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A New Method of Computing the Sums of Certain Series; By Mr. John Landen: Communicated by Mr. Thomas Simpson, F. R. S.

John Landen and Thomas Simpson

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The afhes of this foffil, when burnt, being boiled in water, and the water evaporated, there remained no falt behind.

I am, my Lord, &c.

Grofvenor-Street, Feb. 28, 1760.

LIV. *A new Method of computing the Sums of certain Series; by Mr.* John Landen: *Communicated by Mr.* Thomas Simpfon, *F. R. S.*

Read Feb. 28, 1760.

AS the improving the analytic art, efpecially any branch of it that relates to the fummation of feries, may, by facilitating computations, conduce to the improvement of feveral branches of fcience; it is prefumed, that this paper, which exhibits a new and eafy method of computing the fums of a great number of infinite feries, may be acceptable to the mathematical world, and deemed worthy to be inferted in the Britifh Philofophical Tranfactions.

1.

Suppofing x to be the fine of the circular arc z, whofe radius is 1, $\frac{\dot{x}}{\sqrt{1-x^2}}$ will be $= \dot{z}$; and, confequently, $\frac{\dot{x}}{\sqrt{x^2-1}} = \frac{\dot{z}}{\sqrt{-1}}$. From whence, by taking the correct fluents, we have hyp. log. $\frac{x+\sqrt{x^2-1}}{\sqrt{-1}} = \frac{z}{\sqrt{-1}}$.

Hence,

Hence, writing a for one fourth of the periphery of the circle whose radius is 1, and taking x equal to the said radius, we find hyp. log. $\frac{1}{\sqrt{-1}} = \frac{a}{\sqrt{-1}}$; and, consequently, hyp. log. $\sqrt{-1} = \frac{-a}{\sqrt{-1}}$, and hyp. log. $-1 = \pm \frac{2a}{\sqrt{-1}}$.

2.

The hyp. log. of $\frac{1}{1-x}$ being $= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$, &c.

$\overset{\text{I}}{F}$ = fluent of $\frac{\dot{x}}{x}$ hyp. log. $\frac{1}{1-x}$, is $= x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2}$, &c.

$\overset{\text{II}}{F}$ = fluent of $\frac{\dot{x}}{x} \overset{\text{I}}{F} = x + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \frac{x^4}{4^3}$, &c.

$\overset{\text{III}}{F}$ = fluent of $\frac{\dot{x}}{x} \overset{\text{II}}{F} = x + \frac{x^2}{2^4} + \frac{x^3}{3^4} + \frac{x^4}{4^4}$, &c.

$\overset{\text{IV}}{F}$ = fluent of $\frac{\dot{x}}{x} \overset{\text{III}}{F} = x + \frac{x^2}{2^5} + \frac{x^3}{3^5} + \frac{x^4}{4^5}$, &c.

&c. &c. &c.

3.

By writing, in the first equation in the preceding article, $\frac{1}{x}$ instead of x, we have

Hyp. log. $\frac{1}{1-\frac{1}{x}} = x^{-1} + \frac{x^{-2}}{2} + \frac{x^{-3}}{3}$, &c.

But the hyp. log. of $\frac{1}{1-\frac{1}{x}}$ is = hyp. log. $\frac{x}{x-1}$ =

hyp.

hyp. log. $\frac{1}{1-x}$ + hyp. log. x + hyp. log. $-1 = \pm 2b + X +$ hyp. log. $\frac{1}{1-x}$, b being put for $\frac{a}{\sqrt{-1}}$, and X for the hyp. log. of x.

It is evident, therefore, that

$$\text{Hyp. log. } \frac{1}{1-x} \text{ is} = \mp 2b - X + x^{-1} + \frac{x^{-2}}{2} + \frac{x^{-3}}{3}, \text{ \&c.}$$

where, of the two ſigns prefixed to $2b$, the upper one takes place, when the hyp. log. of -1 is taken equal to $\frac{-2a}{\sqrt{-1}}$, likewiſe when x is taken equal to $\sqrt{-1}$; and the lower one takes place, when the hyp. log. of -1 is taken equal to $\frac{2a}{\sqrt{-1}}$, alſo when x is taken equal to $\frac{1}{\sqrt{-1}}$: wherefore, if we obſerve to take the value of hyp. log. of -1, as laſt mentioned, and x equal to $\frac{1}{\sqrt{-1}}$, inſtead of $\sqrt{-1}$, we need retain only the lower of the ſaid ſigns.

4.

For brevity ſake, we ſhall, in what follows, put the ſeries $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$, &c. $= \overset{II}{P}$,

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4}, \text{ \&c.} = \overset{IV}{P},$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6}, \text{ \&c.} = \overset{VI}{P},$$

&c. &c.

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}, \text{ \&c.} = \overset{II}{Q}$$

1 +

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} +, \text{ \&c.} = \overset{\text{III}}{q},$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4}, \text{ \&c.} = \overset{\text{IV}}{Q}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} +, \text{ \&c.} = \overset{\text{V}}{q},$$

&c. &c.

5.

Multiplying the laſt equation in art. 3. by $\frac{\dot{x}}{x}$, and taking the correct fluents, we have

$$\overset{\text{I}}{F} = 2\overset{\text{II}}{P} + 2bX - \frac{X^2}{2} - x^{-1} - \frac{x^{-2}}{2^2} - \frac{x^{-3}}{3^2}, \text{ \&c.}$$

From whence, by multiplying by $\frac{\dot{x}}{x}$, and taking the fluents, we get

$$\overset{\text{II}}{F} = 2\overset{\text{II}}{P}X + bX^2 - \frac{X^3}{2.3} + x^{-1} + \frac{x^{-2}}{2^3} + \frac{x^{-3}}{3^3}, \text{ \&c.}$$

Again, multiplying the laſt equation by $\frac{\dot{x}}{x}$, and taking the correct fluents, we find

$$\overset{\text{III}}{F} = 2\overset{\text{IV}}{P} + \overset{\text{II}}{P}X^2 + \frac{bX^3}{3} - \frac{X^4}{2.3.4} - x^{-1} - \frac{x^{-2}}{2^4} - \frac{x^{-3}}{3^4}, \text{ \&c.}$$

And, by proceeding in the ſame manner, we find

$$\overset{\text{IV}}{F} = 2\overset{\text{IV}}{P}X + \frac{\overset{\text{II}}{P}X^3}{3} + \frac{bX^4}{3.4} - \frac{X^5}{2.3.4.5} + x^{-1} + \frac{x^{-2}}{2^5} + \frac{x^{-3}}{3^5}, \text{ \&c.}$$

&c. &c.

6.

Now, it is obvious, that $x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. the value of $\overset{\text{I}}{F}$ in art. 2. muſt be equal to $2\overset{\text{II}}{P} + 2bX$

$-\frac{X^2}{2} - x^{-1} - \frac{x^{-2}}{2^2} - \frac{x^{-3}}{3^2}$, &c. the value of F in art. 5. when both ſeries converge.

Therefore, $\frac{x + x^{-1}}{1^2} + \frac{x^2 + x^{-2}}{2^2} + \frac{x^3 + x^{-3}}{3^2}$, &c. is then $= 2\overset{II}{P} + 2bX - \frac{X^2}{2}$.

From which equation, by taking x equal to -1, we have $-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} -$, &c. $= \overset{II}{P} + b^2 = \overset{II}{P} - a^2$; and, by taking x equal to $\frac{1}{\sqrt{-1}}$, we have $-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} -$, &c. $= 4\overset{II}{P} + 3b^2 = 4\overset{II}{P} - 3a^2$.

Therefore $4\overset{II}{P} - 3a^2$ is $= \overset{II}{P} - a^2$:

Hence $\overset{II}{P}$ is found $= \frac{2a^2}{3}$.

Moreover $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$, &c. being $= \overset{II}{P}$, by ſuppoſition; and $-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} -$, &c. $= \overset{II}{P} - a^2$, as found above; we, by ſubtraction, get $\frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2}$, &c. $(= 2\overset{II}{Q}) = a^2$, and, conſequently $\overset{II}{Q} = \frac{a^2}{2}$.

SCHOLIUM.

The hyp. log. of $\frac{1}{1-x}$ being $= x + \frac{x^2}{2} + \frac{x^3}{3}$, &c. we, by writing $1 - x$ inſtead of x, have

 Hyp.

Hyp. log. of $\frac{1}{x} = 1 - x + \frac{\overline{1-x}|^2}{2} + \frac{\overline{1-x}|^3}{3}$, &c.
and consequently $X = -1 - x - \frac{\overline{1-x}|^2}{2} - \frac{\overline{1-x}|^3}{3}$, &c.

Moreover the fluent of $\frac{\dot{x}}{x} \times$ hyp. log. of $\frac{1}{1-x}$ is $= x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. which vanishes when x vanishes; and the fluent of $\frac{\dot{x}}{1-x} \times X$ is $= \overline{1-x} + \frac{\overline{1-x}|^2}{2^2} + \frac{\overline{1-x}|^3}{3^2}$, &c. $- \overset{\text{II}}{P}$, being corrected so as to vanish when x vanishes.

But the fluent of $\frac{\dot{x}}{x} \times$ hyp. log. of $\frac{1}{1-x}$ + fluent of $\frac{\dot{x}}{1-x} \times X$ is $= X \times$ hyp. log. of $\frac{1}{1-x}$, which also vanishes when x vanishes.

Therefore $X \times$ hyp. log. of $\frac{1}{1-x}$ is $= x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. $+ 1 - x + \frac{\overline{1-x}|^2}{2^2} + \frac{\overline{1-x}|^3}{3^2}$, &c. $- \overset{\text{II}}{P}$.

From whence, by taking x equal to $\frac{1}{2}$, we find $-$ square of hyp. log. of $2 = 2 \times \frac{1}{1^2 . 2^1} + \frac{1}{2^2 . 2^2} + \frac{1}{3^2 . 2^3}$, &c. $- \overset{\text{II}}{P}$: hence, $\overset{\text{II}}{P}$ being before found $= \frac{2a^2}{3}$, it appears that, when x is $= \frac{1}{2}$, the series $x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. is $= \frac{a^2}{3} - \frac{1}{2} \times \overline{\text{hyp. log. of } 2}|^2$.

7.

Furthermore, $x + \frac{x^2}{2^3} + \frac{x^3}{3^3}$, &c. the value of $\overset{\text{II}}{F}$ in art. 2. must be equal to $2\overset{\text{II}}{P}X + bX^2 - \frac{X^3}{2.3} + x^{-1}$

+ &c.

$+ \frac{x^{-2}}{2^3} + \frac{x^{-3}}{3^3}$, &c. the value of $\overset{\text{II}}{F}$ in art. 5. when both ſeries converge.

Therefore $\frac{x - x^{-1}}{1^3} + \frac{x^2 - x^{-2}}{2^3} + \frac{x^3 - x^{-3}}{3^3}$, &c. is then $= 2\overset{\text{II}}{P}X + bX^2 - \frac{X^3}{2.3}$.

From whence, by taking x equal to -1, we have $4b\overset{\text{II}}{P} + 4b^3 - \frac{8b^3}{2.3} = 0$; and, conſequently, $\overset{\text{II}}{P} = \frac{2a^2}{3}$, as before found.

And, by taking x equal to $\frac{1}{\sqrt{-1}}$, we find

$$\frac{2}{\sqrt{-1}} \times \overset{\text{III}}{q} = 2b\overset{\text{II}}{P} + b^3 - \frac{b^3}{2.3} = \frac{4a^3}{3\sqrt{-1}} - \frac{a^3}{\sqrt{-1}} + \frac{a^3}{2.3\sqrt{-1}}$$
$$= \frac{a^3}{2\sqrt{-1}}.$$

Therefore $\overset{\text{III}}{q}$ is $= \frac{a^3}{4}$.

8.

From what is done above, it evidently follows, that

$$-\overset{\text{IV}}{P} \text{ is} = \frac{2b^2\overset{\text{II}}{P}}{3} + \frac{2.8b^4}{3.4.5},$$
$$-\overset{\text{VI}}{P} = \frac{2b^2\overset{\text{IV}}{P}}{3} + \frac{8b^4\overset{\text{II}}{P}}{3.4.5} + \frac{3.32b^6}{3.4.5.6.7},$$
&c. &c.
$$-\overset{\text{IV}}{Q} = b^2\overset{\text{II}}{P} + \frac{3.2b^4}{3.4},$$
$$-\overset{\text{VI}}{Q} = b^2\overset{\text{IV}}{P} + \frac{4b^4\overset{\text{II}}{P}}{3.4} + \frac{5.8b^6}{3.4.5.6},$$
&c. &c.

$$\frac{\overset{\text{V}}{q}}{\sqrt{-1}} \text{ is } = b\overset{\text{IV}}{P} + \frac{b^3\overset{\text{II}}{P}}{2.3} + \frac{9b^5}{2.2.3.4.5},$$

$$\frac{\overset{\text{VII}}{q}}{\sqrt{-1}} = b\overset{\text{VI}}{P} + \frac{b^3\overset{\text{IV}}{P}}{2.3} + \frac{b^5\overset{\text{II}}{P}}{2.3.4.5} + \frac{13b^7}{2.2.3.4.5.6.7},$$

&c. &c.

From whence the values of $\overset{\text{IV}}{P}$, $\overset{\text{VI}}{P}$, &c. $\overset{\text{IV}}{Q}$, $\overset{\text{VI}}{Q}$, &c. $\overset{\text{V}}{q}$, $\overset{\text{VII}}{q}$, &c. may be eafily obtained, in terms of a.

9.

Hyp. log. $\left.\overline{\frac{1+x}{1-x}}\right|^{\frac{1}{2}}$ being $= x + \frac{x^3}{3} + \frac{x^5}{5}$, &c.

$\overset{\text{I}}{G}$ = fluent of $\frac{\dot{x}}{x}$ hyp. log. $\left.\overline{\frac{1+x}{1-x}}\right|^{\frac{1}{2}}$ is $= x + \frac{x^3}{3^2} + \frac{x^5}{5^2}$, &c.

$\overset{\text{II}}{G}$ = fluent of $\frac{\dot{x}}{x}\overset{\text{I}}{G} = x + \frac{x^3}{3^3} + \frac{x^5}{5^3}$, &c.

$\overset{\text{III}}{G}$ = fluent of $\frac{\dot{x}}{x}\overset{\text{II}}{G} = x + \frac{x^3}{3^4} + \frac{x^5}{5^4}$, &c.

&c. &c. &c.

10.

By writing, in the firft equation in the preceding article, $\frac{1}{x}$ inftead of x, we have

$$\text{Hyp. log.} \left.\overline{\frac{1+\frac{1}{x}}{1-\frac{1}{x}}}\right|^{\frac{1}{2}} = x^{-1} + \frac{x^{-3}}{3} + \frac{x^{-5}}{5}, \text{ \&c.}$$

But the hyp. log. of $\left.\overline{\frac{1+\frac{1}{x}}{1-\frac{1}{x}}}\right|^{\frac{1}{2}}$ is = hyp. log. $\left.\overline{\frac{x+1}{x-1}}\right|^{\frac{1}{2}}$ = hyp. log. $\left.\overline{\frac{1+x}{1-x}}\right|^{\frac{1}{2}} \mp$ hyp. log. $\sqrt{-1} = \pm b +$ hyp. log. $\left.\overline{\frac{1+x}{1-x}}\right|^{\frac{1}{2}}$.

It

It is manifeſt, therefore, that

$$\text{Hyp. log. } \overline{\frac{1+x}{1-x}}\Big|^{\frac{1}{2}} \text{ is } = \mp b + x^{-1} + \frac{x^{-3}}{3} + \frac{x^{-5}}{5}, \text{ \&c.}$$

where, with reſpect to the two ſigns prefixed to b, the ſame obſervation may be made as in art. 3.

11.

Multiplying the laſt equation by $\frac{\dot{x}}{x}$, and taking the correct fluents, we have

$$\overset{\text{I}}{G} = 2\overset{\text{II}}{Q} + bX - x^{-1} - \frac{x^{-3}}{3^2} - \frac{x^{-5}}{5^2}, \text{ \&c.}$$

From whence, by multiplying by $\frac{\dot{x}}{x}$, and taking the fluents, we get

$$\overset{\text{II}}{G} = 2\overset{\text{II}}{Q}X + \frac{bX^2}{2} + x^{-1} + \frac{x^{-3}}{3^3} + \frac{x^{-5}}{5^3}, \text{ \&c.}$$

Again, multiplying the laſt equation by $\frac{\dot{x}}{x}$, and taking the correct fluents, we find

$$\overset{\text{III}}{G} = 2\overset{\text{IV}}{Q} + \overset{\text{II}}{Q}X^2 + \frac{bX^3}{2.3} - x^{-1} - \frac{x^{-3}}{3^4} - \frac{x^{-5}}{5^4}, \text{ \&c.}$$

And, by proceeding in the ſame manner, we find

$$\overset{\text{IV}}{G} = 2\overset{\text{IV}}{Q}X + \frac{\overset{\text{II}}{Q}X^3}{3} + \frac{bX^4}{2.3.4} + x^{-1} + \frac{x^{-3}}{3^5} + \frac{x^{-5}}{5^5}, \text{ \&c.}$$

&c. &c.

12.

Now, it is obvious, that $x + \frac{x^3}{3^2} + \frac{x^5}{5^2}$, &c. the value of $\overset{\text{I}}{G}$ in art. 9. muſt be equal to $2\overset{\text{II}}{Q} + bX - x^{-1} - \frac{x^{-3}}{3^2} - \frac{x^{-5}}{5^2}$, &c. the value of $\overset{\text{I}}{G}$ in art. 11. when both ſeries converge

There-

4

Therefore $\frac{x + x^{-1}}{1^2} + \frac{x^3 + x^{-3}}{3^2} + \frac{x^5 + x^{-5}}{5^2}$, &c. is then $= 2\overset{\text{II}}{Q} + bX$.

From whence, by taking x equal to $\frac{1}{\sqrt{-1}}$, we have $2\overset{\text{II}}{Q} + b^2 = 0$; and, consequently, $\overset{\text{II}}{Q} = \frac{a^2}{2}$, as in art. 6.

13.

Likewise $x + \frac{x^3}{3^3} + \frac{x^5}{5^3}$, &c. the value of $\overset{\text{II}}{G}$ in art. 9. must be equal to $2\overset{\text{II}}{Q}X + \frac{bX^2}{2} + x^{-1} + \frac{x^{-3}}{3^3}$. $+ \frac{x^{-5}}{5^3}$, &c. the value of $\overset{\text{II}}{G}$ in art. 11. when both series converge.

Therefore $\frac{x - x^{-1}}{1^3} + \frac{x^3 - x^{-3}}{3^3} + \frac{x^5 - x^{-5}}{5^3}$, &c. is then $= 2\overset{\text{II}}{Q}X + \frac{bX^2}{2}$.

Hence, by taking $x = \frac{1}{\sqrt{-1}}$, we find $\frac{2}{\sqrt{-1}} \times \overset{\text{III}}{q} = 2b\overset{\text{II}}{Q} + \frac{b^3}{2} = \frac{a^3}{2\sqrt{-1}}$; and, consequently, $\overset{\text{III}}{q} = \frac{a^3}{4}$, as in art. 7.

14.

From what is done in the last five articles, it evidently follows, that

$$-\overset{\text{IV}}{Q} \text{ is } = \frac{b^2\overset{\text{II}}{Q}}{2} + \frac{b^4}{2.2.3},$$

$$\frac{\overset{\text{V}}{q}}{\sqrt{-1}} = b\overset{\text{IV}}{Q} + \frac{b^3\overset{\text{II}}{Q}}{2.3} + \frac{b^5}{2.2.3.4},$$

—

$$-\overset{\text{VI}}{Q} \text{ is } = \frac{b^2 \overset{\text{IV}}{Q}}{2} + \frac{b^4 \overset{\text{II}}{Q}}{2.3.4} + \frac{b^6}{2.\ldots3.4.5},$$

$$\frac{\overset{\text{VII}}{q}}{\sqrt{-1}} = b\overset{\text{VI}}{Q} + \frac{b^3 \overset{\text{IV}}{Q}}{2.3} + \frac{b^5 \overset{\text{II}}{Q}}{2.3.4.5} + \frac{b^7}{2.2.3.4.5.6},$$

&c. &c.

From whence (as well as from the theorems in art. 8.) may the values of $\overset{\text{IV}}{Q}$, $\overset{\text{V}}{q}$, $\overset{\text{VI}}{Q}$, $\overset{\text{VII}}{q}$, &c. be readily found, in terms of a.

15.

$\overset{\text{I}}{G}$ being $= x + \frac{x^3}{3^2} + \frac{x^5}{5^2}$, &c. by art. 9.

$\overset{\text{I}}{H} =$ fluent of $x\dot{x}\overset{\text{I}}{G}$ is $= \frac{x^3}{1^2.3} + \frac{x^5}{3^2.5} + \frac{x^7}{5^2.7}$, &c.

$\overset{\text{II}}{H} =$ fluent of $\frac{\dot{x}}{x}\overset{\text{I}}{H} = \frac{x^3}{1^2.3^2} + \frac{x^5}{3^2.5^2} + \frac{x^7}{5^2.7^2}$, &c.

$\overset{\text{III}}{H} =$ fluent of $x\dot{x}\overset{\text{II}}{H} = \frac{x^5}{1^2.3^2.5} + \frac{x^7}{3^2.5^2.7} + \frac{x^9}{5^2.7^2.9}$, &c.

$\overset{\text{IV}}{H} =$ fluent of $\frac{\dot{x}}{x}\overset{\text{III}}{H} = \frac{x^5}{1^2.3^2.5^2} + \frac{x^7}{3^2.5^2.7^2} + \frac{x^9}{5^2.7^2.9^2}$, &c.

&c. &c. &c.

16.

Moreover, $\overset{\text{I}}{G}$ being $= 2\overset{\text{II}}{Q} + bX - x^{-1} - \frac{x^{-3}}{3^2} - \frac{x^{-5}}{5^2}$, &c. by art. 11. we, by multiplying by $x\dot{x}$, and taking the correct fluents, get $\overset{\text{I}}{H} = x^2\overset{\text{II}}{Q} - \overset{\text{II}}{Q} + \frac{bx^2X}{2} - \frac{bx^2}{4} + \frac{b}{4} - x + 1 + 2\overset{\text{II}}{S} + \frac{x^{-1}}{1.3^2} + \frac{x^{-3}}{3.5^2} + \frac{x^{-5}}{5.7^2}$, &c. $\overset{\text{II}}{S}$ being put for the series $\frac{1}{1^2.3^2} + \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2}$, &c.

 Now,

Now, it is obvious, that this value of $\overset{\text{I}}{H}$ muſt be equal to the value of $\overset{\text{I}}{H}$ in the preceding article, when both ſeries converge.

Therefore $\frac{3x^3 - x^{-1}}{1^2.3^2} + \frac{5x^5 - 3x^{-3}}{3^2.5^2} + \frac{7x^7 - 5x^{-5}}{5^2.7^2}$, &c. is then $= x^2\overset{\text{II}}{Q} - \overset{\text{II}}{Q} + \frac{bx^2X}{2} - \frac{bx^2}{4} + \frac{b}{4} - x + 1 + 2\overset{\text{II}}{S}$.

Hence, by taking x equal to -1, we find $-2\overset{\text{II}}{S} = b^2 + 2 + 2\overset{\text{II}}{S}$; and, conſequently, $\overset{\text{II}}{S} = \frac{a^2}{4} - \frac{1}{2}$.

And, by taking x equal to $\frac{1}{\sqrt{-1}}$, we find

$$-\frac{2}{\sqrt{-1}} \times \overline{\frac{1}{1^2.3^2} - \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2} -, \&c.} = -2\overset{\text{II}}{Q} - \frac{b^2}{2} + \frac{b}{2} - \frac{1}{\sqrt{-1}} + 1 + 2\overset{\text{II}}{S} = \frac{a}{2\sqrt{-1}} - \frac{1}{\sqrt{-1}};$$

and, conſequently, $\frac{1}{1^2.3^2} - \frac{1}{3^2.5^2} + \frac{1}{5^2\,7^2} -$, &c. $= \frac{1}{2} - \frac{a}{4}$.

17.

Seeing that $\overset{\text{II}}{Q}$ is $= \frac{a^2}{2}$, and $\overset{\text{II}}{S} = \frac{a^2}{4} - \frac{1}{2}$, it follows, from the laſt article, that

$$\overset{\text{I}}{H} \text{ is } = x^2\overset{\text{II}}{Q} + \frac{bx^2X}{2} - \frac{bX^2}{4} + \frac{b}{4} - x + \frac{x^{-1}}{1.3^2} + \frac{x^{-3}}{3.5^2} + \frac{x^{-5}}{5.7^2}, \&c.$$

From whence, by multiplying by $\frac{\dot{x}}{x}$, and taking the correct fluent, we get

$\overset{\text{II}}{H}$

$$\overset{II}{H} = \frac{x^2 \overset{II}{Q}}{2} + \frac{b x^2 X}{4} - \frac{b x^2}{4} + \frac{b}{4} + \frac{b X}{4} - x + \frac{a^2}{4} - \frac{x^{-1}}{1^2.3^2} - \frac{x^{-3}}{3^2.5^2} - \frac{x^{-5}}{5^2.7^2}, \text{ \&c.}$$

And hence, by multiplying by $x \dot{x}$, and taking the correct fluents, we have

$$\overset{III}{H} = \frac{x^4 \overset{II}{Q}}{8} + \frac{b x^4 X}{16} - \frac{5 b x^4}{64} + \frac{b x^2}{16} + \frac{b x^2 X}{8} + \frac{b}{64} - \frac{x^3}{3} - \frac{x}{9} + \frac{4}{9} + \frac{a^2 x^2}{8} - \frac{3 a^2}{16} + 4 \overset{III}{S} + \frac{x^{-1}}{1.3^2.5^2} + \frac{x^{-3}}{3.5^2.7^2} + \frac{x^{-5}}{5.7^2.9^2}, \text{ \&c.}$$

$\overset{III}{S}$ being put for the series

$$\frac{1}{1^2.3^2.5^2} + \frac{1}{3^2.5^2.7^2} + \frac{1}{5^2.7^2.9^2}, \text{ \&c.}$$

Now, this value of $\overset{III}{H}$ being equal to the value of $\overset{III}{H}$ in art. 15. when both series converge, it follows, that

$$\frac{5 x^5 - x^{-1}}{1^2.3^2.5^2} + \frac{7 x^7 - 3 x^{-3}}{3^2.5^2.7^2} + \frac{9 x^9 - 5 x^{-5}}{5^2.7^2.9^2}, \text{ \&c. is then}$$

$$= \frac{x^4 \overset{II}{Q}}{8} + \frac{b x^4 X}{16} - \frac{5 b x^4}{64} + \frac{b x^2}{16} + \frac{b x^2 X}{8} + \frac{b}{64} - \frac{x^3}{3} - \frac{x}{9} + \frac{4}{9} + \frac{a^2 x^2}{8} + \frac{3 a^2}{16} - 4 \overset{III}{S}.$$

Hence, by taking x equal to -1, we find $-4 \overset{III}{S} = \frac{3 b^2}{8} + \frac{8}{9} + 4 \overset{III}{S}$; and, consequently, $\overset{III}{S} = \frac{3 a^2}{64} - \frac{1}{9}$.

Many other instances of the use of this method might be given; but these may suffice to enable the intelligent reader to pursue the speculation farther, at his pleasure.