

Surface Area Calculations

We give formulas and examples of surface area calculations chosen from various sources.

1. First the **parametric** formula.

Suppose that a surface has a parametric form

$$P = \langle x, y, z \rangle \text{ where } x, y, z \text{ are functions of } u, v$$

and u, v vary over some region D in the uv plane.

Then the surface area is given by the formula $\iint_D dS$ where

$$dS = \|P_u \times P_v\| du dv.$$

2. Now the **equational** formula.

If the surface is given by an equation $z = f(x, y)$ then we can take $P = \langle x, y, f(x, y) \rangle$ with x, y as parameters and the above formula transforms to

$$\iint_D \sqrt{1 + f_x^2 + f_y^2} dx dy$$

where D is the region of integration in the $x - y$ plane.

Later, we shall derive a more general formula when the equation is not explicitly solved for any of the variables, i.e. is the form $F(x, y, z) = 0$. Then the formula takes on the shape

$$\iint_D \sqrt{F_x^2 + F_y^2 + F_z^2} \frac{dx dy}{|F_z|}$$

provided $F_z \neq 0$ in D . Clearly, it works if one of the other partials are non zero in a similar manner.

3. Example of parametric form. A variant of Q.6 on the old exam.

$$P = \langle u + v, u - v, u^2 - v^2 \rangle$$

where the points (u, v) are inside the disk $D = \{(u, v) \mid u^2 + v^2 \leq 4\}$.

We calculate $P_u = \langle 1, 1, 2u \rangle$ and $P_v = \langle 1, -1, -2v \rangle$. Thus $P_u \times P_v = \langle 2(u-v), 2(u+v), -2 \rangle$.

Thus, we want

$$\iint_D \sqrt{4((u-v)^2 + (u+v)^2 + 1)} du dv.$$

This simplifies as:

$$\iint_D 2\sqrt{2u^2 + 2v^2 + 1} du dv.$$

As usual, we convert to the polar coordinates and evaluate:

$$\int_{\theta=0}^{2\pi} \int_{r=0}^2 2\sqrt{1 + 2r^2} r dr d\theta = 52\pi/3.$$

4. Example of solvable equation.

Determine the area of the paraboloid $z = 12 - x^2 - y^2$ above the plane $z = 3$ and lying in the first octant.

The two surfaces meet when $3 = 12 - x^2 - y^2$ or $x^2 + y^2 = 9 = 3^2$.

Thus, we want the area of the paraboloid lying above the disc $\{x^2 + y^2 \leq 3^2\}$ in the first quadrant of the xy plane.

Taking $f(x, y) = 12 - x^2 - y^2$ and the equational formula, we calculate:

$$\iint_D \sqrt{1 + (-2x)^2 + (-2y)^2} dx dy.$$

Again, in polar coordinates, we get

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^3 \sqrt{1 + 4r^2} r dr d\theta.$$

This evaluates to: $\left(\frac{37\sqrt{37}-1}{24}\right)\pi$.

5. Area of a piece of a plane.

Consider the plane $3x + 4y + 5z = 10$. Given a region D in the xy plane, determine the area of the plane lying above it in terms of the area of D .

We use the equational form again and note that $f(x, y) = \frac{10-3x-4y}{5}$.

Thus, we integrate $\sqrt{1 + (-3/5)^2 + (-4/5)^2}$. The square root simplifies to $\sqrt{50/25} = \sqrt{2}$.

Since this is a constant, it comes out of the integral and we have: $\sqrt{2} \iint_D dx dy$. This is clearly $\sqrt{2}|D|$ if $|D|$ denotes the area of D .