
Your task is to prove the following theorems completely. Be sure to write full sentences and make sure that your argument can be understood by straight reading.

1. Let $f(X)$ denote a polynomial $a_0X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n$ where a_0, a_1, \dots, a_n are integers and $a_0 \neq 0$. Suppose that α is a rational root of the polynomial, i.e. $f(\alpha) = 0$ and $\alpha = \frac{p}{q}$ where p, q are integers and $\gcd(p, q) = 1$. Prove Newton's Theorem that p divides a_n and q divides a_0 .
2. Deduce that if α is a rational root of $X^2 - a$ where a is an integer, then α is an integer.
3. Deduce that $X^2 - 90$ has no rational root. Be careful to discuss all possibilities and clarify how the above theorem is used.
4. More generally, show that $X^n - 90$ has no rational root for any integer $n > 1$.