Bhāskarācārya's Treatment of the Concept of Infinity[1]

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1. INTRODUCTION

Even a child learns to count 1, 2, 3 and so on. The idea of infinity arises when one starts wondering whether this sequence ends or not, and how does it end if it does. This idea needs greater sophistication.

In the same fashion, the idea to extend the count backwards and imagine a 0 or even a -1, -2, and so on needs a different imaginative process.

The ideas of zero and negative numbers are, of course old and by now, quite familiar to everybody. One has, thus become used to these numbers called the "Integers" which include zero and are unbounded on both ends.

This is the so-called Hindu Arabic system, developed in India and propagated through the Arabic Mathematical sources across Europe.

Many of the standard techniques of algebraic calculations with integers written as decimal numbers are routinely taught in elementary schools. One no longer thinks about their power or significance, mainly due to their familiarity. Not surprisingly, the methods were developed in India, since they crucially depended on the place value system of number representation.

The idea of infinity is a different story. If you open mathematical books of today, you will find the idea of infinity mentioned in somewhat higher level courses. In Calculus related courses, you will find the idea of the "real'' (positive or negative)

[1]Dedicated to one of the greatest Indian Mathematicians on the 900th anniversary of his birth year 2014.

Gaṇita Bhāratī Vol. 37, No. 1-2 (2015) pages 55-67
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infinity, where the variables take on larger and larger (positive or negative) values. The resulting analysis of dependent expressions leads to the notion of limits. It is at the heart of Calculus and thus also at the heart of Modern Analysis.

Similarly, the idea of a variable getting infinitesimally close to a finite number, like 0, is equally important in Calculus. You won't, however, find either the infinite or the infinitesimal in an elementary book on algebra, let alone a book on arithmetic!

The only thing you may find in an elementary algebra book is a very stern warning: "Thou shalt not divide by zero!"[2] Indeed the warning persists in all higher Mathematics.

On the other hand, in the algebra books of old times in India, we find both the infinite and the infinitesimal treated routinely. In particular, Bhāskarācārya's books Līlāvatī [BLI] and Bījagaṇita [BBI] include some rather intriguing exercises based on these ideas.[3]

These appear to be sheer nonsense, if one approaches them armed with modern con-ventions without recognizing the novelty of approach used in these ancient books. Often, these exercises are discredited (even dropped), calling them unfortunate blemishes on the otherwise brilliant achievements of Bhāskarācārya.

My aim in this short essay is to analyze these exercises by Bhāskarācārya in detail and propose that they might have been based on an algebraic idea.

[2]A Google search for "ten commandments of Mathematics" will produce numerous posters with this commandment included, often on top!

[3]There is yet another facet of the idea of the infinite, which arises from enumerating infinitely many objects, like natural numbers, for example. In Modern Mathematics, this leads to the concepts of ordinal and cardinal numbers. In ancient Indian Mathematics, we find Jain texts discussing various such concepts of infinities. These texts are mainly religious or philosophical, but often carry a healthy amount of serious mathematics. They seem to introduce formal concepts of finite or enumerable, innumerable (very large but still finite) and infinite. They even classify multidimensional concepts for infinity. It is possible that they might have come close to the ideas of modern cardinal (or at least ordinal) numbers.

However, I have not yet succeeded in finding explicit pointers to their advanced ideas similar to the algebraic ideas discussed here. So a similar evaluation of Jain theories of infinities will have to await further evidence. One introductory reference for Jain concepts is [JES]. Discussion of infinity types is in Vol. 1 Chap. 3(b).

I am not suggesting that some old books had developed the full algebraic machinery. The ideas certainly never got elaborated in either the original mathematical texts or their subsequent commentaries.[4]

Some of the ideas about infinity that I am proposing to discuss were already stated by Brahmagupta (seventh century). The listed exercises below are to be found only in the works of Bhāskarācārya II in the twelfth century. It is quite likely that Bhāskarācārya's novel ideas about infinity were not fully appreciated by his commentators, perhaps due to the lack of explanations of subtle algebraic concepts.

The likely reason for this is, perhaps, as follows. The first of the three problems being discussed is from Līlāvatī and it can be comfortably explained using the natural ideas of limits. It was also declared to be of great use in Astronomical and other calculations.

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[4]In Bhāskarācārya's case his idea may have been ignored due to a lack of appreciation or comprehension. It could also have been ignored because the last two exercise went against the natural notion of limits.

There is an interesting parallel story in Indian Astronomy which had a very different outcome. The Indian astronomers treated the earth as a globe, unlike the descriptions in scriptures called purāṇas or other proponents of "flat earth" around the world. However, they described it as a globe fixed in space, neither moving nor spinning. To explain the apparent westward movement of the stars, they stated on scriptural authority that the creator (Brahmā) put the whole star-frame (bhapañjara) in a westward motion by a special wind called pravaha (great flow). The planets being closer to earth moved at a slower speed and thus appeared to be generally moving eastward. Āryabhaṭa, the great mathematician and astronomer of the fifth century, proposed instead that the earth actually rotated daily from towards east and the stars stayed fixed. [AGO] verse 9. His idea was ridiculed and rejected on several grounds by all his commentators and subsequent astronomers. The arguments included appeal to common sense that such a massive motion, if it really took place, must have been felt and would cause many disasters.

To counteract other common sense arguments by flat earthers, Indian astronomers introduced a gravitational force, but attributed it to earth only and declared it to be another endowment from the creator. This theory naturally led to idea that earth is fixed in place. Even commentators like Pṛthūdakasvāmī, who saw that Āryabhaṭa's theory had the merit of simplifying the mathematical model of planetary motions, did not endorse or pursue the idea further. [PVB] (commentary on Golādhyāya, chapter 21, verse 30 of Brahmagupta's Brāhmasphuṭasiddhānta [BBS]). In addition, the very next verse [AGO] verse 10 of Āryabhaṭa seems to contradict his own rotation theory! The full story, however, should be discussed elsewhere.

The next two problems are from Bījagaṇita. They **appear incorrect** when viewed as usual limits. They are usually criticized and sometimes ignored or even dropped.

It is noteworthy that the greatest Indian mathematician Ramanujan is reported to have thought about the ideas of zero and infinity similar to what is presented in our formalism. This was brought to our attention by Parthasarathi Mukhopadhyay and is originally reported in a book by S. R. Ranganathan entitled "Ramanujan: The Man and the Mathematician". [RRM] pages 82,83.

See the details below in the section 3 on Formalism.

2. BASIC DEFINITIONS

In the following pages, I will give all the citations from Bhāskarācārya's Bījagaṇita (his book on Algebra) with some cross reference from his Līlāvatī (his book on Arithmetic). There is a small difference in the numbering of these verses in different editions, but the reader should be able to locate them near the indicated citations.

First, I collect the various defining properties of multiplication and division by zero.

1. वधादौ वियत् खस्य खं खेन घाते खहारो भवेत् खेन भक्तश्च राशिः॥ बीज २.१८

vadhādau viyat khasya khaṁ khena ghāte khahāro bhavet khena bhaktaśca rāśiḥ || bīja 2.18

A zero results when multiplied by zero, a "khahara'' (zero-divided) results when a (natural) number (rāśi) is divided by zero.

2. In Līlāvatī , he gives more instruction about multiplying by zero.

योगे खं क्षेपसमं वर्गादौ खं खभाजितो राशिः।

yoge khaṁ kṣepasamaṁ vargādau khaṁ khabhājito rāśiḥ |

खहरः स्यात् खगुणः खं खगुणश्चिन्त्यश्च शेषविधौ॥ लीला.४६

khaharaḥ syāt khaguṇaḥ khaṁ khaguṇaścintyaśca śeṣavidhau || līlā. 46

Zero plus (minus) zero is zero and powers of zero are zero. A number divided by zero is "khahara'' (zero-divided i.e. having zero as a divider). A number multiplied by zero is zero (but this) khaguṇa must be paid attention to in the rest of the calculation.

In other words, Bhāskarācārya recommends that one should wait to finish all operations before evaluating the khaguṇa.

3. शून्ये गुणके जाते खं हारश्चेत् पुनस्तदा राशिः।

śūnye guṇake jāte khaṁ hāraścet punastadā rāśiḥ ।

अविकृत एव ज्ञेयस्तथैव खेनोनितश्च युतः॥ लीला. ४७

avikṛta eva jñeyastathaiva khenonitaśca yutaḥ ।। līlā. 47

If a zero becomes a multiplier and a number turns into zero, it should (really) be considered as unchanged if it is again divided by zero. Similarly, if a zero is subtracted off or added to (a number, then the number is considered unchanged.)

4. Effectively, Bhāskarācārya is proposing two special terms to be called khaguṇa and khahara which require specific algebraic manipulations.

For khahara, he explicitly adds a colorful description:

अस्मिन् विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु।

asmin vikāraḥ khahare na rāśāvapi praviṣṭeṣvapi niḥsṛteṣu ।

बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत्॥ बीज. २।२०

bahuṣvapi syāllayasṛṣṭikāle'nante'cyute bhūtagaṇeṣu yadvat ।। bīja. 2।20

There is no change in this khahara by adding or subtracting (quantities), just like the infinite Immutable (Brahma or Viṣṇu) which is not affected by the living beings entering or leaving it at the time of dissolution or creation of the world respectively.

3. FORMALISM

For convenience, let us make our own formal definitions using modern terminology. It is clear that if we want to multiply or divide by zero then we need a place holder formal symbol for it, so that it does not get evaluated to zero until the formal rules have been properly applied.

So, we choose a distinctive symbol.

Definition: Let $\mathfrak{R}$ stand for the usual set of real numbers and let ϵ stand for the multiplier zero.

If is any number in $\mathfrak{R}$, then by $x\epsilon$ we shall denote the corresponding khaguṇa. The set of all khaguṇa can be denoted as $\mathfrak{R}\epsilon$.

Similarly, we write ∞ to denote the specific khahara $1/0$. Naturally, in our formalism this should be defined as $1/\epsilon$.

Thus every khahara can be represented as $x\infty$ as x varies over $\mathfrak{R}$. Thus, the set of khaharas can be written as $\mathfrak{R}\infty$.

We now have three kinds of numbers: ordinary ($\mathfrak{R}$), khaguṇa and khahara. The additional facts about the khahara can be presented thus.

- We may explain Bhāskarācārya's calculations in bīja. 2|20 above by proposing the following rule:

$$x\infty + y = x\infty \text{ for any } x, y \in \mathfrak{R}$$

Let us call this the **winning rule** for infinity.

- We will make the winning rule also applicable for a khaguṇa thus:

$$x\infty + \epsilon y = x\infty \text{ for any } x, y \in \mathfrak{R}$$

Thus when a khahara is added to a non-khahara number, then only the khahara survives!

- Now we define a product structure among the various numbers.

We propose the following rules, which are natural for modern algebraic structures. Let $x, y \in \Re$.

1. $x\epsilon \,.\, y\epsilon = xy\epsilon$.

2. $x\infty \,.\, y\infty = xy\infty$.

3. $x\infty \,.\, y\epsilon = xy = y\epsilon \,.\, x\infty$.

The third rule is related to thoughts of Ramanujan expressed during private discussions with P. C. Mahalanobis (the "Father of Indian Statistics"). Here is a summary of their reported conversation:

Ramanujan spoke of zero as representative of the Absolute (nirguṇa brahma), something which has no attributes and no description. Infinity, on the other hand was totality of all possibilities capable of being manifest in reality. Further, the product of zero and infinity would supply the whole set of finite numbers.

In other words, he was probably thinking of infinity as khahara or $\Re\infty$ in our notation, which contains a "copy" of all real numbers. These real numbers become manifest upon multiplication by zero (i.e. ϵ in our notation).

Later on, we will explain why these rules are necessary. However, it is useful to note that in Modern Algebra, an entity e is called **an idempotent** if it satisfies $e^2 = e$.

Our entities ϵ and ∞ are defined to be idempotent and that would turn out to be crucial later.[5]

Since ϵ represents zero, it is natural to expect it to be idempotent. Since ∞ represents a fraction $\frac{1}{0}$, it is natural to make it an idempotent also. Indeed we find that in the commentary by Kṛṣṇadaivajña on the Bījagaṇita [KDJ], we find on page 142 in the discussion of verse 120 the following discussion. We record the changes of notational conventions first:

- For convenience of *typing*, we have replaced his variable **या** by y and **याव** by *ys*. which denotes the square y^2 in modern notation.

[5]Usually, 0 and 1 are the only two idempotents. More idempotents are possible (as in a Boolean Algebra which is full of idempotents), but the resulting algebraic system gets too far away from the classical notions of number systems.

- We have replaced १ by 1 and note that this represents the coefficient in his notation, written after the variable.
- A fraction was denoted by simply stacking symbols above each other and the addition sign + was usually dropped. We have supplied the +.

अत्र राशिः This number $\frac{y\,1}{0}$ **वर्गितः** squared $\frac{ys\,1}{0}$ **स्वपदेन युक्तः** added by itself $\frac{ys\,1}{0} + \frac{y\,1}{0}$

अयं खगुणो जातः this multiplied by 0 has become *ys* 1 + *y*1.

Now we shall take up the discussion of three "exercises" from Bhāskarācārya's works to illustrate how our stipulation of these rules give answers which are consistent with the "reported" answers in the original text. We shall also see how the traditional interpretation of the algebraic rules (which is probably a misinterpretation of Bhaskaracharya's approach) will make the exercises as either nonsense, or, at best, mysterious!

4. THE EXERCISES

Here are the three exercises. We give the original formulation as well as a translation using modern terminology. Each exercise has one special equation to solve which is of interest to us.

4.1 Problem 1

खं पञ्चयुग्भवति किं वद खस्य वर्गं मूलं घनं घनपदं खगुणाश्च पञ्च।

khaṁ pañcayugbhavati kiṁ vada khasya vargaṁ mūlaṁ ghanaṁ ghana-padaṁ khaguṇāśca pañca ।

Meaning: (1.1) What is 0 plus 5? Tell what is the square, square root, cube and cube root of zero and (what is) 5 times 0.

खेनोद्धृता दश च कः खगुणो निजार्धयुक्तस्त्रिभिश्च गुणितः खहृतस्त्रिषष्ठिः ॥ लीला. ४८

khenoddhṛtā daśa ca kaḥ khaguṇo nijārdhayuktastribhiśca guṇitaḥ khahṛ-tastriṣaṣṭmhiḥ ।।līlā. 48

Meaning: (1.2) What is 10 divided by 0?

(1.3)	What is the number	x
	which, when multiplied by 0	$x\epsilon$
	combined with its half	$x\epsilon + (1/2)x\epsilon$ i.e. $(3x/2)\epsilon$
	then multiplied by 3	$3 \times (3x/2)\epsilon = (9x/2)\epsilon$
	then divided by 0	$(9x/2)\epsilon \cdot \infty = 9x/2$
	equals 63 ?	

The answers:

(1.1) 5, 0, 0, 0, 0, 5 $\times$ ϵ = 0.

(1.2) 10∞

(1.3) Solution of $(9/2)x$ = 63, so x =14.

4.2 Problem 2

कः खेन विहृतो राशिः कोट्या युक्तोऽथवोऽनितः।

kaḥ khena vihṛto rāśiḥ koṭyā yukto'thavo'nitaḥ ।

Meaning:

What is the number (positive by convention)	x
divided by 0	$x\infty$
augmented or reduced by 10,000,000	$x\infty \pm 100000000$ still $x\infty$ by the winning

वर्गितः स्वपदेनाढ्यः खगुणो नवतिर्भवेत्॥ बीज. १२०

vargitaḥ svapadenāḍyaḥ khaguṇo navatirbhavet । । bīja. 120

Meaning:

squared	$x\infty \cdot x\infty = x^2\infty$
and then augmented by its own square root:	$x^2\infty + x\infty = (x^2 + x)\infty$
multiplied by 0	$(x^2 + x)\infty \cdot \epsilon = (x^2 + x)$
becomes 90?	

The answer: The equation is now reduced to $(x^2 + x) = 90$, so x= 9.

We discard the second answer −10 since the question asks for a rāśī and as noted by conventional wisdom we keep it non negative.

4.3 Problem 3

कः सार्धसहितो राशिः खगुणो वर्गितो युतः।

kaḥ sārdhasahito rāśiḥ khaguṇo vargito yutaḥ ।

स्वपदाभ्यां खभक्तश्च जातः पञ्चदशोच्यताम्॥ बीज. १२१

svapadābhyāṁ khabhaktaśca jātaḥ pañcadaśocyatām ।। bīja. 121

Meaning:

What is the (positive) number	x
combined with its own half	$x + x/2 = 3x/2$
multiplied by 0	$(3/2)x\epsilon$
squared	$(9/4)x^2\epsilon$
augmented by twice its square root	$((9/4)x^2 + 2(3/2)x)\epsilon$
divided by 0:	$((9/4)x^2 + 2(3/2)x)\epsilon \cdot \infty$
	$= (9/4)x^2 + 2(3/2)x$
becomes 15?	

The Answer: The equation is now reduced to $((9/4)x^2 + 2(3/2)x) = 15$, which simplifies to $9x^2 + 12x = 60$.

As above, we accept the positive solution $x = 2$ discarding the negative $-10/3$.

5. DISCUSSION OF THE PROBLEMS

1. For problem 1, if we consider a variable t to replace our ϵ then our final expression may be calculated as $\frac{9x}{2}\frac{t}{t}$ and clearly we can deduce the limit as t goes to zero to be $\frac{9x}{2}$.

Thus, the equation makes perfect sense as a usual limit.

Bhāskarācārya has stated in his own commentary on Līlāvatī that such calculations are of immense use in Astronomical calculations.

2. For problem 2, if we attempt a similar plan, we get the following expression:

$$\frac{(xT+c)^2+(xT+c)}{T}=\frac{x^2T^2+2xcT+c^2+xT+c}{T}$$

where we are letting T to be a variable going to infinity and c denotes the large arbitrary number $\pm$ 1000000.

By conventional Mathematics, this expression would have an infinite limit.

Only by the winning rule, we can simplify $xT + c$ to xT and then by the idempotence of T we get the final simplification $x^2 + x$.

We note that if the winning rule is not applied first, but only after idempotence is used, then the expression reduces to $\frac{\left(x^2+(2xc+x)\right)T+c^2+c}{T}$ which leads to $x^2 + 2xc + x$ which is no longer meaningful, since c is random.

Thus, there is no doubt that Bhāskarācārya was serious about using the winning rule as soon as it becomes applicable.

Note that Bhāskarācārya states this problem in his Algebra text and does not suggest an applicability to practical problems. This is in contrast with problem 1 from Līlāvatī .

Thus, he probably had a different scheme of calculations in mind. We are simply trying to guess his intended mechanism.

3. For problem 3, we may take a variable t going to zero as in problem 1 and this time the resulting expression becomes

$$\left(\frac{9x^2}{4}t^2+\frac{6x}{2}t\right)\frac{1}{t}$$

and the "usual" limit would simply be $(3x)/2$. This results in a very different solution. Here, the winning rule is not needed, but idempotence must be used.

6. MODERN ANALOGS

We now describe some constructions in modern mathematics which show how division by zero is made to be logically meaningful. This should illustrate that

Bhāskarācārya's attempts, though only partially developed, were not totally out of place.

1. **Inverting zero divisors** A zero divisor is usually defined as a non zero element x such that $xy = 0$ for some non zero y. Naturally, for such a zero divisor, we may not wish to define $\frac{1}{x}$ for otherwise we will have to admit $\frac{1}{x} = \frac{y}{xy} = \frac{y}{0}$, a potential problem.

However, in Modern Algebra, this is systematically done thus:

Let $R = \{f(x) + g(y) + c\}$
where $f(x), g(y)$ are polynomials with coefficients in a field, say $\mathbb{Q}$ the field of rational numbers, $c \in \mathbb{Q}$ and $f(0) = g(0) = 0$.

Thus, $xy = 0$, yet x, y are non zero in R.

Modern algebra will then produce a new ring, say S with a homomorphism from R to S so that the image of x has a meaningful reciprocal in S. The ring S can be identified with $\{\frac{u(x)}{x^r}\}$ where $u(x)$ is any polynomial in x over $\mathbb{Q}$ and r is a non negative integer.

The main idea of this operation can be described thus:

Say, you wish to invert a set Λ of quantities. If you have any element such that $xy = 0$ for some $x \in \Lambda$, then you first set all such y to be zero. After this, you have no problem inverting elements of Λ!

2. Idempotents The above two examples illustrate two of the three principles invoked by Bhāskarācārya. The third notion of using idempotents is encountered in the study of Boolean algebras; these, however, are rarely encountered in usual numerical calculations.

The concept of idempotents is, however, very useful in the study of rings in Modern Algebra.

7. CONCLUSION

We see that Bhāskarācārya certainly had a novel calculation scheme introduced in his exercises and might have intended further developments. However, he seems to have worked more extensively on astronomy and perhaps did not return to these ideas again.

The idea about infinity and especially the idea of using this extended number system does seem to point to possible new algebraic concepts.

Whether these ideas can create new useful Mathematical Systems remains an open question.

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