution of the dual minimization problem from the same tableau, then its variables would correspond to s, t, u with values $0, 1, 9$.

Thus the minimization problem solution is:

Minimum value of $C = 7$ with $s = 0, t = 1, u = 9$.

10. Set this problem up, by stating the chosen variables, the equations to be solved and **the initial augmented matrix. Do not solve the problem.**

A toymaking company "Plaything" has a contract to supply 340 toys. It can supply any combination of three different types of toys with model names A,B,C. The cost of materials for the three types of toys is \$40,\$50, \$60 respectively.

Plaything has a total of \$ 17,400. available for buying the materials.

Each toy of type A requires 40 labor-hours, each toy of type B requires 25 labor-hours and each toy of type C requires 20 labor-hours.

Plaything has a total of 9,300. labor-hours available. Plaything wants to decide how many toys of each type should be manufactured to use all the resources and also fulfill the contract.

Use x,y,z to denote the number of toys produced of each of the types A,B,C respectively.

The equations to be solved are:

Answer:

$$x + y + z = 340,\ 40x + 50y + 60z = 17400,\ 40x + 25y + 20z = 9300.$$

The augmented matrix is:

Answer:

x	y	z	RHS
1	1	1	340
40	50	60	17400
40	25	20	9300

11. (i) Consider the following system of linear equations.

$$x + 2\,y + z = 2$$

$$2\,x + 5\,y + 3\,z = 7$$

Write down the augmented matrix for this system of equations.

x	y	z	RHS
1	2	1	2
2	5	3	7

Reduce the augmented matrix to REF (the row echelon form). It is essential to show the steps of row reductions and explicitly write the row operations used.

$R_2 - 2R_1$ gives:

x	y	z	RHS
1	2	1	2
0	1	1	3

The p.p. are now $(1, 2)$ so this is REF.

(ii) Using above calculations, determine all the solutions to the system of equations in x, y, z given above.

Use back substitutions. Note that x, y are pivot variables and z is free.

Solve the second for y. $y = 3 - z$.

Now solve the first for x. $x = 2 - z - 2y = 2 - z - 2(3 - z) = -4 + z$.

The answer is: $(x, y, z) = (-4 + z, 3 - z, z)$.

12. A group of 153 people were asked if they drink Coke or Pepsi.

It was found that 68 people drink Coke and 78 people drink Pepsi, while 25 people announced that they drink no soft drink!

Note: Let C, P be the set of the coke and Pepsi drinkers. The sample space S has $n(S) = 153$. $n(C) = 68$ and $n(P) = 78$. $n(S - C \bigcup P) = 25$.

(i) Estimate the probability that a random person drinks at least one of the two soft drinks.

We want

$$\frac{n(C \bigcup P)}{n(S)} = \frac{153 - 25}{153} = 0.8366$$

or 83.66%.

(ii) Estimate the probability that a random person drinks both Coke and Pepsi.

Answer: $n(P \bigcap C) = 68 + 78 - (153 - 25) = 18$. So the answer is $\frac{18}{153} = 0.1176$ or 11.76%.

(iii) Estimate the probability that a random person drinks only Pepsi.

Answer: We have $n(P - P \bigcap C) = 78 - 18 = 60$.

So the answer is $\frac{60}{153} = 0.3922$ or 39.22%.

13. An experiment consists of casting a die and observing the number on top. It is found that 54% of the time the number on top is 1 or 3 or 5 or 6.

It is also observed for the same die that 63% of the time the number on top less than or equal to 4. Answer the following questions **based on these observations.**

Note: Let A be the event that $1, 3, 5, 6$ show up and B the event that numbers less than or equal to 4 show up (i.e. $1, 2, 3, 4$ show up.)

(i) What is the probability that the number on top is 1 or 3 ?

Answer; This event is $A \bigcap B$.

Note that $P(A \bigcup B) = 1$ since this event covers all the six numbers.

So,

$$P(A \bigcap B) = P(A) + P(B) - P(A \bigcup B) = 0.54 + 0.63 - 1 = 0.17.$$

Thus 17% is the answer.

(ii) What is the probability that the number on top is 2 or 4 ?

Answer This event is $B - (A \bigcap B)$ so the probability is $0.63 - 0.17 = 0.46$ or 46%.

It is important to note that $A \bigcap B$ is a subevent of B and hence this formula works.

(iii) The experimenter concludes that this die must be loaded since a fair die would show a number less than or equal to 4 on top for ____% of the time.

The desired percentage is the mathematical probability of $1, 2, 3, 4$ appearing is $4/6 = 0.67$ i.e. 67%. Since this is not the same as the observed 63%, we make the conclusion.

14. Two fair dice are tossed, one red and one blue.

Let A be the event that the sum of the numbers on top is 7.

Let B be the event that the number on top of the red die is **not** 2. Answer the following:

(i) What is the probability that the sum of the numbers on top is 7, i.e. what is $P(A)$?

Answer: The sample space is the set of pairs (a, b) where $1 \leq a, b \leq 6$.

The sample points with sum 7 are $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$. Note that the dice have different colors, so we must keep the order. We are letting a be the number on the red die and b the number on the blue die.

Thus the probability is $\frac{6}{36} = \frac{1}{6}$.

(ii) What is the probability that the sum of the numbers on top is 7, given that the number on top of the red die is **not** 2, i.e. what is $P(A|B)$?

Answer: The event $A \bigcap B$ has 5 sample points since it omits $(2, 5)$.

The required probability is $\frac{5}{5 \cdot 6} = \frac{1}{6}$ again!

(iii) Are the events A and B independent? Explain your reasoning.

Suggestion. Be sure to keep at least four digits of accuracy.

Answer: We know $P(A \bigcap B) = \frac{5}{36}$ and $P(A) \cdot P(B) = \frac{6}{36} \cdot \frac{5 \cdot 6}{6 \cdot 6} = \frac{5}{36}$ also.

These are equal and hence the events are independent.

Note; We could have deduced this from $P(A|B) = P(A)$ as well, especially since the numbers are precisely equal. When the two are nearly equal we must test it by inspecting the equation $P(A \bigcap B) = P(A) \cdot P(B)$.