chords AP, BQ and CR are their sines and OP, OQ and OR their cosines. BS is drawn perpendicular to CR. Since $BQRS$ is a rectangle, we have then $CS = \sin(\theta+\phi) - \sin(\theta-\phi)$ and $BS = \cos(\theta-\phi) - \cos(\theta+\phi)$. Now, the triangles BCS and AOP are similar since the pairs of lines (BC, AO), (CS, OP) and (BS, AP) are mutually orthogonal. Hence,

$$\frac{BS}{BC} = \frac{AP}{OA}, \qquad \frac{CS}{BC} = \frac{OP}{OA}.$$

But $OA = 1$, $AP = \sin\theta$, $OP = \cos\theta$ and $BC = 2\sin\phi$ and we have immediately the familiar formulae for the symmetric differences of sine and cosine which I write again:

$$\sin(\theta+\phi) - \sin(\theta-\phi) = 2\sin\phi\cos\theta,$$

$$\cos(\theta+\phi) - \cos(\theta-\phi) = -2\sin\phi\sin\theta.$$

Specialise now to $\theta = (n-1/2)\epsilon$ and $\phi = (1/2)\epsilon$ to get the canonical differences

$$\delta s_m = s_m - s_{m-1} = 2\sin\frac{\epsilon}{2}\cos\left(m - \frac{1}{2}\right)\epsilon,$$

and

$$\delta c_m = c_m - c_{m-1} = -2\sin\frac{\epsilon}{2}\sin\left(m - \frac{1}{2}\right)\epsilon,$$

where $c_m = \cos m\epsilon$.

Faced with this pair of coupled linear inhomogeneous equations, the modern reader will know what to do: substitute one equation into the other. In other words, take the second difference of the sines

$$\delta s_m - \delta s_{m-1} = 2\sin\frac{\epsilon}{2}\left(\cos\left(m - \frac{1}{2}\right)\epsilon - \cos\left(m - \frac{3}{2}\right)\epsilon\right)$$

and use the cosine difference formula. The result is

$$\delta s_m - \delta s_{m-1} = -4\sin^2\frac{\epsilon}{2}s_{m-1}.$$

That is what Nilakantha (and probably Aryabhata) did except in one respect. Instead of breaking up the geometric reasoning separately for sine and cosine and then bringing them together algebraically, he does the substitution geometrically so to say by deriving the cosine difference from the beginning in parallel with the sine difference, not for the right triangle with vertices at $(m+1/2)\epsilon$ and $(m-1/2)\epsilon$ as I have done here, but for the one with vertices at $(m-1/2)\epsilon$ and $(m-3/2)\epsilon$ (Nilakantha's geometric substitution is described in [AB-S]). It is also easy to check, though unnecessary for the sine table, that the cosines satisfy the identical equation

$$\delta c_m - \delta c_{m-1} = -4\sin^2\frac{\epsilon}{2}c_{m-1}.$$

Nilakantha now uses the lowest second difference $\delta s_2 - \delta s_1 = -4\sin^2(\epsilon/2)s_1$ to eliminate the half-angle factor on the right:

$$\delta s_m - \delta s_{m-1} = \frac{s_{m-1}}{s_1}(\delta s_2 - \delta s_1)$$

for $m > 1$ (for $m = 2$ it is empty). This equation is exact and it is valid for any step size of the form $\pi/2n$.

There are several ways in which the equation can be reexpressed. The one Aryabhata chooses relies on the fact that the sum of the successive differences of any ordered sequence, not necessarily equally spaced, is the difference between the two extreme elements: $\delta s_{m-1} + \delta s_{m-2} + \cdots + \delta s_1 = s_{m-1} - s_0 = s_{m-1}$ in our particular case, so that the exact sine-difference equation can be rewritten as

$$\delta s_m = \delta s_{m-1} - \frac{(\delta s_1 - \delta s_2)}{s_1} \sum_{i=1}^{m-1} \delta s_i.$$

Once the values of s_1 and $\delta s_1 - \delta s_2 = 2s_1 - s_2$ are known, the equation determines each δs_m recursively from the lower differences and, thereby, each s_m recursively from the lower sines. The recursive structure is independent of these 'initial values' as is, equally obviously, the linear structure, the latter being a consequence of the fact that the first differences of sines and cosines are proprtional respectively to the cosines and sines at the mean points. A final remark is that this as well as the apparently trivial but very general property of differences, that the parts so defined sum up to the whole, are the seeds which Madhava and the Nila school nurtured into the infinitesimal calculus of trigonometric functions.

How did Aryabhata go from the exact formula to the rule as given in the previous section? The rule itself, confirmed by the numbers in *Gītikā* 12, tells us that $\sin\epsilon$ was taken to be ϵ. The only question is about how it was verified to be a good approximation and the unanimously accepted answer is that it was computed by applying the half-angle formula to go from $s_8 = 1/2$ to s_4 to s_2 to s_1 which last is 225 minutes, equal to ϵ to within 1 minute. Since one has to compute s_2 along the way, the other constant $\delta s_1 - \delta s_2 = 2s_1 - s_2$ is also thereby determined and it turns out to be 1 minute and is therefore absent in the numerator of the second term. This accidental simplification due to the use of minutes to measure chords can be and has been a source of occasional confusion.

It is clear at the end of it all that the only place the rule deviates from exactness, in fact the only place it is sensitive to the choice of the step size, is in the constants specifying the initial conditions, the two adjustments referred to earlier.

The position Aryabhata's table came to acquire as an integral part of Indian astronomy has few parallels in any branch of learning. Virtually every subsequent work, beginning with Varahamihira and until Shankara Varman (the last descendent of the Nila lineage, early 19th century, different from the 16th

11.3 Integration: The Power Series

Yuktibhāṣā describes every step in the summation of the linearised arc segments

$$I_n = \sum_{i=1}^{n} \sin \delta\theta_i = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + i^2/n^2}$$

and the passage to the limit

$$I = \lim_{n \to \infty} I_n$$

in elaborate detail. Considering how many of the ideas involved were entirely new to the mathematical culture of the times, it had good reasons for doing so. It is not necessary or practical to go over all of it here in equal detail; we have met some of the steps earlier and others involve operations that have become standard since then and are very familiar to a modern student. The account in the present section is therefore somewhat streamlined[2]: in particular the occasionally tedious case-by-case analyses – the price paid for the absence of an efficient notation – of some of the novel points that come up are dealt with below directly in the general case, without deviating, it is hoped, from the logical and mathematical line of thought of the text.

Preliminary to the summation is the expansion of $(1 + i^2/n^2)^{-1}$:

$$\frac{1}{1 + i^2/n^2} = 1 - \frac{i^2}{n^2} + \frac{i^4}{n^4} - \cdots .$$

The demonstration is the same as the *saṃskāram* of Nilakantha (see Chapter 10.4) though the terminology is slightly different: successive terms are called *śodhyaphalam*, literally "the result of purification", reminiscent of the terminology of the Bakhshali manuscript in the recursive refining of the square root. Along the way, the question of the negligibility of the error after a large but finite number of terms, convergence in modern parlance, is addressed and illustrated with a numerical example. Disappointingly, no motivation for resorting to the expansion is provided, nothing about the difficulty in integrating $(1 + t^2)^{-1}$ or the irrationality of π.

The resulting infinite series for I_n is thus

$$I_n = I_{n,0} - I_{n,2} + I_{n,4} - \cdots ,$$

with

$$I_{n,k} = \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k$$

for $k = 0, 2, 4, \cdots$. It is assumed in this step that the (infinite) sum over k and the (as of now finite) sum over i can be interchanged; this is perhaps one of

[2] A reading of the relevant sections, 6.3.3 to 6.4.4 primarily, of Sarma's translation ([YB-S]) is, nevertheless, very rewarding.

the reasons for insisting on the negligibility of the error in the k-summation after a large enough number of terms. Eventually, the limit $n \to \infty$ will have to be taken and it is another of the gains from postponing that limit until all computations are finished that we do not have to worry about the legitimacy of interchanging infinite sums and integrals.

The expansion reduces the problem to the evaluation of the quantities

$$J_k = \lim_{n\to\infty} I_{n,k} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{i^k}{n^k},$$

for $k = 0, 2, 4, \cdots$ and to adding them up with the apprpriate signs:

$$I = J_0 - J_2 + J_4 - \cdots .$$

In terms of the original continuous variable t with the identifications $t = i/n$ and $\delta t = 1/n$, J_k as the limit of the sum over i is precisely the modern definition of the integral of t^k:

$$J_k = \int_0^1 t^k dk.$$

I have compressed several of *Yuktibhāṣā*'s careful and precise explanations, both conceptual and methodological, but this in all essentials is how it reduces the problem finally to an infinite series of definite integrals of even powers. It is also not significantly different from the textbook reduction to integrals of powers as summarised in section 1 of this chapter, allowing, of course, for the fact that the notion of an integral was assumed there to be already acquired. The other, consequent, difference from the standard treatment is, as already noted, in the actual determination of the general power integral from first principles, as the asymptotic limit of a finite sum over i.

A small note of caution. *Yuktibhāṣā* uses the term *saṃkalitam*, with various qualifiers, for the sum (over i) whose limit is the integral (it is not used for the infinite sum over k) both before and after the limit, i.e., both for the discrete integral and the 'true' integral: thus *ghana-saṃkalitam* is the sum of cubes of natural numbers as well as the integral of the third power, etc., and *saṃkalita-saṃkalitam* is both a sum over sums and a multiple integral. Where appropriate, I will feel free to use the word 'integral' for *saṃkalitam* without further explanation.

The first surprise in the working out of the integrals of powers is that they are sought for all (non-negative) integral k, not just for the even k that occur in the expansion: " Even though it is not useful here, [I am] describing also the integrals of equals multiplied three, five, etc. times among themselves, as they occur in the midst of those which are useful" (6.4, opening paragraph). The reason for the broadening of the problem becomes evident in the following sections: J_k (or $I_{k,n}$ for large n) will be related to J_{k-1}; in other words, mathematical induction will play an indispensable part in the integration of the general power. In what is probably its most carefully written part, *Yuktibhāṣā* describes the procedure in several steps. First J_1 is evaluated from J_0 and J_2 from J_1 with equal attention to detail; then J_3 from J_2 and J_4 from J_3 progressively more briefly. The general rule is then given: "To produce integrals of higher and higher powers, multiply the given integral by the radius and remove from it itself divided by the number which is one greater". (6.4.4). Altogether, it is difficult to escape the feeling that Jyeshthadeva is trying to convey to his disciples an unfamiliar and particulary subtle line of thought – recall his quasi-axiomatic treatment of based natural numbers through the property of succession (Chapter 4.2).

For the unit circle, the general inductive prescription above amounts to

$$J_k = J_{k-1} - \frac{J_{k-1}}{k+1} = \frac{k}{k+1} J_{k-1}$$

which implies by iteration (with $J_0 = 1$ as input)

$$J_k = \frac{1}{k+1}.$$

The *Yuktibhāṣā* proof of this fundamental result – the first ever example of the rigorous working out of a nontrivial integral – is both subtle and unexpectedly original to a modern student. Rather than follow it literally and describe first the special cases of small values of k individually I will, in the next section, do the case of general k, following the same arguments as used in *Yuktibhāṣā* but exploiting the flexibility and efficiency of present-day notation. Apart from saving space, it will also bring to light the modern analogues of some of what may at first sight appear to be just ingenious 'tricks'.

But, before that, two general remarks. The first is a historical-epistemic point we have encountered earlier (Chapter 7.1). The exact expressions for $I_{n,k}$ for $k = 1, 2, 3$ and for any n were known to Aryabhata (*Gaṇita* 19, 21 and 22) and were almost surely derived by geometric, 'building block', methods; at least that was how it was done in the Nila school if we go by Nilakantha in the *Āryabhaṭīyabhāṣya* (see Chapter 7.1). But Chapter 6 of *Yuktibhāṣā* does not refer to these exact low k results at all (except, implicitly, for $I_{n,0} = n$; $I_{n,1}$ is derived geometrically in Chapter 7) possibly because, as suggested by Sarasvati Amma in a related context, geometric imagination could not break through the barrier of the dimension of physical space. Inductive proofs seem to have been thought of as a substitute – from solid, down-to-earth, geometry to logical abstraction as it were. The idea would have found ready acceptance if not, necessarily, immediate comprehension; mathematical induction is after all the elevation to the status of a proof-device of another of the hallmarks of the Indian mathematical mind, the attachment to recursive reasoning.

The other remark is that induction is applied not to the exact sum of powers of natural numbers, but only to its dominant term in the large n approximation. Sums of general powers higher than 3 would not have been easy to work out (they involve the Bernoulli numbers). Working with the asymptotically dominant term not only met the needs of the problem at hand. The judicious neglect of subdominant terms also simplified the proof enormously.

11.4 Integrating Powers; Asymptotic Induction

What is required now is the asymptotic behaviour in n of the sums of powers of positive integers:

$$n^{k+1} I_{n,k} = S_n^k = \sum_{i=1}^{n} i^k,$$

where I have reverted to the notation of Chapter 7.1) for all k (keep in mind that the superscript k in S_n^k is not an exponent). Gathering together the common threads running through *Yuktibhāṣā*'s individual treatment of low values of k as well as its explanation of the general case, here is how it is done.

Replace one factor of i in i^k by n, thereby changing S_n^k to $n\sum_{i=1}^{n} i^{k-1} = nS_n^{k-1}$. It is natural to think of this substitution as the first guess in a process of *saṃskāram*: $S_n^k = nS_n^{k-1}+$ a (negative) correction to be determined; that is how *Yuktibhāṣā* proceeds (though without explicitly using the word *saṃskāram*). The error introduced by the substitution is

$$nS_n^{k-1} - S_n^k = \sum_{i=1}^{n-1} (n-i) i^{k-1},$$

the coefficient of the $i = n$ term being 0. The ingenious step now is to reorder the sum on the right as

$$\sum_{i=1}^{n-1} (n-i) i^{k-1} = \sum_{i=1}^{n-1} \sum_{j=1}^{i} j^{k-1} = \sum_{i=1}^{n-1} S_i^{k-1}.$$

The proof is a matter of enumeration of terms: the right side is, explicitly, $1^{k-1}+(1^{k-1}+2^{k-1})+\cdots+(1^{k-1}+2^{k-1}+\cdots+(n-1)^{k-1})$; collecting coefficients of i^{k-1}, the expression becomes $(n-1)1^{k-1}+(n-2)2^{k-1}+\cdots+1\cdot(n-1)^{k-1}$ which is the left side. The rearrangement has thus resulted in a recursion relation in k,

$$S_n^k = nS_n^{k-1} - \sum_{i=1}^{n-1} S_i^{k-1},$$

for the power sum.

In principle, one can iterate the step, reducing k by unity successively until $k = 0$ is reached. But the n-dependece coming from the second term on the right will be quite involved. *Yuktibhāṣā* chooses to circumvent such complications by taking advantage of the knowledge that n is eventually going to be made to tend to infinity and keeping only the dominant terms in the relevant n-dependent quantities. Start the induction with the trivial observation $S_n^0 = n$ for all n. We have then the recursion relation for $k = 1$:

$$S_n^1 = nS_n^0 - \sum_{i=1}^{n-1} S_i^0$$

in which the first term on the right is n^2 and the second term is $\sum_{i=1}^{n-1} i = S_{n-1}^1$:

$$S_n^1 = n^2 - S_{n-1}^1.$$

Now, for large n, ignore the difference $(= n)$ between S_n^1 and S_{n-1}^1 (justified since S_n^1 increases quadratically with n). From the resulting approximate equation we get

$$S_n^1 \sim \frac{n^2}{2}$$

where, and in the following few equations, $\sim$ denotes the dominant term in the limit $n \to \infty$, including the coefficient.

Rather than repeat the steps above for $k = 2, 3$, and so on, let us incorporate them in a general inductive proof in the modern manner. Accordingly, suppose $S_n^{k-1} \sim n^k/k$ for a given $k > 0$. The first term on the right in the recursion relation is dominated by n^{k+1}/k. The crucial point in evaluating the second term is that it is legitimate to extend the induction ansatz to all i: $S_i^{k-1} \sim i^k/k$, when $n \to \infty$ and i is summed over. The reason is that the sum is dominated by terms corresponding to large n and the error introduced in the lower terms will sum to a finite quantity; stated otherwise, $\sum_{i=1}^{n-1} S_i^{k-1}$ is a polynomial one degree higher in n than S_{n-1}^{k-1}. The asymptotically dominant part of the second term is therefore

$$\sum_{i=1}^{n-1} S_i^{k-1} \sim \sum_{i=1}^{n-1} \frac{i^k}{k} = \frac{1}{k} S_{n-1}^k \sim \frac{1}{k} S_n^k,$$

where we have once again ignored the difference between S_n^k and S_{n-1}^k $(= n^k)$. The recursion relation then gives, for the dominant terms,

$$\frac{k+1}{k} S_n^k \sim \frac{n^{k+1}}{k}$$

or

$$S_n^k \sim \frac{n^{k+1}}{k+1}$$

which is what is needed to be shown. Alternatively, we can leave the first term as it is and determine the asymptotic values of S_n^k recursively:

$$S_n^k \sim n \frac{k}{1+k} S_n^{k-1}$$

which is what *Yuktibhāṣā*, not surprisingly, prefers.

Finally, for the quantities $I_{n,k}$ themselves (whose limits are the integrals of powers) we have

$$I_{n,k} = \frac{1}{n^{k+1}} S_n^k \sim \frac{1}{k+1}$$

which is the same as

$$J_k = \int_0^1 t^k dt = \lim_{n \to \infty} I_{n,k} = \frac{1}{k+1}.$$