KHA IN ANCIENT INDIAN MATHEMATICS

AMARTYA KUMAR DUTTA

It was in ancient India that zero received its first clear acceptance as an integer in its own right. In 628 CE, Brahmagupta describes in detail rules of operations with integers — positive, negative and zero — and thus, in effect, imparts a ring structure on integers with zero as the additive identity. The various Sanskrit names for zero include *kha*, *śūnya*, *pūrṇa*.

There was an awareness about the perils of zero and yet ancient Indian mathematicians not only embraced zero as an integer but allowed it to participate in all four arithmetic operations, including as a divisor in a division. But division by zero is strictly forbidden in the present edifice of mathematics. Consequently, verses from ancient stalwarts like Brahmagupta and Bhāskarācārya referring to numbers with "zero in the denominator" shock the modern reader. Certain examples in the Bījagaṇita (1150 CE) of Bhāskarācārya appear as absurd nonsense.

But then there was a time when square roots of negative numbers were considered non-existent and forbidden; even the validity of subtracting a bigger number from a smaller number (i.e., the existence of negative numbers) took a long time to gain universal acceptance. Is it not possible that we too have confined ourselves to a certain safe convention regarding the zero and that there could be other approaches where the ideas of Brahmagupta and Bhāskarācārya, and even the examples of Bhāskarācārya, will appear not only valid but even natural?

Enterprising modern mathematicians have created elaborate legal (or technical) machinery to overcome the limitations imposed by the prohibition against use of zero in the denominator. The most familiar are the methods of calculus with its concept of limit, results like l'Hôpital's rules, and a language which enables one to express intuitive ideas like $\frac{1}{0} = \infty$ through legally permitted euphemisms. Less well-known are the devices of commutative algebra, algebraic geometry and algebraic number theory like "localisation" which describes a legal structure for directly writing fractions with zero in the denominator without any subterfuge, and the more sophisticated ideas of "valuation theory" which admit multiple levels of infinities and thereby provide higher-dimensional algebraic analogues of l'Hôpital's rules.

In this talk we shall highlight an algebraic model proposed by Prof. Avinash Sathaye for understanding Bhāskarācārya's treatment of *khahara*, (numbers with) zero in the denominator, including the apparently erroneous examples in the algebra treatise Bījagaṇita. A crucial ingredient of this model is the ubiquitous concept of "idempotent" in modern algebra (elements e satisfying $e^2 = e$). The commentary by Kṛṣṇadaivajña (c.1548) indicates that idempotence was indeed envisaged as a natural property of numbers like zero and its reciprocal, the *khahara*. While historians of mathematics have tried to analyse Bhāskarācārya's *khahara* in the framework of calculus, the difficulties with his examples disappear in the algebraic interpretation based on idempotents. Prof. Sathaye's interpretation of Bhāskarācārya's *khahara* also gives a new meaning to certain mysterious utterances of Ramanujan recorded by P.C. Mahalanobis.

Towards the beginning of the talk, we shall make a brief discussion on the role of zero as a place-holder in the written decimal notation and how the exclusive emphasis on the importance of zero has hindered our appreciation of other sophisticated aspects of the decimal system which, in its verbal form, occurs in all Vedic treatises including the oldest — the Ṛgveda.