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RATIONALE OF THE CHAKRAVĀLA PROCESS OF JAYADEVA AND BHĀSKARA II

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SUMMARIES

The old Indian chakravāla *method for solving the Bhāskara-Pell equation or* varga-prakṛti $x^2 - Dy^2 = 1$ *is investigated and explained in detail. Previous misconceptions are corrected, for example that* chakravāla, *the short cut method* bhāvanā *included, corresponds to the quick-method of Fermat. The* chakravāla *process corresponds to a half-regular best approximating algorithm of minimal length which has several deep minimization properties. The periodically appearing quantities* (jyeṣṭha-mūla, kaniṣṭha-mūla, kṣepaka, kuṭṭakāra, *etc.) are correctly understood only with the new theory.*

Den fornindiska metoden cakravāla *att lösa Bhāskara-Pell-ekvationen eller* varga-prakṛti $x^2 - Dy^2 = 1$ *detaljundersöks och förklaras här. Tidigare missuppfattningar rättas, såsom att* cakravāla, *genvägsmetoden* bhāvanā *inbegripen, motsvarade Fermats snabbmetod. Cakravālaprocessen motsvarar en halvregelbunden bäst-approximerande algoritm av minimal längd med flera djupt liggande minimeringsegenskaper. De periodvis uppträdande storheterna* (jyeṣṭha-mūla, kaniṣṭha-mūla, kṣepaka, kuṭṭakāra, *osv.) blir förståeliga först genom den nya teorin.*

Die alte indische Methode cakravāla *zur Lösung der* Bhāskara-Pell-*Gleichung oder* varga-prakṛti $x^2 - Dy^2 = 1$ *wird hier im einzelnen untersucht und erklärt. Frühere Missverständnisse werden aufgeklärt, z.B. dass* cakravāla, *einschliesslich der Richtwegmethode* bhāvanā, *der Fermatschen Schnellmethode entspreche. Der* cakravāla-*Prozess entspricht einem halbregelmässigen bestapproximierenden Algorithmus von minimaler Länge und mit mehreren tiefliegenden Minimierungseigenschaften. Die periodisch auftretenden Quantitäten* (jyeṣṭha-mūla, kaniṣṭha-mūla, kṣepaka, kuṭṭakāra, *usw.) werden erst durch die neue Theorie verständlich.*

I. *Introduction*

The old Hindu *chakravāla* method of solving the indeterminate "equation of the multiplied square" or *varga-prakṛti* (= square nature) [1]

(1) $$x^2 - Dy^2 = q,$$

especially the case $q = 1$

(2) $$x^2 - Dy^2 = 1,$$

is first mentioned in connection with the algebraist Ācārya Jayadeva, who lived about 1000 or earlier, at least 100 years before Bhāskara II. His work is quoted and explained with illustrations in the *Sundarī* [Shankar Shukla 1954, 1-4], written in 1073 by Srīmad Udayadivākara as a commentary on the *Laghu-Bhāskarīya* of Bhāskara I (629). The *chakravāla* method was first ascribed to Jayadeva by Kripa Shankar Shukla [1954, 1].

In fact, Bhāskara in his *Līlāvatī* and *Bīja-gaṇita* (1150) ascribed the solution of *varga-prakṛti* to earlier writers [Colebrooke 1817, 1-276; Strachey 1813]. Moreover, a special method of solution, that is, "the principle of composition" or *bhāvanā* goes back to Brahmagupta's *Brahma-spuṭa-siddhānta* (628) [Colebrooke, xviii, 64-65]. On the other hand, Bhāskara mentions Brahmagupta, Śrīdhara (1039) and Padmanābha, but gives no references to Jayadeva [Shankar Shukla 1954, 20]. For the general chronology see [Balagangadharan 1947].

The method of Jayadeva (comprising 15 of 20 stanzas in his *Sundarī*) is essentially the same as in Bhāskara. However, for the general case (1) the method expounded in stanzas 16-20 differs from those of Brahmagupta, Bhāskara II and Nārāyaṇa (1350). We here restrict ourselves to the special case (2).

For equation (2) I have used the name Bhāskara-Pell equation [Selenius 1963]. Srinivasiengar [1967, 110] suggests the name Brahmagupta-Bhāskara equation. Perhaps Jayadeva-Bhāskara equation would be the best.

II. *The Chakravāla Process*

As is well known, Brahmagupta by means of his composition method *bhāvanā* (or *samāsa*, *samāsabhāvanā*) was able to solve the *varga-prakṛti* equation (2) provided that he already had a solution of (1) with $q = -1, +2, -2, +4$, or -4, and then he could obtain an infinite number of solutions. Otherwise he was limited to trial-processes. The *bhāvanā* composition according the rule that $x_1^2 - Dy_1^2 = q_1$ and $x_2^2 - Dy_2^2 = q_2$ imply that $(x_1x_2 + Dy_1y_2)^2 - D(x_1y_2 + x_2y_1)^2 = q_1q_2$ was rediscovered by Euler and called *theorema elegantissimum*. It is often incorrectly stated that Fermat was the first in 1657 to note infinitely many solutions [Dickson 3, ch. 12; Datta and Singh Pt. 2, 141ff.; Srinivasiengar ch. 10; Shankar Shukla 1954; Selenius 1963].

On the other hand, the *chakravāla* of Jayadeva and Bhāskara II is a general method for solving the *varga-prakṛti*. It is called *chakravāla* (= circle) because of its iterative character: "It proceeds as in a circle, the same sets of operations being applied again and again in a continuous round" [Datta and Singh Pt. 2, 162] and it reveals thereby a resemblance to the continued fraction process of Euler.

In historical analyses of the *chakravāla* process (e.g. in those just cited, except the last), it is assumed that the only goal is to find solutions for the "interpolators" $q = \pm 1, \pm 2$, or ± 4, which are needed for the *bhāvanā* short cut method. Indeed, this immediate use of *bhāvanā* greatly shortens the task of attaining $q = +1$.

However, we can in fact continue the *chakravāla* process without use of the *bhāvanā* jumps starting in $q = -1, +2, -2, +4$, or -4, since it always leads to $q = +1$ or the solution of (2) [Ayyangar 1929; Selenius 1963]. This corresponds to an iterated continued fraction process where every cycle gives one denominator via four steps (see below §VI), one of them involving a noteworthy interaction between the second degree *varga-prakṛti* (2) and a first degree indeterminate equation *kuṭṭaka*

$$\frac{aX + c}{b} = Y, \tag{3}$$

solvable by the rules of Āryabhaṭa I (499), Bhāskara I (522), Brahmagupta, Mahāvīra (850), Āryabhaṭa II (950), Śrīpati (1039), Bhāskara II, or Nārāyaṇa (1350) [Datta and Singh Pt. 2, 162-175; Srinivasiengar, 112; Selenius 1963, 10-20].

III. *Questions*

Now the following questions arise: A. Is there a well-defined algorithm equivalent to the *chakravāla* process? If so, B. What type of algorithm is it? How is it defined? Is it a continued fraction algorithm? C. What is the meaning of the quantities in *chakravāla*? D. What mathematical relations correspond to the rules (to the four steps in every cycle)? E. Does *chakravāla* always lead to $q = +1$, and in some cases also to the values $q = -1, \pm 2, \pm 4$? F. Do the "*chakravāla*-type" continued fractions give new information about the Indian method of solution?

IV. *Previous Conceptions and Misconceptions*

As is well known, the classical theory (Euler, Lagrange,...) of the fundamental equation (2) is based on the *regular* continued fraction expansion of the number $\sqrt{D}$:

$$(4) \qquad \sqrt{D} = [b_0, \overline{b_1, \ldots, b_k}] = b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \ldots ,$$

where $b_1, \ldots, b_k$ is the primitive period ($b_k = 2b_0$) and $b_1, \ldots, b_{k-1}$ is a palindrome. The first (least) non-trivial solution to (2) is given by the penultimate convergent A_{k-1}/B_{k-1} for k even (or A_{2k-1}/B_{2k-1} for k odd), where $x = A$, $y = B$. The equation can in this way be solved in k (or $2k$) "steps" [Dickson ch. 12; Perron 1954, §27].

Now there exist more general types of continued fraction expansions of real numbers, called semiregular (*halbregelmässige*) [Perron ch. 5]. We can replace some unit numerators in (4) by -1. Thereby the expansion is "contracted," and some convergents are annihilated. However, the penultimate convergents remain intact, and (2) can be solved more easily.

For example, for $D = 58$. The regular expansion $\sqrt{58} = [7, \overline{1,1,1,1,1,1,14}]$, $A_0/B_0 = 7/1$, $A_1/B_1 = 8/1$, $A_2/B_2 = 15/2, \ldots, A_{13}/B_{13} = 19603/2574 = x/y$. Halfregular expansions (denominators with negative numerator are underlined): $\sqrt{58} = [8, \overline{\underline{2},1,1,1,1,15}]$, $A_0/B_0 = 8/1, A_1/B_1 = 15/2$, $A_2/B_2 = 23/3, \ldots, A_{11}/B_{11} = 19603/2574 = x/y$; $\sqrt{58} = [8, \overline{\underline{3},\underline{2},1,1,15}]$, $A_0/B_0 = 8/1$, $A_1/B_1 = 23/3$, $A_2/B_2 = 38/5, \ldots, A_9/B_9 = 19603/2574 = x/y$; and so on.

In the history of mathematics one of the greatest and most often repeated misapprehensions is the underrating of the *chakravāla* method, especially the general opinion that Lagrange [1766; 1769] in his proofs of the existence of solutions of (2) rediscovered the "cyclic method" (meaning the whole process including *bhāvanā*, if used). [2]

Strachey, in his translation into English of the Persian manuscript [1813, 42], had already asserted that the cyclic method was "in principle the same as that for solving the problem in integers by the application of continued fractions, which was at first given in Europe by DeLaGrange." The second translator Colebrooke [1817, xviii-xix] stated also that Euler and Lagrange had rediscovered the Hindu methods.

The first real interpretation of the whole of Hindu mathematics, in Arneth's *Geschichte* [1852, 140ff., esp. 162-164], does not refer at all to Lagrange. However, Arneth says that the *chakravāla* method "so sicher und schnell sie auch zum Ziele führt, stellt jedoch manches dem Ermessen des Rechners anheim, aber Bhaskara bemerkt auch: Algebra ist Scharfsinn, der Rechner muss sich selbst zu helfen wissen."

Hankel [1870, 200] realizes that *chakravāla* is "ohne Zweifel der Glanzpunkt ihrer gesammten Wissenschaft" and [ibid., 202] "Diese Methode ist über alles Lob erhaben; sie ist sicherlich das Feinste, was in der Zahlenlehre vor Lagrange geleistet worden ist." However, he believes that "sie ist merkwürdiger Weise genau dieselbe, welche Lagrange in einer 1768 erschienenen Abhandlung vortrug und erst nachträglich auf den Kettenbruchalgorithmus ... reduzierte, den Euler im Jahre 1767 auf dies Problem angewandt hatte." Yet Hankel proposes the name "die indische Gleichung," since "in der That fehlt der 'cyklischen Methode' der Brahmanen nichts als ... der Beweis ihrer Richtigkeit ... und ferner der Nachweis, dass sie ... zum Ziele führt." These proofs he ascribes to Lagrange. He proves [ibid., 201] that the produced quantities really are integers.

In his *Vorlesungen* Cantor [1880, ch. 29, p. 538] argued without any motivation that the cyclic method was not a general method: "Allerdings wird dieses indische Verfahren nicht stets zum Ziele führen, namentlich nicht nach ganz vorschriftsmässigen Regeln die Wurzeln der Gleichung $ax^2 + 1 = y^2$ finden lassen. Dieses bleibt dem Takte der auflösenden überlassen."

Tannery [1882, 325] naturally ascribed the whole Hindu indeterminate analysis to the Greeks. With the aid of the Greek rules for approximation of $\sqrt{D}$ one could "by simple steps" pass to the Indian method. Tropfke [Pt. 3, 183] as late as 1937 embraces this opinion: "Was Diophant ... noch geleistet, ..., das alles ist uns in dem Dunkel des ... Niedergangs der damaligen Kultur entschwunden; nur Reflexe sind es, die die indische Mathematik zurückwirft." About the method of Lagrange Tropfke [Pt. 3, 193] states: "Lagranges Verfahren von 1769 deckt sich mit dem der Inder."

Very curious explanations were given by Gunther [1882, 40; for details, see Selenius, 1963, 5] and by Knirr [1889].

Konen [1901] gave a thorough investigation of the Hindu methods, following up the interpretations of Hankel. He vacillates between Hankel and Tannery. However, he recognizes the originality of the Hindus: "... dass die Inder jedenfalls um 600 n.Chr. im Besitz einer Methode zur ganzzahligen Lösung der Gleichung $t^2 - Du^2 = 1$ waren, welche, der Sache nach eine Kettenbruchmethode, nichts zu wünschen übrig liess, als den Beweis, dass sie auch stets zum Ziele führt" [Konen, 28].

Heath [1910] is the first of the many modern repeaters of Hankel's interpretations. Another compilation is Whitford's [1912] history of our equation.

Even today misunderstanding or underrating of the Hindu achievements is common in textbooks. For example, even Srinivasiengar in his excellent little book [1967, 110] states that the Indian method involves "an element of trial-process." He certainly demonstrates the efficiency of the Hindu method compared with "Lagrange's method" [p. 115] but does not realize, despite the results of Ayyangar (see below in section V), that the former really is also a continued fraction method.

As is known, the "Pellian" equation once was a main object of the interest to Fermat, Brouncker, Wallis et al. [Fermat 1896; about his "second défi" and correspondence see [Dickson 2, 351ff.]. The "method of Brouncker" mentioned in Fermat [1896, 494-498] corresponds really in some cases to a semiregular continued fraction expansion but this does not "imitate" the *chakravāla* method, as can easily be verified. The imitation does occur in one special case, $D = 109$, but this is mere chance. Indeed, for the analogous $D = 433$ Brouncker's method produces an expansion of the singular type (see section V below). The efforts of Fermat to solve the "Pellian" equation reached at no point the fundamental content of *chakravāla*, and moreover treated only special cases, such as the primes $D = 61, 109, 127, 149$ and 433 [Fermat 2, 334-335]. Incorrect are the statements by Hofmann [1951, 123 and 193] that "die elastische Schnellmethode" of Fermat, described and reconstructed by him in [1944, 6-18], should be related to the *chakravāla* method. Hence the allegations in his review [1964] are quite incorrect (in contrast to the review by Bruins [1965]). As I emphasized in the discussion after my lecture at the *17. Tagung über Problemgeschichte der Mathematik at Oberwolfach* in 1972, *the non-general "elastic rapid method" of Fermat has nothing to do with the Hindu* chakravāla *method.*

V. *The Real Nature of Chakravāla*

Our first question is: Is there a well-defined (not regular) continued fraction type corresponding to the *chakravāla* process?

Lagrange had committed a serious blunder in his *Additions* [Lagrange 1898, 151] to Euler's algebra. With the aid of Euler's example $x^2 - 6y^2 = 1$ in [Euler 1911, II 1, ch. 7, art. 101], Lagrange intended to prove the incorrectness of Wallis' and Euler's proposal that also negative numerators (-1) occasionally could be used. Later this possibility was shown by Minnigerode [1873] and Stern [1886].

The main types of semiregular continued fraction expansions are: (a) "reduced-regular" [Möbius 1830; Zurl 1935]. (b) "nach nächsten Ganzen" [Hurwitz 1889]. (c) singular [Hurwitz 1889]. (d) Minkowski's diagonal [1901]. (e) ideal [Selenius 1960]. Of these, (a), (b), (c) and (e) are "shortest continued fractions" [Perron 1954; Selenius 1960]. Checking shows that (a) - (d) do not correspond to the *chakravāla* process.

An attempt to "imitate" the *chakravāla* in the form of a continued fraction process was made by Krishnaswami Ayyangar [1929, 1938a, 1938b, 1940a, 1940b, 1941]. After all the misunderstandings by European authorities during 150 years, Ayyangar was the first to "appreciate the distance between Lagrange's simple continued fraction and that one discussed" in [1940a, 21]. Though fairly reviewed, Ayyangar's works attracted very little attention, even in India.

However, Ayyangar's continued fractions ("Bhāskara continued fractions") were defined only for numbers of the form $\sqrt{D}$ and for no other real numbers. Though constructed for imitating *chakravāla*, their theory, strangely enough, did not at all explain the rules of the *chakravāla* process, especially not its ingenious core: the interaction between *varga-prakṛti* (2) and *kuṭṭaka* (3). This is remarkable, and so is the fact that he did not study more closely the connection with the regular (and any semiregular) expansion of $\sqrt{D}$. Nor have other Indian mathematicians (Datta, Singh, Shukla, Srinivasiengar, etc.) explained the recursive character of the *chakravāla* process. [See also Chandrasekharan 1967, 4.]

In my papers [1960, §10; 1962a; and especially in the extensive 1963] was shown that *the chakravāla process corresponds to the generally defined continued fractions of type (e) above.* Because of this fact the old Indian methods were shown to have many remarkable properties (see below section VII).

VI. *Interpretation of the Chakravāla Rules*

For the quantities of the *varga-prakṛti* (2) we use the following terms [Colebrooke; Datta and Singh; Selenius 1963; Srinivasiengar; Shankar Shukla; Strachey]: x = greater root (*jyeṣṭha-mūla*); y = lesser root (*kaniṣṭha-mūla*, *hrasva-mūla*); D = multiplier or *prakṛti*; q = interpolator or additive (*kṣepaka*).

For the quantities of the *kuṭṭaka* (3) we use the terms: X = multiplier, pulverizer, or *kuṭṭakara* (*guṇaka*), generally positive; Y = quotient (*phala*), generally positive; a = dividend (*bhājya*); b = divisor (*bhāgahāra*); c = interpolator or additive (*kṣepa*, *kṣepaka*).

We give a lucid interpretation of the four steps of a cycle as follows:

1. The first step is the marvellous "nucleus" of the process. Given $x_n^2 - Dy_n^2 = q_n$ (positive or negative), we form the *kuṭṭaka*-equation $(y_nX + x_n)/|q_n| = Y$ (originally q_n instead of the absolute value): "Make [in $x^2 - Dy^2$ in (1)] the lesser root $[y_n]$, greater root $[x_n]$ and interpolator $[q_n]$ [in a *kuṭṭaka* (3)] the dividend, addend, and divisor, respectively" [Datta and Singh, 164; Shankar Shukla, 10].

2. From the solutions X, Y of this *kuṭṭaka*, by the "method of Āryabhaṭa," we choose the one for which $|X^2-D|$ is minimal: "...the multiplier $[X]$ of it so taken as will make the residue of the *prakṛti* $[D]$ diminished by the square of that multiplier or the latter minus the *prakṛti*, as the case may be, the least" [ibid., 162-163 and 11, respectively]. This X may be called p_{n+1}; the Y so appearing will be the next lesser root y_{n+1}: "The quotient $[Y]$ (corresponding to that value of the multiplier $[X]$) is the new lesser root" [ibid., 165 and 11, resp.].

3. The difference X^2-D or p_{n+1}^2-D divided by q_n gives the new interpolator q_{n+1}, that is $(p_{n+1}^2 - D)/q_n = q_{n+1}$: "That residue $[D-X^2$ or $X^2-D]$ divided by the (original) interpolator $[q_n]$ is the [new] interpolator $[q_{n+1}]$; it should

be reversed in sign in case of subtraction from the *prakṛti* [case X^2-D]" (so originally take $-(D-X^2)$ if $D>X^2$ but X^2-D if $D<X^2$) [Datta and Singh, 163; Selenius 1963, 11; Shankar Shukla, 11].

4. Finally x_{n+1} is to be found. Remarkably enough, the Hindus had three mathematically different paths to tread:

4a. On account of equation (1), the values of $Y = y_{n+1}$ and q_{n+1} produce the new greater root x_{n+1} , according to $x^2_{n+1} = Dy^2_{n+1} + q_{n+1}$, which is the natural interpretation of "...thence the greater root" [Datta and Singh, 163; Shankar Shukla, 11].

4b. Quite different is Nārāyaṇa's rule $x_{n+1} = p_{n+1}y_{n+1} - y_nq_{n+1}$, which surely is "original" since we find it in *Jayadeva*, stanza 15: "...and that [the new lesser root] multiplied by the multiplier [$X = p_{n+1}$] and diminished by the product of the previous lesser root [y_n] and (new) interpolator [q_{n+1}] will be its greater root" [ibid., 165 and 11 resp., where "added" appears in place of "diminished"].

4c. The third way is given by the Bhāskara commentator Krishna Bhaṭṭa's rule $x_{n+1} = (x_np_{n+1} + y_nD/q_n)$. "The original 'greatest' root [x] multiplied by the multiplier [$X=p_{n+1}$], is added to the 'least' root [y_n] multiplied by the given coefficient [=*prakṛti* D]; and the sum is divided by the additive [q_n]" [Colebrooke, 175, fn. 6]. Starting from the quantities x_n, y_n and q_n in (2), we thus have reproduced the "same" quantities: x_{n+1} (step 4), y_{n+1} (step 2) and q_{n+1} (step 3). The play can go on.

VII. The Rationale of Chakravāla

We are now going to answer the questions formulated in section III. Together the answers form, mathematically, the rationale of the old *chakravāla* process.

Question A has already been answered in Section V. So has *Question B*, since the ideal continued fractions of type (e) were able to explain the *chakravāla* [Selenius 1963].

How is this type of continued fraction defined? The definition is much more complicated than those of the previously known types (a) - (d), and this explains the difficulties, and perhaps excuses some of the misapprehensions mentioned in section IV. However, the plain construction (*Übergang*) of the ideal expansion originating from the corresponding regular one is very simple [Selenius 1960, §6, *zweiter Hauptsatz*]. In my table [Selenius 1962a] of ideal expansions of $\sqrt{D}$ for $D = 2,\ldots,1000$ all *chakravāla* cycles leading to solutions for the corresponding equations are given implicitly.

Roughly speaking, the ideal expansion (of a real number ξ_0) is so defined that the regular expansion during the *Übergang* is maximally contracted (section IV above), i.e. cleared of all partial denominators $b_n = 1$; and the values of the quantity

(5) $$B_{n-1}|B_{n-1}\ \xi_0 - A_{n-1}|$$

are minimal. It is noteworthy that we already here have two "minimizing properties" (to say nothing of *bhāvanā*) securing a certain "economization" of the calculations. We will see (below, question E) that the plain *chakravāla* method in several respects is characterized by this minimization.

Question C. What sense do the *chakravāla* quantities x, y, p and q have? It is rather striking that all who have "explained" the *chakravāla* as a common (regular) continued fraction algorithm never observed that not all of the quantities occurring belong to the regular process. Now the theory of the ideal expansion type showed that x/y by turns represent the convergents of $\sqrt{D}$, developed in the ideal expansion.

The role of p and q appears best from an example. Let D be 58 (see section IV), that is, $\xi_0 = \sqrt{58}$, which number we develop in its ideal expansion according to $\xi_n = b_n \pm 1/\xi_{n+1}$ (a periodic chain). In this way we get successively:

$$\begin{array}{lll}
\xi_0 = \sqrt{58} = 8 - 1/\xi_1 , & \xi_1 = (P_1+\sqrt{58})/Q_1 = (8+\sqrt{58})/6 , \\
\xi_1 = 3 - 1/\xi_2 , & \xi_2 = (P_2+\sqrt{58})/Q_2 = (10+\sqrt{58})/7 , \\
\xi_2 = 2 + 1/\xi_3 , & \xi_3 = (P_3+\sqrt{58})/Q_3 = (4+\sqrt{58})/6 , \\
\xi_3 = 2 - 1/\xi_4 , & \xi_4 = (P_4+\sqrt{58})/Q_4 = (8+\sqrt{58})/1 , \\
\xi_4 = 16 - 1/\xi_5 , & \xi_5 = (P_5+\sqrt{58})/Q_5 = (8+\sqrt{58})/6 = \xi_1 ,
\end{array}$$

etc.

Now the successive p and $|q|$ are just the intergers P_n and Q_n respectively. So we get in the example $p_1 = 8$, $p_2 = 10$, $p_3 = 4$, $p_4 = 8$, and $|q_1| = 6$, $|q_2| = 7$, $|q_3| = 6$, $|q_4| = 1$, wholly in accordance with the *chakravāla* process (where q_3 and q_4 are negative). Strictly following the *chakravāla* rule one has to continue until $q_8 = +1$ appears, but the Hindus naturally used the short cut *bhāvanā* for jumping from $x_4^2 - 58y_4^2 = 99^2 - 58 \cdot 13^2 = q_4 = -1$ to $x_8^2 - 58y_8^2 = 19603^2 - 58 \cdot 2574^2 = q_8 = +1$.

We see that *all quantities produced in the chakravāla process have simple counterparts in the ideal continued fraction process.*

Question D. Since the *chakravāla* rules, especially the noteworthy rules in steps 1 and 4a and 4c are quite sophisticated, it is not surprising that no interpretations along these lines were made prior to my results dating from 1959 [1960, 1962b, 1962c, 1963].

The most common interpretation of the rule in step 1, based on Brahmagupta's lemma [Datta and Singh, 162; and later authors], is that

$$\left.\begin{array}{l} x_n^2 - Dy_n^2 = q_n \\ p^2 - D\cdot 1^2 = p^2 - D \end{array}\right\}$$

(6) $$\Longrightarrow \underbrace{\left(\frac{x_n p + D y_n}{q_n}\right)}_{x_{n+1}}^2 - D \underbrace{\left(\frac{y_n p + x_n}{q_n}\right)}_{y_{n+1}}^2 = \underbrace{\frac{p^2 - D}{q_n}}_{q_{n+1}}$$

does not refer at all to continued fractions. However, in my 1963 article I identified the *kuṭṭaka* formula (3), or $aX - bY = -c$, with the recursive identity

(7) $$Q_n P_n - B_{n-1} P_{n+1} = A_{n-1} ,$$

first given in my 1960 article. This formula is valid not only for a regular expansion but also for an ideal one. It is the clue to the marvellous first step in *chakravāla* [Selenius 1963, 13-19].

The rules in step 1, for example the minimization of $|X^2-D|$, where X is a solution of (3), exactly correspond to the theory [Selenius 1963, 14-19]. The rule in step 3 is a simple consequence of an old formula and one derived by the author [Selenius 1960, 62].

The formula in step 4a is clear, but our interpretation of Jayadeva's and Nārāyaṇa's formula in 4b is quite new. It can be derived from the theory in [Selenius 1963] and its mathematical substance is

(8) $$A_{n-1} = Q_n B_{n-2} + B_{n-1} P_n ,$$

in other words, a "simple" consequence of (7) above. In a sense, this is also the case of Bhaṭṭa's formula in 4c, since it corresponds to a formula $A_n Q_n - A_{n-1} P_{n+1} = DB_{n-1}$, an analogue of (7) [Selenius, 32-33].

Historically seen, the facts mentioned are very remarkable. First, the formula in 4a, self-evident for calculating x_{n+1}, was replaced by the Hindus by the more sophisticated formulas in 4b and 4c. (We ignore the chronological order of the formulas, since here this is not relevant.) The reason for this was probably the desire to avoid square root extraction and large numbers. Secondly, the common interpretation (with the aid of Brahmagupta's lemma) of the famous rule in step 1 given above in (6) elucidates the rule in 4c [see e.g. Datta and Singh, 162] *but not at all the rule in 4b.* Now, the vital point is this: from the standpoint of the theory of continued fractions both formulas have the same theoretical "background," and so their coherence and equal applicability is clear.

We have here an interesting historical problem: Is the interpretation (6) only a "strained" reconstruction (I have always nearly "felt" it so!) of the original "thoughts," knowledge and experience amongst the Hindus? How were they capable of "inventing" such an ingenious rule as Nārāyaṇa's 4b? (Observe that y_{n+1} and q_{n+1}, but not D, enter into the formula.)

Question E. Of course, the assertion that *chakravāla* always leads to $q = +1$ (and so for all values of D to a solution) was wholly tentative. The proof of it is not simple at all [Srinivasiengar, 114] and is never given in the textbooks! However, it is a consequence of the theory of continued fractions [Selenius 1963, 20]. If interpolators $q = -1$, ± 2 or ± 4 appear, the short cut rules (*bhāvanā*), if used, directly lead [ibid., 21-26] to the *same* solution as the alternative strict continuation of *chakravāla*.

Question F. In the answer to question A we have mentioned the (defining) "minimization property" for the (real) quantities (5). Therefore the *chakravāla* quantities x_n, y_n have best possible approximation properties, that is, x_n/y_n approximate $\sqrt{D}$ very well ("ideal approximation") [Selenius 1960, Kap. 3]. The constants γ in $|x/y - \sqrt{D}| = (\gamma y^2)^{-1}$, where x/y are general convergents to $\sqrt{D}$, are in a sense the best possible for all x/y equal to the ideal *chakravāla* convergents x_n/y_n.

Chakravāla represents [Selenius 1963, 35-36] a shortest possible continued fraction algorithm: the number of the cycles is minimal (naturally use of the short cut *bhāvanā* effects a further shortening of the calculations). For example, in the case $D = 61$ (*Bhāskara*) the European solution of the equation requires calculation of 22 convergents, while the Hindu solution used only five or six.

Another property explained by the theory is the fact that *chakravāla* always produces the least (positive) solution of (2), and starting from this one, all solutions. The least solution is obtained with or without use of *bhāvanā*.

The sequences of the appearing p_n *and* q_n *are symmetrical* [Selenius 1963, 37-38]. The set $\{p_n\}$ is in general not a subset of the corresponding set for Euler's "regular" process. For $D = 58$ the two sets $\{p_n\}$ are:

Euler:	7	2	4	3	4	2	7	7	2	4	3	4	2	7
Chakravāla:	8		10		4	8		8		4	10		8	

corresponding to two primitive periods of the regular expansion.

A very interesting property of *chakravāla* is: *all appearing interpolators* q_n *are less than* $\sqrt{D}$, are in fact less than or or equal to $[\sqrt{D}]$ = the denominator b_0 in the regular expansion (4). This is not the case when one makes use of regular fractions. For the example $D = 58$, we have $b_0 = 7$ and values of q as follows:

Chakravāla: $1, 6, 7, 6, 1$

Euler: $1, 9, 6, 7, 7, 6, 9, 1.$

Again *chakravāla* ingeniously avoids large numbers in the calculations.

A remarkable geometrical property of *chakravāla* may also be noticed. Compared with the regular expansion, the ideal one has the property that all sequences of denominators $b_n = 1$ are contracted to sequences of denominators 2 or 3. The way in which this contraction happens has a very beautiful and simple geometrical configuration [Selenius 1962b]. When $D = 2081$, we have $\sqrt{D} = [45,\overline{1,1,1,1,1,1,1,1,1,1,90}]$ (Euler) $= [46,\overline{\underline{3},\underline{3},\underline{2},2,\underline{3},92}]$ (*chakravāla*). (Compare the example $\sqrt{58}$ in section IV.) Here the ten denominators $b_n = 1$ are illustrated by the points $A_n - 1,2,\ldots,10$ on a circle and ordered according to decreasing magnitude of the approximating quantities in equation (5). For $An = 1, 10, 3, 8,$ and 5 the convergents A_{n-1}/B_{n-1} are annihilated, for $n = 6, 7, 4, 9,$ and 2 they remain (see Figure 1).

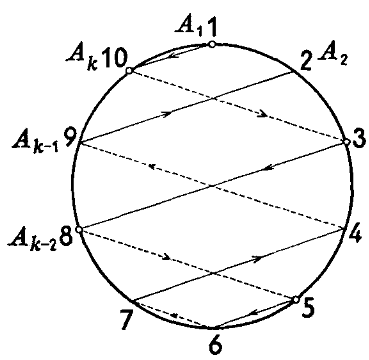

Figure 1

VIII. Illuminating Examples

A full appreciation of Indian indeterminate analysis presupposes a close and thorough knowledge of the original works, the efforts to interpret them, the corresponding European work, and the theories of (regular and semiregular) continued fractions. Nevertheless, illustrative examples may elucidate the effectiveness of the *chakravāla* method for solving (2). Some classical examples, more or less completely carried out, are:

$D = 97$ [Colebrooke, 176; Strachey, 43-46; Euler; Hankel, 202; Konen, §87; Dickson 2, 348; Brun, 6-7; Datta and Singh, 166; Heath, 284; Srinivasiengar, 115; Juschkewitsch, 149; Selenius 1963, 26].

$D = 61$ [Strachey, 46; Datta and Singh, 168; Srinivasiengar, 114; Selenius 1963, 31]. In a letter to Freniele, Fermat [1894, vol.2, 334] asked for the least solution in this difficult case. Bhāskara, 500 years earlier, had found $x = 1{,}766{,}319{,}049$, $y = 226{,}153{,}980$! [Strachey, 47].

$D = 13$ [Strachey, 48]. Wallis' example [Fermat 1896, vol.3, 464,490].

$D = 103$ and 97 . Examples from Nārāyaṇa [Datta and Singh, 169-171; Srinivasiengar, 116-117].

Here we give only a few comparative examples showing the differences between *chakravāla* and other methods. We choose $D = 88$, 97, and 433. Of course "Euler" and "*chakravāla*" refer to results of the two methods and not to values actually found by Euler or ancient Hindu mathematicians.

$D = 88$

Convergents

Euler:	9/1	19/2	28/3	47/5	75/8	197/21	= x/y
Chakravāla:	9/1		28/3		75/8	197/21	= x/y

q-values

Euler:	1	7	9	8	9	7	1
Chakravāla:	1	7		8		7	1

$D = 97$

Convergents

Euler:	9/1	10/1	59/6	69/7	128/13	197/20
Chakravāla:		10/1		69/7		197/20
Euler:	325/33	522/53	847/86	4757/483	5604/569	
Chakravāla:	325/33		847/86		5604/569	

After *11* and *6* steps respectively one gets the solution $x/y = 62809633/6377352$.

q-values

Euler:	1	16	3	11	8	9	9	8	11	3	16	1
Chakravāla:	1		3		8		9	8		3		1

$D = 433$

(a prime to which the interpretation in [Hofmann 1944] is applicable)

q-values

Euler:	*1*	*33*	*8*	*9*	*16*	*13*	*24*	*11*	*3*	*24*	*17*	*etc.*
Chakravāla:	*1*		*8*	*9*	*16*	*13*		*11*	*3*		*17*	*etc.*
Brouncker:	*1*		*8*	*9*	*16*	*13*		*11*	*3*	*24*	*17*	*etc.*
Hofmann:	*1*		*8*		*16*		*24*					*etc.*

("Brouncker" means here the method of Brouncker, described by Wallis in [Fermat 1896, vol.3, 498].)

IX. Remarks

The full generality of the ideal continued fraction definition renders it possible that *chakravāla*, even though practiced by the Hindus only for solution of the equation (2), can be generalized to the equation $x^2 + xy - Gy^2 = 1$, and to other types [Selenius 1966a]. Much more remarkable is the ability of the Hindu methods, in their entirety, to permit a generalization to the cubic case(!) $x^3 - Dy^3 = 1$ [Selenius 1970 and lecture at the 17. Tagung über Problemgeschichte der Mathematik in Oberwolfach 1972], as well as to equations over the domain $Z[i]$ [ibid. and Selenius 1971].

X. Summary

The old Indian *chakravāla* method for solving the mathematically fundamental indeterminate *varga-prakṛti* equation (2) was a very natural, effective and labour-saving method with deep-seated mathematical properties.

The method represents *a best approximation algorithm of minimal length* that, owing to *several minimization properties, with minimal effort* ("economization") and *avoiding large numbers,* always *automatically* (without trial processes) *produces the least solutions* to the equation, and thereby *the whole set of solutions.*

Since the *chakravāla* method, and the other Hindu methods for solving this Jayadeva-Bhāskara equation (2), did not occur in China at all [Needham, 119ff.], it must be regarded as *a purely Indian creation.* More than ever are the words of Hankel (mentioned above in section IV) valid, that the *chakravāla* method was the absolute climax ("ohne Zweifel der Glanzpunkt") of old Indian science, and so of all Oriental mathematics.

It is accepted that the *chakravāla* method here explained anticipated the European methods by more than a thousand years. But, as we have seen, no European performances in the whole field of algebra at a time much later than Bhāskara's, nay nearly up to our times, equalled the marvellous complexity and ingenuity of *chakravāla.*

NOTES

1. Some scholars (e.g., Datta and Singh) use the English transliteration *cakravāla*, others (e.g. Srinivasiengar) use *chakravāla*. For the origin of the name *varga-prakṛti*, e.g. by Brahmagupta, Bhāskara II, Kṛṣṇa (1580) and Kamalākara (1658), see [Datta and Singh Pt.2, 141]. For the technical terms used by Hindu algebraists in connection with *varga-prakṛti*, explained e.g. by Pṛthūdakasvāmī (860), Śrīpati, Bhāskara II, Nārāyaṇa and Kamalākara, see [ibid., 143].
2. It is interesting to recall the quite kindred misapprehension of Kaye [1908], Heath [1910] and others regarding *kuṭṭaka*, the other fundamental indeterminate equation. In his *Notes* Kaye maintained an analogy between Āryabhaṭa's rule and "an easy development" of Euclid's analysis of the G.C.D. According to Heath [1910, 281] the solution "given by Āryabhaṭa, as well as by Brahmagupta and Bhāskara, though it anticipated Bachet's solution ... is an easy development from Euclid's method of finding the G.C.M. ... and it would be strange if the Greeks had not taken this step." Here the correct rationale of *kuṭṭaka* was given by Mazumdar [1911]. Even Colebrooke [1817, xv] had realized that "the general character of the Diophantine problems and of the Hindu unlimited ones is by no means alike." Two significantly titled articles germane to the misunderstanding of Hindu mathematics are [Eneström 1912] and [Bruins 1970]. The words of Marshall [1890, 379] are applicable: "Facts are suggestive in their similarities, but are still more suggestive in the differences that peer out through those similarities." An extreme example of the underrating attitude is Kaye's [1910, 298-299] final verdict on Hindu mathematical methods. Yet Cantor [1877, 20] admitted simply "dass die Inder Lehrer der Griechen in arithmetischen und algebraischen Dingen" [but not in geometrical and astronomical things] "gewesen sein können."

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