

CHAPTER IX

Brahmagupta as an Algebraist

Ancient Indian name for algebra is *Bijaganita* where *bija* means *element* or *analysis* and *ganita* stands for the *science of calculation*. As early as 860 A.D., Pṛthūdaka Svāmi used this epithet for algebra in his commentary. Brahmagupta calls algebra as *Kuṭṭakaganita* or merely *kuṭṭaka*, a term which was later on used for "*pulveriser*" which deals with that special section of algebra which is connected with indeterminate equations of the first degree. Algebra is often also known as *avyakta-ganita* or the calculations with *unknowns*, in contrast to arithmetic which was known as *vyakta-ganita* or the calculations with *knowns*.

Algebra goes to Europe from India

In the history of mathematical sciences, as Colebrooke rightly remarks, it has long been a question to whom the invention of algebraic analysis is due. There is no doubt that Europe got algebra from Arabs mediately or immediately. But the Arabs themselves scarcely pretend to the discovery of algebra. Colebrooke says that they were not in general inventors but scholars during the short period of their successful culture of the sciences; and the germ at least of the algebraic analysis is to be found among the Greeks in an age not precisely determined, but more than probably anterior to the earliest dawn of civilisation among the Arabs; and this science in a more advanced state subsisted among the Hindus prior to the earliest disclosure of it by the Arabians to modern Europe. (Colebrooke: *Dissertation on the Algebra of the Hindus*)¹.

Colebrooke based his observations on the texts he could procure for his studies. These were : Bhāskara II's *Bijaganita* or *Vijaganita* (1150 A.D.) and *Līlāvati* (1150 A.D.), the *Gaṇitādhyāya* and *Kuṭṭakādhyāya* of Brahmagupta in his famous treatise the *Brahma Siddhānta* or rather the *Brahmasphuṭasiddhānta* (628

1. Colebrooke, H. T., *Miscellaneous Essays*, Vol. II, 1872, p. 418.

A.D.). There can be no doubt regarding the age of these two authors. Bhāskara II completed his great work on the *Siddhanta-siromani* in 1072 Śaka, and *Karaṇa-kutūhala* a practical astronomical treatise in 1105 Śaka; these dates are based on the passages given by Bhāskara himself in his works. The *Bija-ganita* and the *Lilāvati* form parts of the great treatise, the *Siddhanta-siromani*. The genuineness of the text is established, as Colebrooke says, with no less certainty by numerous commentators in Sanskrit, besides a Persian version of it. Those commentaries comprise a perpetual gloss, in which every passage of the original is noticed and interpreted : and every word of it is repeated and explained. From comparison and collation of various texts, it appears then that the work of Bhāskara, exhibiting the same uniform text which the modern transcripts of it do, was in the hands of both Muhammedans and Hindus, between two or three centuries ago : and numerous copies of it having been diffused throughout India, at an earlier period, as of a performance held in high estimation, It was the subject of study and habitual reference in countries and places so remote from each other as the north and west of India and the Southern Peninsula.

This though not marking any extraordinary antiquity, nor approaching to that of the author himself, was a material point to be determined : as there will be in the sequel, so says Colebrooke, occasion to show, that modes of analysis, and in particular, general methods for the solution of indeterminate problems both of the first and second degrees, are taught in the *Bija-ganita*, and those for the first degrees repeated in the *Lilāvati*, which were unknown to the mathematicians of the West, until invented anew in the last two centuries by algebraists of France and England.¹ Bhāskara who himself flourished more than six hundred and fifty years ago, was in this respect a compiler and took those methods from Indian authors as much more ancient than himself.

Regarding the age of the precursors of Bhāskara II, Colebrooke says : The age of his precursors cannot be determined with equal precision. He then proceeds to examine the evidence as follows :

¹ Colebrooke, H. T., *Miscellaneous Essays*, p. 421.

Towards the close of his treatise on Algebra, Bhāskara II informs us, that it is compiled and abridged from the more diffuse works on the same subject, bearing the names *Brāhme* (meaning no doubt Brahmagupta), Śrīdhara and Padmanābha; and in the body of his treatise, he has cited a passage of Śrīdhara's algebra and another of Padmanābha. He repeatedly adverts to preceding writers and refers to them in general terms, where his commentators understand him to allude to Āryabhaṭa, to Brahmagupta to the latter's scholiast Caturveda *Prthūdaka Svāmī* and to the other writers above mentioned.

Most, if not all, of the treatises, to which he thus alludes, must have been extant, and in the hands of his commentators, when they wrote; as appears from their quotations of them; more especially those of Brahmagupta and Āryabhaṭa, who are cited, and particularly the first mentioned, in several instances. A long and diligent research in various parts of India, has, however, failed of recovering any part of the *Padmanābha Bija* (or the algebra of Padmanābha) and of the algebraic and other works of Āryabhaṭa.

But the translator has been more fortunate in regard to the works of Śrīdhara and Brahmagupta, having in his collection Śrīdhara's compendium of arithmetic, and a copy incomplete however, of the text and scholia of Brahmagupta's *Brahmasiddhānta* comprising among other no less interesting matter, a chapter treating of arithmetic and mensuration; and another, the subject of which is algebra: both of them fortunately complete. The commentary is a perpetual one; successively quoting in length each verse of the text; proceeding to the interpretation of it, word by word; and subjoining elucidations and remarks; and its colophon, at the close of each chapter, gives the title of the work and the name of the author. Now the name which is there given, Caturveda *Prthūdaka Svāmī*, is that of a celebrated scholiast of Brahmagupta, frequently cited as such by the commentaries of

Bhāskara and by other astronomical writers; and the title of the work, *Brahmasiddhānta* or sometimes *Brahmasphuṭasiddhānta*, corresponds, in the shorter form, to the known title of Brahmagupta's treatise in the usual references to it by Bhāskara's commentators, and answers, in the longer form, to the designation of it, as indicated in an introductory couplet which is quoted from Brahmagupta by Lakṣmīdāsa, a scholiast of Bhāskara II. Remarking this coincidence, the translator proceeded to collate, with the text and commentary, numerous quotations from both, which he found in Bhāskara's writings or in those of his expositors. The result confirmed the indication and established the identity of both text and scholia as Brahmagupta's treatise, and the gloss of Pṛthūdaka. The authenticity of the *Brahmasiddhānta* is further confirmed by numerous quotations in the commentary of Bhaṭṭotpala on the *Samhitā* of Varāhamihira : as the quotations from the *Brahmasiddhānta*, in that commentary, (which is the work of an author who flourished eight hundred and fifty years ago) are verified in the copy under consideration. A few instances of both will suffice, and cannot fail to produce conviction.

It is confidently concluded, that the chapters on arithmetic and algebra, fortunately entire in a copy in many parts imperfect, of Brahmagupta's celebrated work as here described, are genuine and authentic. It remains to investigate the age of the author.

Mr. Davis, who first opined to the public a correct view of the astronomical computations of the Hindus, is of opinion, that Brahmagupta lived in the seventh century of the Christian era. Dr. William Hunter, who resided for some time with a British Embassy at Ujjayinī, and made diligent researches into the remains of Indian science at that ancient seat of Hindu astronomical knowledge, was there furnished, by the learned astronomers whom he consulted, with the ages of the principal ancient authorities. They assigned to Brahmagupta the date of 550 Saka; which answers to A.D.

628. The grounds on which they proceeded are unfortunately not specified: but as they gave Bhāskara's age correctly, as well as several other dates right, which admit of being verified; it is presumed that they had grounds, though unexplained, for the information which they communicated.

Mr. Bentley, who is little disposed to favour the antiquity of an Indian astronomer, has given his reasons for considering the astronomical system which Brahmagupta teaches, to be between twelve and thirteen hundred years old (1263 years in A.D. 1799). Now as the system taught by this author is professedly one corrected and adapted by him to conform with the observed positions of the celestial objects when he wrote, the age, when their positions would be conformable with the results of computations made as by him directed, is precisely the age of the author himself: and so far as Mr. Bentley's calculations may be considered to approximate the truth, the date of Brahmagupta's performance is determined with like approach to exactness, within a certain latitude however of uncertainty for allowance to be made on account of the inaccuracy of Hindu observations.

The translator has assigned on former occasions the grounds upon which he sees reason to place the author's age, soon after the period when the vernal equinox coincided with the beginning of the lunar mansion and zodiacal asterism *Āśvini*, where the Hindu ecliptic now commences. He is supported in it by the sentiments of Bhāskara and other Indian astronomers, who infer from Brahmagupta's doctrine concerning the solstitial points, of which he does not admit a periodical motion, that he lived when the equinoxes did not, sensibly to him, deviate from the beginning of *Āśvini* and middle of *citra* on the Hindu sphere. On these grounds it is maintained, that Brahmagupta is rightly placed in the sixth or beginning of the seventh century of the Christian era, as the subjoined calculations will more particularly show. The age when Brahmagupta

flourished, seems then, from the concurrence of all these arguments, to be satisfactorily settled as antecedent to the earliest dawn of the culture of sciences among the Arabs; and *consequently establishes* the fact, that *the Hindus were in possession of algebra before* it was known to the Arabians.

Brahmagupta's treatise, however, is not the earliest work known to have been written on the same subject by an Indian author. The most eminent scholiast of Bhāskara II (Gaṇeśa) quotes a passage of Āryabhaṭa specifying algebra under the designation of *Bija*, and making separate mention of *Kuṭṭaka*, which more particularly intends a problem subservient to the general method of resolution of indeterminate problems of the first degree : he is understood by another of Bhāskara's commentators to be at the head of the elder writers, to whom the text then under consideration adverts, as having designated by the name of *Madhyamāharaṇa* the resolution of affected quadratic equations by means of the completion of the square. It is to be presumed, therefore, that the treatise of Āryabhaṭa then extant, did extend to quadratic equations in the determinate analysis, and to indeterminate problems of the first degree; if not to those of the second likewise, as most probably it did.

This ancient astronomer and algebraist, so says Colebrooke, was anterior to both Varāhamihira and Brahmagupta, being repeatedly named by the latter; and the determination of the age when he flourished is particularly interesting as his astronomical system, though on some points agreeing, essentially disagreed on others, with that which the Hindu astronomers still maintain.

He, as Colebrooke says, is considered by the commentators of the *Śūryasiddhānta* and *Śiromaṇi*, as the earliest of uninspired and mere human writers on the science of astronomy, as having introduced requisite corrections into the system of Parāśara, from whom he took the numbers for the planetary mean motions; as

having been followed in the tract of emendation, after a sufficient interval to make further correction requisite, by Durgāsinha and Mihira; who were again succeeded after a further interval by Brahmagupta, son of Jīṇu.

In short, says Colebrooke, Āryabhaṭa was founder of one of the sets of Indian astronomers, as Puliśa, an author likewise anterior to both Varāhamihira and Brahmagupta, was of another : which were distinguished by names derived from the discriminative tenets respecting the commencement of planetary motions at sunrise according to the first, but at midnight according to the latter, on the meridian of Lāṅkā, at the beginning of the great astronomical cycle. A third sect began the astronomical day, as well as the great period, at noon.

Āryabhaṭa's name accompanied the intimation which the Arab astronomers (under the Abbasside Khalifs, as it would appear,) received, that three distinct astronomical systems were current among the Hindus of those days : and it is but slightly corrupted, certainly not at all disguised, in the Arabic representation of it *Arjabahar*, or rather *Arjabhar*, (corrupted form of Āryabhaṭa). The two other systems were, first, Brahmagupta's *Siddhānta* which was the one they became best acquainted with, and to which they apply the denomination of the *sind-hind*; and second, that of *Arca*, the Sun, which they write *Arcand* a corruption still prevalent in the vulgar Hindi.

Āryabhaṭa appears to have had more correct notions of the true explanation of celestial phenomena than Brahmagupta himself, so says Colebrooke; who in a few instances, correcting errors of his predecessor, but oftener deviating from that predecessor's juster views, has been followed by the herd of modern Hindu astronomers, in a system not improved, but deteriorated, since the time of the more ancient author.

Considering the proficiency of Āryabhaṭa in astronomical science, and adverting to the fact of his having

written algebra, as well as to the circumstance of his being named by numerous writers as the founder of a sect, or author of a system in astronomy, and being quoted at the head of algebraists, when the commentators of extant treatises have occasion to mention early and original writers on this branch of science, it is not necessary to seek further for a mathematician qualified to have been the great improver of the analytic art, and likely to have been the person by whom it was carried to the pitch to which it is found to have attained among the Hindus, and at which it is observed to be nearly stationary through the long lapse of ages which have since passed : the later additions being few and unessential in the writings of Brahmagupta, of Bhāskara and of Jñānarāja, though they lived at intervals of centuries from each other.

Āryabhaṭa, Colebrooke rightly says, then being the earliest author known to have treated of Algebra among the Hindus, and being likely to be, if not the inventor, the improver of that analysis, by whom too it was pushed nearly to the whole degree of excellence which it is found to have attained among them; it becomes in an especial manner interesting to investigate any discoverable trace in the absence of better and more direct evidence, which may tend to fix the date of his labours; or to indicate the time which elapsed between him and Brahmagupta, whose age is more accurately determined.

Taking Āryabhaṭa, for reasons given, to have preceded Brahmagupta and Varāhamihira by several centuries; and Brahmagupta to have flourished more than twelve hundred years ago, and Varāhamihira, concerning whose works and age, Colebrooke has given a few notes, and has placed him at the beginning of the sixth century after Christ, it appears probable that this earliest of known Hindu algebraists wrote as far back as the fifth century of the Christian era; and perhaps in an earlier age. Hence it is concluded that he is nearly as ancient as the Grecian algebraist Diophantus, sup-

posed on the authority of Abulfaraj, to have flourished in the time of the Emperor Julian or about A. D. 360.

Colebrooke further says : Admitting the Hindu and Alexandrian authors to be nearly equally ancient, it must be conceded in favour of the Indian algebraist, that he was more advanced in the science; since he appears to have been in possession of the resolution of equations involving several unknowns, which it is not clear, nor fairly presumable, that Diophantus, knew; and a general method of indeterminate problems of at least the first degree, to a knowledge of which the Grecian algebraist had certainly not attained; though he displays infinite sagacity and ingenuity in particular solutions; and though a certain routine is indiscernible in them.

Colebrooke appears to be of the view that Greeks were the first to discover the solution of equations involving one unknown; and this knowledge was passed to ancient Indians by their Greek instructors in improved astronomy. But "by the ingenuity of the Hindu-scholars, the hint was rendered fruitful and the algebraic method was soon ripened from that slender beginning to the advanced state of a well arranged science, as it was taught by Āryabhata, and as it is found in treatises compiled by Brahmagupta and Bhāskara."

We do not agree with this analysis in entirety. Indian algebra is entirely of Indian roots. It had its beginning in the times of Samhitās and Brāhmaṇas. Some of the equations and problems were solved by geometric methods. It must have had its origin in the Śulba period if not before. Āryabhata undoubtedly was the discoverer of many algebraic solutions of equations of the first and higher order with one and more unknowns. It is rather too much to trace the influence of Diophantus on Indian algebra which developed in this country independently. Brahmagupta is one of the most brilliant algebraists we ever had in the entire history of mathematics.

Technical Terms

Coefficient—

In the ancient Indian algebra, there is no systematic term for the coefficient. Usually, the power of the unknown is mentioned when the reference is to the coefficient of that power. At one place, for example, we find Pṛthūdaka Svāmī (the commentator of Brahmagupta's *Brāhmasphuṭasiddhānta*) writing "the number (*aṅka*) which is the coefficient of the square of the unknown is called the 'square' and the number which forms the coefficient of the (simple) unknown is called the 'unknown quantity' (*avyakta-māna*)."¹ However, at many places, we find the use of a technical term also. Brahmagupta once calls the coefficient *saṃkhyā*² (number) and on several other occasions *guṇaka*³ or *guṇakāra*⁴ (multiplier). Pṛthūdaka Svāmī (860 A.D.) calls it *aṅka* (number) or *prakṛti* (multiplier). These terms may also be seen in the works of Śrīpati⁵ (1039) and Bhāskara II⁷ (1150 A.D.). The former also used the word *rūpa* for the same purpose.⁸

Unknown Quantity

The unknown quantity has been termed as *yāvat-tāvat* (meaning so-much-as or as-many-as) in literature as early as 300 B. C. (vide the *Sthānāṅga-sūtra*⁹). In the *Bakhasālī Manuscript*, it has been termed as *yadycchā*, *vāñchā* or *kāmikā* (or any desired quantity)¹⁰. Āryabhaṭa I in one of his verses calls the unknown as *gulikā*¹¹ (literally meaning a shot) From the early seventh century A.D., the word *avyakta* was used for unknown quantities. Brahmagupta uses this term in his *Brāhmasphuṭasiddhānta*¹²

1. BrSpSi. XVIII. 44 (Com.)

2. वर्षप्रमाण भावितघातो भवतीष्ट वर्ष संख्यैवम् ।

—BrSpSi. XVIII 63

3. मूलं द्विषेष्ट वर्गाद् गुणक गुणादिष्ट युत विहीनाच्च ।

—BrSpSi. XVIII 64

वर्गच्छिन्ने गुणके प्रथमं तन्मूल भाजितं भवति ।

—BrSpSi. XVIII 70

4. प्रथमोन्यमूलमन्यो गुणकार पदोद्धृतः प्रथमः ।

—BrSpSi. XVIII 69

5. BrSpSi. XVIII. 44 (Com.)

6. Siśe XIV. 33-5.

7. Bijaganita

8. Siśe XIV. 33-5.

9. Sūtra 747.

10. BMs. Folio 22, verso; 23, recto and verso.

11. गुलिकान्तरेण विभजेद् द्वयोः ।

12. अन्यक्तव्यं धनवर्गं वर्गपङ्क्तं षड्गतादीनाम् ।

Since long, the word *varga* has been used for the second power; the word also stands for *square* (*Uttarādhyayana Sūtra*¹, B. C. c. 300). The third power is similarly known as *ghana*, the fourth power as *varga-varga* (square-square), the sixth power as *ghana-varga* (cube-square) and the twelfth power as *ghana-varga-varga* (cube-square-square). In later days, the fifth power was called *vargaghana ghāta* (here the word *ghāta* means product; the term means product of cube and square). The former system was multiplicative, rather than additive; whereas the latter was on the additive system. The seventh power on the additive system was known as *varga-varga-ghana-ghāta* (product of square-square and cube). Brahmagupta, however, uses a more scientific system for expressing the powers more than four. He calls the fifth power as *Pañca gata* (literally meaning, raised to the fifth), the sixth power as *ṣaḍ-gat* (raised to the sixth) and so on, thus adding the suffix *gata* to the name of the number indicating that power.² Bhāskara II has followed the system of Brahmagupta almost consistently for powers one and upwards.

Perhaps Brahmagupta has for the first time used the term *samakarāṇa* or *samīkarāṇa* (literally meaning making equal) or simply *sama* (equal or equation)³. Pṛthudaka Svāmī (860) employs the term *sāmya* (equality or equation) for equation⁴. The equation is said to possess two *Pakṣas*⁵ (sides) *Itara-pakṣa* and *apara-pakṣa*.

Brahmagupta uses the term *rūpa* (literally meaning appearance) for an absolute term. It represents the visible or known

1. Chapter XXX, 10, 11.
2. अश्वक्त्वर्गं वनवर्गं-वर्गं-यंचगत-वड्गतादीनाम् ।
सदृश दिवधो वर्गस्यादि वधस्तद्गतोऽन्य जातिवधः ॥
3. वर्णं प्रमाणं भावित-वातो भवतीष्टवर्णं संख्यैवम् ।
सिध्यति विनाऽपि भावित-उमकरणात् किं कृतं तदतः ॥
अश्वक्तात्तरं भक्तं व्यस्तं रूपान्तरं समेऽश्वक्तः ।
4. *Side*. XIV. 19.
5. *Bhāṣanī*.

portion of the equation whilst its other part is practically invisible or unknown¹.

Unknowns and Symbolism

Āryabhaṭa I (499 A. D.) probably used coloured *gulikas* or shots for representing different unknowns. Brahmagupta mentions *varṇa* as the symbols for unknown. He has, however, not indicated how these *varṇas* or colours were used as symbols for unknowns. Perhaps we might conclude from this that the method of using colours as symbols for unknown quantities was very common and familiar to the algebraists. Datta and Singh say that the Sanskrit word *varṇa* denotes colour as well as a letter of alphabet, and therefore, letters of alphabet came into use for unknown quantities: *kālaka* (black), *nilaka* (blue), *pūtaka* (yellow), *lohita* (red), *haritaka* (green), *śvetaka* (white), *citraka* (variegated), *kapilaka* (tawny), *piṅgalaka* (reddish-brown), *dhūmraka* (smoke-coloured), *pāṭalaka* (pink), *śaralaka* (spotted), *śyāmalaka* (blackish), *mecaka* (dark blue) etc².

It should be further noted that the first unknown quantity *yāvat-tāvat* is not a *varṇa* or colour. It thus clearly indicates that the use of colours as symbols came at a later stage, whilst the word *yāvat-tāvat* was in currency from much earlier times. Some authorities think that the term *yāvat-tāvat* is a corrupted form of *yāvakaśtāvat* (where *yāvaka* means red). Pṛthūdaka Svāmī has sometimes used the term *yavaka* for an unknown quantity³.

Laws of Signs

Brahmagupta has in his Chapter XVIII devoted a special section entitled "*Dhanarṇa Śūnyānam Samkalanam*" or calculations dealing with quantities bearing positive and negative signs and zero,

1. अव्यक्तान्तरं भक्तं व्यस्तं रूपान्तरं समेऽव्यक्तः । —BrSpSi XVIII. 43
वर्गं चतुर्गुणितानां रूपाणां मध्यर्कं सद्वितानाम् ॥ —BrSpSi XVIII. 44
2. यावत्-तावत् कालको नीलकोऽन्यो वर्णः पीतो लोहितश्चैतदाद्याः ।
अव्यक्तानां कल्पिता मानसंज्ञास्तत्संख्यानं कर्तुं भावार्थवर्धैः ॥
यावत्-तावत् कालक नीलक पीताश्च लोहितो हरितः ।
श्वेतक चित्रक कपिलक पाटलकाः पण्डु धूम्र शक्लाश्च ॥
श्यामलक-मेचक-धवलक-पिशङ्ग-शारङ्ग-वज्र-गौराद्याः । —Nārāyaṇa, *Bijaganita*
3. BrSpSi XII. 15, (Com.) ; XII 18.

Regarding *addition*, Brahmagupta says :

The sum of two positive numbers is positive, of two negative numbers is negative; of a positive and negative number is the difference¹.

Regarding *subtraction*, Brahmagupta further says :

From the greater should be subtracted the smaller; (the final result is) positive, if positive from positive, and negative, if negative, from negative. If, however, the greater is subtracted from the less, that difference is reversed (in sign), negative becomes positive and positive becomes negative. When positive is to be subtracted from negative or negative from positive, then they must be added together².

Mahāvira (850 A. D.), Bhāskara II (1150 A.D.) and Nārāyaṇa (1350 A. D.) have also given similar rules regarding addition (*Samkalanam*) and the subtraction (*vyavakalanam*).

Again, the rule given by Brahmagupta regarding *Multiplication* is as follows :

The product of a positive and a negative (number) is negative; of two negatives is positive; positive multiplied by positive is positive³.

His rule regarding *division* is as follows ;

Positive divided by positive or negative divided by negative becomes positive. But positive divided by negative and negative divided by positive remains negative⁴.

Similar rules for multiplication and division were provided by later authorities as Mahāvira and Bhāskara II.

1. *BrSpSi XII. 15. (Com.); XII 18. (Com.)*
2. धनयोर्वैनमृणमृणयोर्वैनर्णं योरन्तरं समैवयं खम् ।
ऋणमैवयं च धनमृणधन शून्ययोः शून्ययोः शून्यम् ॥ —*BrSpSi. XVIII. 30*
3. ऊनमधिकादिशोध्यं धनं धनादृणमृणादधिकमूनात् ।
व्यस्तं तदन्तरं स्यादृणं धनं धनमृणं भवति ॥
शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम् ।
शोध्यं यदा धनमृणादृणं धनाद्वा तदा क्षेप्यम् ॥ —*BrSpSi XVIII. 31-32*
4. ऋणमृणधनयोर्धातो धनमृणयोर्वैनवधो धनं भवति ।
शून्यर्णयोः ख धनयोः ख शून्ययोर्वा वधः शून्यम् ॥ —*BrSpSi XVIII. 33*
5. धनभक्तं धनमृणहृतमृणं धनं भवति खं खभक्तं खम् ।
भक्तमृणेन धनमृणं धनेन हृतमृणमृणं भवति ॥ —*BrSpSi. XVIII. 34*

Brahmagupta lays down the rules regarding *evolution* and *involution* as follows :

The square of a positive or a negative number is positive
.....The (sign of the) root is the same, as was that from
which the square was derived¹.

As regards the latter portion of this rule, Pṛthūdaka Svāmī has the following comment to make : "The square-root should be taken either negative or positive, as will be most suitable for subsequent operations to be carried on."

It will be interesting to observe the following observation of Mahāvīra (850 A. D.) regarding square-root of a negative quantity "Since a negative number by its own nature is not a square, it has no square-root."² So says Śrīpati : "A negative number by itself is non-square, so its square-root is unreal; so the rule (for the square-root) should be applied in the case of a positive number."³

Algebraic Operations

Brahmagupta and other algebraists recognise six operations as fundamental in algebra : addition, subtraction, multiplication, division, squaring and the extraction of the square-root.

Regarding *addition* and *subtraction* Brahmagupta says :

Of the unknowns, their squares, cubes, fourth powers, fifth powers, sixth powers, etc., addition and subtraction are (performed) of the like; of the unlike (they mean simply their) statement severally.⁴

In place of "of the like", Bhāskara II uses the term "of those of the same species (*jāti*) amongst unknowns" :

Addition and subtraction are performed of those of the same species (*jāti*) amongst unknowns; of different species they mean their separate statement.⁵

1. खोद्घृत मृणं धनं वा तच्छेदं खमृणधनविभक्तं वा ।

ऋणधनयोर्वर्गः स्वं स्वं स्वस्य पदं कृतिर्यत् तत् ॥

—BrSpSi, XVIII. 35

2. GSS, I, 52.

3. Śīśe, XIV, 5.

4. अव्यक्तवर्गं धनवर्गं वर्गं पंचगतं षड्गतादीनाम् ।

तुल्यानां संकलितं व्यवकलिते पृथगतुल्यानाम् ॥

—BrSpSi. XVIII. 41.

5. योगोऽन्तरं तेषु समानं जाल्योर्विभिन्नं जाल्योश्च पृथक् स्थितिश्च ।

—Bhāskara II, Bijaganita.

This means that the numerical coefficients of x cannot be added to or subtracted from the numerical coefficients of y or x^2 or x^3 or xy and so on because these terms belong to different *jāti* or they do not belong to the category of the "like".

Again, regarding *multiplication*, Brahmagupta says :

The product of the two like unknowns is a square; the product of three or more like unknowns is a power of that designation. The multiplication of unknowns of unlike species is the same as the mutual product of symbols; it is called *bhāvita* (product or factum).¹

Having given the rules of the operations for addition, subtraction and multiplication, Brahmagupta does not think it necessary to deal with other operations. His section on the calculations with zero, negative and positive quantities ends here.

How is an Equation Formed?

Prthūdaka Svāmī while commenting on a verse in *Brāhma-sphuṭasiddhānta* speaks as follows :

In this case, in the problem proposed by the questioner, *yāvat-tāvat* is put for the value of the unknown quantity. Then performing multiplication, division etc. as required in the problem the two sides shall be carefully made equal. The equation being formed in this way, then the rule (for its solution) follows.²

Plan for Writing Equations

When in regards to a given problem, an equation has been formed, it has to be written down for further operations. This writing down of an equation is technically known as *nyāsa*. Perhaps the oldest record of *nyāsa* is to be found in the *Bakhaṣālī Manuscript*. According to the procedure prescribed in this work, the two sides of an equation are put down one after the other in the same line without any sign of equality being interposed. Thus the equations :

$$\sqrt{x+5}=s, \sqrt{x-7}=t$$

appear as

1. सदृशदिवधो वर्गस्यादिवधस्तद् गतोऽन्यजातिवधः ।

अन्योऽन्यवर्णधातो भावितकः पूर्ववच्छेषम् ॥

2. *BrSpSi*. XVIII. 43 (com)

—*BrSpSi*. XVIII. 42.

$$\begin{array}{|c|c|c|c|c|} \hline 0 & 5 & yu & m\bar{u} & 0 \\ \hline 1 & 1 & & & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline sa & 0 & 7 \\ \hline 1 & 1 & \\ \hline \end{array} + \begin{array}{|c|} \hline m\bar{u} \\ \hline 0 \\ \hline \end{array}$$

Here *yu* (यु) stands for *yuta* (युत), meaning added, subtraction is + sign, derived from *Kṣaya* or (क्षय) meaning diminished, *gu* (गु) for *guṇa* or *guṇita*, meaning multiplied; *bhā* (भा) for division from *bhājita* and *mū* (मू) for square-root, from *mūla* meaning root; zero (०) was used to mark a vacant place.

Again, the following equation

$$x+2x+3 \times 3x+12 \times 4x=300$$

is represented as ;

$$\begin{array}{|c|c|c|c|c|c|} \hline 0 & 2 & 1 & 3 & 12 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \quad dṛśya \ 300$$

There is no sign for unknown in the *Bakhaśālī Manuscript*.

Later on this plan of writing equations as adopted in *Bakhaśālī Manuscript* was abandoned in India; a new one was adopted in which the two sides are written one below the other without any sign of equality. It must be stated that in this new plan the term of similar denominations are usually written one below the other and even the terms of absent denominations on either side are clearly indicated by putting zeros as their coefficients. We find a reference to this new plan in the algebra of Brahmagupta.

From which the square of the unknown and the unknown are cleared, the known quantities are cleared (from the side) below that¹.

Here in this verse, the words “*adhastāt*” clearly indicate that one side of the equation is written below the other., As an illustration, Pṛthūdaka Svāmī represented the equation² :—

$$10x-8=x^2+1$$

as :

$$y\dot{a} \ va \ 0 \ y\dot{a} \ 10 \ r\bar{u} \ \dot{8} \quad (x^2.0+x.10-8)$$

$$y\dot{a} \ va \ 1 \ y\dot{a} \ 0 \ r\bar{u} \ 1 \quad (x^2.1+x.0+1)$$

which means, x^2 was written as *yāvat-varga* (*yā va*) and x was written as *yāvat* or *yā*. The minus sign was represented by a dot at the top of the number. (—8 was written as $\dot{8}$). We shall take another illustration from Pṛthūdaka Svāmī.

He would write the equation

$$197x-1644 \ y-z=6302$$

as

1. *BrSpSi*. XVII. 43; compare also Bhāskara II, *Bijaganita*.

2. *BrSpSi*. XVIII. 49 (*com.*)

$$\begin{array}{rccccccc} y\bar{a} & 197 & k\bar{a} & 1644 & n\bar{i} & \dot{1} & r\bar{u} & 0 \\ y\bar{a} & 0 & k\bar{a} & 0 & n\bar{i} & 0 & r\bar{u} & 6302 \end{array}$$

Here the first unknown x is represented by $y\bar{a}(vat)$, the second unknown y by $k\bar{a}(laka)$ and the third unknown z by $n\bar{i}$ ($laka$) and the term without unknown, a mere number is written by $r\bar{u}(paka)$. The two sides, one written below the other if written in the present form, would appear as :

$$197x - 1644y - z + 0 = 0x + 0y + 0z + 6302.$$

The *Bijaganita* of Bhāskara II also follows the same procedure. One instance from it would be quoted here to illustrate the method of expressing equations.

$$8x^3 + 4x^2 + 10y^2x = 4x^3 + 12y^2x$$

$$\text{or } 8x^3 + 4x^2 + 10y^2x = 4x^3 + 0x^2 + 12y^2x$$

is written as follows on Bhāskara's or Brahmagupta's plan :

x^3 is *ghana* of $y\bar{a}vat$ (abbreviated as $y\bar{a} gha$)

x^2 is *varga* of $y\bar{a}vat$ (abbreviated as $y\bar{a} va$)

y^2 is *varga* of $k\bar{a}laka$ (abbreviated as $k\bar{a} va$)

the coefficients 10 and 12 are *bhāvita* (abbreviated as *bhā*).

The equation is :

$$y\bar{a} gha \ 8 \ y\bar{a} va \ 4 \ k\bar{a} va \ y\bar{a}. \ bh\bar{a} \ 10$$

$$y\bar{a} gha \ 4 \ y\bar{a} va \ 0 \ k\bar{a} va \ y\bar{a}. \ bh\bar{a} \ 12$$

Datta and Singh state that the use of the old plan of writing equations is sometimes met with in later works also. For instance, in the MS. of Pṛthūdaka Svāmī's commentary¹ on the *Brahmasphuṭasiddhānta*, an incomplete copy of which is preserved in the library of the Asiatic Society of Bengal (No. I B6), we find a statement of equations thus : "first side $y\bar{a}vargaḥ \ 1 \ y\bar{a}vakāḥ \ 200 \ r\bar{u} \ 0$; second side $y\bar{a}vargaḥ \ 0 \ y\bar{a}vakāḥ \ 0 \ r\bar{u} \ 1500$.

Śodhana or Clearance of an Equation

After *nyāsa* or statement of an equation, the operation to be performed is known as *śodhana* (clearance) or *samśodhana* (equi-clearance or complete clearance). The nature of this clearance varies according to the kind of equation. In the case of an equation in one unknown only, whether linear,

1. *BrSpSi*. XII. 15 (*com.*)

quadratic or of higher powers, one side of it is cleared of the unknowns of all denominations and the other side of it of the absolute terms, so that the equation is ultimately reduced to one of the form

$$ax^2+bx=c,$$

where a, b, c may be positive or negative; some of them may even be zero. Thus Brahmagupta observes :

From which the square of the unknown and the unknown are cleared, the known quantities (*rūpāṇi*) are cleared (from the side) *below* that.¹

On this *Prthūdaka Svāmi* comments as follows :

This rule has been introduced for that case in which the two sides of the equation having been formed in accordance with the statement of the problem, there are present the square and other powers of the unknown together with the (simple) unknown. The absolute terms should be cleared off from the side opposite to that from which are cleared the square (and other powers) of the unknown and the (simple) unknown. When perfect clearance (*samsodhana*) has been thus made...²

Śrīdhara and *Bhāskara II* have also given the rules of clearance almost on the same lines. Thus the equation

$$yā\ va\ 0\ yā\ 10\ rā\ 8$$

$$yā\ va\ 1\ yā\ 0\ rā\ 1$$

after perfect clearance having been made will be (according to *Prthūdaka Svāmi*)

$$yā\ va\ 1\ yā\ 10\ rā\ 9$$

i.e. the equation $10x-8=x^2+1$

after clearance would become

$$x^2-10x=-9.$$

Classification of Equations

Usually equations are classified as :

simple equation : *yāvat-tāvat*

quadratic : *varga*

1. कर्माव्यवृत्ताः शोभ्या यस्माद् रूपाणि तदधस्तात् ॥

2. *BrSpSi*. XVIII. 43 (*com.*)

cubic : *ghana*
 biquadratic : *varga-varga*

Brahmagupta classified them as

- (i) equations in one unknown quantity : *eka-varṇa samīkaraṇa*.
- (ii) equations in several unknowns : *aneka-varṇa samīkaraṇa*.
- (iii) equations involving products of unknowns : *bhāvita*.

Eka-varṇa samīkaraṇas (equations with one unknown) are further divided into (i) linear equations, and (ii) quadratic equations (*avyakta-varga samīkaraṇa*).

Prthūdaka Svāmī has classified equations in a different manner as follows :

- (i) linear equations with one unknown : *eka-varṇa samīkaraṇa*.
- (ii) linear equations with more unknowns : *aneka-varṇa samīkaraṇa*.
- (iii) equations with one, two or more unknowns in their second or higher powers : *madhyamāharaṇa*.
- (iv) equations involving products of unknowns : *bhāvita*.

As the method of solution of an equation of the third class (i.e. equations with one or several unknowns in their second or higher powers) is based upon the principle of the elimination of the middle term, that class is called by the term *madhyama* (middle) *āharaṇa* (elimination). The classification of Brahmagupta and Prthūdaka Svāmī more or less received recognition by later writers on algebra as Bhāskara II and others.

Linear Equations with One Unknown and Their Solutions

The first solution of a linear equation with one unknown is obtainable in the *Śulba Sūtras* but not through an algebraic process,—the *Śulba* process is geometrical. It is said that there is a reference in the *Sthānāṅga Sūtra* (c. 300 B.C.) to a linear equation by its name *yāvat-tāvat*. There has been a good deal of

controversy regarding the date of the *Bakhaṣālī Manuscript* where we have definitely a method of solving linear equations by the *Rule of False Position*. It would be interesting to give an account of this rule here by taking an illustration from the *Bakhaṣālī Manuscript*.

Problem :

The amount given to the first is not known. The second is given twice as much as the first; the third thrice as much as the second; and the fourth four times as much as the third. The total amount distributed is 132. What is the amount of the first ?

(BMS. Folio 23, recto)

In modern algebraic language, the solution of the problem would be given by the equation

$$x + 2x + 6x + 24x = 132$$

where x is the amount given to the first.

The solution of this equation is given as follows in the *Bakhaṣālī Manuscript* :

'Putting any desired quantity in the vacant place'; any desired quantity is $\parallel 1 \parallel$, 'then construct the series'

$$\begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 6 & 4 & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & \\ \hline \end{array}$$

'multiplied' $\parallel 1 \parallel 2 \parallel 6 \parallel 24 \parallel$; 'added' 33. 'Divide the visible quantity' $\parallel \begin{array}{c} 132 \\ 33 \end{array} \parallel$; which) on reduction be-

comes $\parallel \begin{array}{c} 4 \\ 1 \end{array} \parallel$. (This is) the amount given (to the first)

(BMS. Folio 23, recto)

The Rule of False Position may be regarded as an early stage of the development of the science of algebra, since no symbol could have been evolved for an unknown quantity. As soon as the system of notations was introduced, the application of this Rule was no longer considered as necessary. Thus we find that Āryabhaṭa I does not mention of this Rule.

Āryabhaṭa I states as follows regarding the solution of linear equations :

The difference of the known 'amounts' (*rūpaka*) relating to two persons should be divided by the difference

of the coefficients on the unknown (*gulikā*). The quotient will be the value of the unknown (*gulikā*), if their possessions be equal.¹

The original verse contains the term "*gulikāntara*" which has been here translated as the difference of the coefficients of the unknowns. We have already stated earlier that Āryabhaṭa uses the term *gulikā* or shot for an unknown quantity. (*gulikāntara* literally means only the difference of unknowns). This practice is also followed by other Indian algebraists. Pṛthūdaka Svāmī rightly observed that according to the usual practice in this country, "the coefficient of the square of the unknown is called the square (of the unknown) and the coefficient of the (simple) unknown is called the unknown."²

The rule given by Āryabhaṭa, then, contemplates a problem of this kind :

Two persons, who are equally rich, possess respectively a, b times a certain unknown amount together with c, d units of money in cash. What is that amount ?

Now if x be the amount unknown, then according to the problem

$$ax + c = bx + d$$

Thence

$$x = \frac{d-c}{a-b}$$

Āryabhaṭa has merely expressed this solution in his language.

Regarding the solution of linear equations, Brahmagupta says :

In a (linear) equation in one unknown, the difference of the known terms taken in the reverse order, divided by the difference of the coefficients of the unknown (is the value of the unknown),³

1. गुलिकान्तरेण विभजेद् द्वयोः पुरुषयोस्तु रूपकविशेषम् ।
लब्धं गुणिकामूल्यं यद्यर्थं कृतं भवति तुल्यम् ॥

—Ārya. II. 30

2. BrSpSi, XVIII. 44 (com.)

3. अव्यक्तान्तरभक्तं व्यस्तं रूपान्तरं समेऽव्यक्तः ।
वर्गाव्यक्ताः शोभ्या यस्माद् रूपाणि तदधस्ताद् ॥

—BrSpSi. XVIII. 43

Similar solutions have been offered by the other Indian algebraists who followed Brahmagupta like Śrīpati, Bhāskara II and Nārāyaṇa. Here again, we take a problem proposed by Brahmagupta in this connection :

Problem :

Tell the number of elapsed days for the time when four times the twelfth part of the residual degrees increased by one, plus eight will be equal to the residual degrees plus one.¹

Prthūdaka Svāmī has solved this problem as follows :

Here the residual degrees are (put as) *yāvat-tāvat*, *yā*;
increased by one, *yā 1 rū 1*; twelfth part of it, $\frac{yā 1 rū 1}{12}$;
four times this, $\frac{yā 1 rū 1}{3}$; plus the absolute quantity
eight, $\frac{yā 1 rū 25}{3}$. This is equal to the residual degrees
plus unity. The statement of both sides tripled is

$$\begin{array}{rcl} yā 1 & rū 25 \\ yā 3 & rū 3 \end{array}$$

This difference between the coefficients of the unknown is 2. By this the difference of the absolute terms, namely 22, being divided, is produced the residual of the degrees of the Sun, 11. These residual degrees should be known to be irreducible. The elapsed days can be deduced then, (proceeding) as before.

If put in the modern notations, it means the solution of the equation :

$$\frac{4}{12} (x+1) + 8 = x + 1,$$

from which we have

$$x + 25 = 3x + 3$$

$$\text{or} \quad 2x = 22$$

$$\text{or} \quad x = 11.$$

Rule of Concurrence or Saṁkramaṇa

Brahmagupta has included this rule in algebra, whereas other Indian mathematicians included it in arithmetic. *Sam-*

1. सैकादशकरोषाद् द्वादशभागश्चतुर्गुणोऽष्टयुतः ।

सैकादशकरोषतुल्यो यदा तदाऽद्वयं कथय ॥

kramaṇa is the solution of the simultaneous equations of the type :

$$x+y=a$$

$$x-y=b$$

Brahmagupta's rule for solution is:

The sum is increased and diminished by the difference and divided by two; (the result will be the two unknown quantities): (this is) concurrence (*Samkramaṇa*).¹

Brahmagupta restates this rule in the form of a problem and its solution :

The sum and difference of the residues of two (heavenly bodies) are known in degrees and minutes. What are the residues? The difference is both added to and subtracted from the sum, and halved; (the results are) the residues.²

Linear Equations with Several Unknowns

The first mention of a solution of the problem with more than one unknown is found in the *Bakhaśālī Manuscript*, and a system of linear equations of this type is solved in the *Bakhaśālī* treatise substantially by the False Position Rule.

A generalised system of linear equations will be

$$b_1 \sum x - c_1 x_1 = a_1, b_2 \sum x - c_2 x_2 = a_2, \dots, b_n \sum x - c_n x_n = a_n$$

$$b_n \sum x - c_n x_n = a_n$$

Therefore

$$\sum x = \frac{\sum(a/c)}{\sum(b/c) - 1}$$

Hence

$$x_r = \frac{b_r}{c_r} \times \frac{\sum(a/c)}{\sum(b/c) - 1} - \frac{a_r}{c_r} \quad r=1, 2, 3, \dots, n$$

One particular case, where $b_1=b_2=b_3=\dots=b_n=1$ and $c_1=c_2=c_3=\dots=c_n=c$ has been treated by Brahmagupta at one place. He gives the rule as follows :

1. योगोऽन्तरयुतहीनो दिहृतः संक्रमणमन्तरविभक्तं वा ।

वर्गान्तरमन्तरयुतहीनं दिहृतं विषयकर्म ॥

— *BrSpSi. XVIII. 36*

2. भागकला विकलैक्यं दृष्ट्वा विकलान्तरं च के शेषे ।

ऐक्यं द्विधाऽन्तराधिक हीनं च द्विभाजितं शेषे ॥

— *BrSpSi. XI'VIII. 96*

The total value (of the unknown quantities) plus or minus the individual values (of the unknowns) multiplied by an optional number being severally (given), the sum (of the given quantities) divided by the number of unknowns increased or decreased by the multiplier will be the total value; thence the rest (can be determined).¹

$$\begin{aligned}\Sigma x \pm cx_1 &= a_1, \Sigma x \pm cx_2 = a_2, \Sigma x \pm cx_3 = a_3, \dots, \\ \Sigma x \pm cx &= a_n\end{aligned}$$

Therefore

$$\Sigma x = \frac{a_1 + a_2 + a_3 + \dots + a_n}{n \pm c}$$

Hence

$$x_1 = \frac{1}{c} \left(\pm a_1 \mp \frac{a_1 + a_2 + a_3 + \dots + a_n}{n \pm c} \right);$$

and so on for x_2, x_3 etc.

Now we shall give the rule enunciated by Brahmagupta for solving linear equations involving several unknowns :

Removing the other unknowns from (the side of) the first unknown and dividing by the coefficient of the first unknown, the value of the first unknown (is obtained). In the case of more (values of the first unknown), two and two (of them) should be considered after reducing them to comon denominators. And (so on) repeatedly. If more unknowns remain (in the final equation), the method of the pulveriser (should be employed). (Then proceeding) reversely (the values of other unknowns can be found).²

Prthūdaka Svāmī has commented on this rule as follows :

In an example in which there are two or more unknown quantities, colours such as *yāvat-tāvat*, etc., should be assumed for their values. Upon them should

1. गच्छधनमिष्ट गुणितैर्धनैर्मु तोनं पृथक् पृथक् सहितम् ।

गुणकयुतेन पदहतं सर्वधनमतोऽवशेषाणि ॥

—*BrSpSi*. XIII. 47

2. आधादकर्णान्यान् वर्णान् प्रो ह्यधिकमानमाद्यहतम् ।

सदशब्देदाक्सकृद् द्वौ व्यस्तौ कुट्टको बहुषु ॥

—*BrSpSi*. XVIII. 51

be performed all operations conformably to the statement of the example and thus should be carefully framed two or more sides and also equations. Equi-clearance should be made first between two and two of them and so on to the last : from one side one unknown should be cleared, other unknowns reduced to a common denominator and also the absolute numbers should be cleared from the side opposite. The residue of other unknowns being divided by the residual coefficient of the first unknown will give the value of the first unknown. If there be obtained several such values, then with two and two of them, equations should be formed after reduction to common denominators. Proceeding in this way to the end find out the value of one unknown. If that value be (in terms of) another unknown then the coefficients of those two will be reciprocally the values of the two unknowns. If, however, there be present more unknowns in that value, the method of the pulveriser should be employed. Arbitrary values may then be assumed for some of the unknowns.

Datta and Singh have said that the above rule of Brahmagupta, and also the one indicated in the commentary of Pṛthudaka Svāmī, embraces the solution of indeterminate as well as the determinate equations. In fact, all the examples given by Brahmagupta in illustration of the rule are of indeterminate character. So far as the determinate simultaneous equations are concerned, Brahmagupta's method for solving them will be easily recognised to be the same as our present one.

Quadratic Equations

The geometrical solution of a quadratic equation in this country would take us to the Vedic Śulba period. The *Bakhaśālī Manuscript* also contains certain problems which need the solving of quadratic equations. I shall quote one out of the numerous available :

A certain person travels s *yojana* on the first day and b *yojana* more on each successive day. Another who travels at the uniform rate of S *yojana* per day, has a start of t days. When will the first man overtake the second ?

This problem would today be expressed in terms of the following equation :

$$S(t+x)=x\left\{s+\left(\frac{x-1}{2}\right)b\right\},$$

where x is the number of days after which the first overtakes the second. We may write this equation as

$$bx^2-\{2(S-s)+b\}x=2tS$$

whence the value x would be after solving the quadratic :

$$x=\frac{\sqrt{\{2(S-s)+b\}^2+8bts}+\{2(S-s)+b\}}{2b}$$

The *Bakhasālī Manuscript* gives this solution as follows :

The daily travel (S) diminished by the march of the first day (s) is doubled; this is increased by the common increment (b). That (*sum*) multiplied by itself is designated (as the *kṣepa* quantity). The product of the daily travel and the start (t) being multiplied by eight times the common increment, the *kṣepa* quantity is added. The square-root of this (is increased by the *kṣepa* quantity; the sum divided by twice the common increment will give the required number of days), (*BMS. Folio 5, recto*)

Āryabhaṭa I (499 A.D.) is regarded as the founder of algebra, since he gives the solutions of a few quadratic problems. For example, to find the number of terms of an arithmetical progression (A.P.), he gives the following rule :

The sum of the series multiplied by eight times the common difference is added by the square of the difference between twice the first term and the common difference: the square-root (of the result) is diminished by twice the first term and (then) divided by the common difference : half of this quotient plus unity is the number of terms.¹

In the modern notations of algebra, the solution would be expressed as follows :

1. गच्छोऽष्टोत्तर गुणिताद् द्विगुणाद्युत्तर विशेषवर्गयुतात् ।

मूलं द्विगुणाद्यूनं स्वीत्तर भजितं सरूपार्थं ॥

—Ārya. II, 20

$$n = \frac{1}{2} \left\{ \frac{\sqrt{8bs + (2a-b)^2} - 2a}{b} + 1 \right\}$$

There is another certain interest problem¹, the solution of which has been provided in the *Āryabhaṭīya* as

$$x = \frac{\sqrt{Apt + (p/2)^2} - p/2}{t}$$

which is the solution of the quadratic equation :

$$tx^2 + px - Ap = 0$$

Āryabhaṭa I has thus given the solutions of a few quadratic equations, but he nowhere gives the procedure of solving these equations.

We give here the Rules of Brahmagupta for the solution of quadratic equations. He undoubtedly is not the discoverer of these rules; but perhaps for the first time in the history of algebra we find the process of solving a quadratic equation so clearly indicated.

First Rule :

The quadratic : the absolute quantities multiplied by four times the coefficient of the square of the unknown are increased by the square of the coefficient of the middle (i.e. unknown); the square-root of the result being diminished by the coefficient of the middle and divided by twice the coefficient of the square of the unknown, is (the value of) the middle.”²

This expressed in the modern notations would mean

$$x = \frac{\sqrt{4ac + b^2} - b}{2a}$$

It would be noted that in this rule, Brahmagupta has employed the term *madhya* (middle) to imply the simple unknown as well as its coefficient. The origin of the term is doubtless connected with the mode of writing the quadratic equation in the form

$$ax^2 + bx + 0 = 0x^2 + 0x + c$$

so that there are three terms on each side of the equation.

1. मूलफलं सफलं काजमूल गुणमधर्ममूल कृति युक्तम् ।

मूलं मूलार्धेनं कालहृतं स्यात्स्वमूलफलम् ॥

—Ārya. II. 25.

2. वर्गं चतुर्गुणितानां रूपाणां मध्यवर्गसहितानाम् ।

मूलं मध्यनोनं वर्गं द्विगुणोद्भूतं मध्यः ॥

—BrSpSi. XVIII. 44.

Second Rule :

The absolute term multiplied by the coefficient of the square of the unknown is increased by the square of half the coefficient of the unknown; the square-root of the result diminished by half the coefficient of the unknown and divided by the coefficient of the square of the unknown is the unknown.²

This when expressed in the modern algebraic notations would be

$$x = \frac{\sqrt{ac + (b/2)^2} - (b/2)}{a}$$

Here if the quadratic equation is

$$ax^2 + bx + c = 0$$

the 'absolute term' is c (the one without the unknown x), 'the coefficient of the square of unknown' means the coefficient of x^2 , i.e. a , and the 'coefficient of the unknown' means the coefficient of x , i.e. b .

The above two methods of Brahmagupta are exactly the same as were suggested by Āryabhaṭa I.

The root of the quadratic equation for the number of terms of an arithmetic progression (A.P.) is given by Brahmagupta according to the first rule² :

$$n = \frac{\sqrt{8bs + (2a - b)^2} - (2a - b)}{2b}$$

Third Rule :

Brahmagupta also suggests a Third Rule which is very much the same as is used commonly now. Though it has not been expressly suggested as a new rule, we find its application in a few instances. For example this rule has been suggested in connection with the following problem on interest :

A certain sum (p) is lent out for a period (t_1); the interest accrued (x) is lent out again at this

1. वर्गाहत रूपाणामव्यक्तार्थकृति संयुतानां यत् ।

पदमव्यक्तार्थोनं तद्वर्गं विभक्तमव्यक्तः ॥

—BrSpSi. XVIII. 45

2. उत्तरहीनदिगुणादि शेषवर्गं धनोत्तराष्टववे ।

अद्विप्य पदं शेषोनं दिगुणोत्तरहस्तं गच्छः ॥

—BrSpSi. XII. 18

rate of interest for another period (t_2) and the total amount is A . Find x .

The equation for determining x is

$$\frac{t_2}{pt_1}x^2 + x = A.$$

The solution of this equation would be :

$$x = \sqrt{\left(\frac{pt_1}{2t_2}\right)^2 + \frac{A}{t_2}} - \frac{pt_1}{2t_2}$$

Brahmagupta has stated the result in exactly the same form. Pṛthudaka Svāmī has illustrated it in solving the following problem of interest :

Problem :

A sum of five hundred *paṇas* (p) is lent out for a period of 4 months (t_1); the interest accrued (x) is lent out again at this rate of interest for another period of 10 months (t_2) and the total amount is 78 (A). Give the *pramāṇa-phala*, i.e., the interest accrued x .

Here *pramāṇa-kāla* (t_1) = 4 months

pramāṇa-dhana (p) = 500 *paṇas*

para-kāla (t_2), the subsequent period = 10 months

miśra dhana or the total interest accrued (A) = 78 *paṇas*.

Brahmagupta states his solution of such quadratics like this :

Take the product of the *pramāṇa-dhana* (p) or the sum originally lent out and *pramāṇa-kāla*, i.e. the period for which originally lent out (t_1); and divide by the *para-kāla* or the subsequent time (t_2); place this result at two places. Multiply the one placed at the first place with the *miśra-dhana* (A), that is with the total interest accrued; in this product add the square of half the one placed in the second place; now take the square-root of it, and from it subtract half of the one placed at the second place.¹

1. कालप्रमाणघातः परकालहतो दिवाऽऽवमिश्रवधात् ।

अन्वार्धकृति युतात् पदमन्यार्धेन प्रमाण फलम् ॥

—BrSpSi. XII. 15:

Thus in the above example the product of *pramāṇa-dhana* and *pramāṇa kālā* divided by *parakālā* is (pt_1/t_2) is $\frac{500 \times 4}{10} = 200$.

This is first multiplied by the total interest accrued (A); it becomes $200 \times 78 = 15600$. To this is now added square of half of 200 (which is 10000); it becomes 15600 plus 10000 = 25600. Its square-root is taken which is 160. From this is subtracted half of the quantity (i.e. half of 200 which is 100). Thus $160 - 100 = 60$, which is the answer. It was the interest which first accrued (x).

Another Quadratic Problem:

Brahmagupta refers to an astronomical problem which involves the quadratic equation

$$(72 + a^2)x^2 \mp 24apx = 144\left(\frac{R^2}{2} - p^2\right),$$

where $a = \text{agra}$ (the sine of the amplitude of the Sun), $b = \text{palabha}$ (the equinoctial shadow of a gnomon 12 anguli long), $R = \text{radius}$, and $x = \text{koṇasāṅku}$ (sine of the altitude of the Sun when his altitude is 45°). Dividing out by $(72 + a^2)$ we have

$$x^2 \mp 2mx = n,$$

where

$$m = \frac{12ap}{72 + a^2}, \quad n = \frac{144(R^2/2 - p^2)}{72 + a^2}$$

Therefore we have

$$x = \sqrt{m^2 + n} \pm m,$$

as stated by Brahmagupta. We find the same result in the *Sūrya-siddhānta* and in the text of Śrīpati. Āryabhaṭa II (1150) also followed the method of Āryabhaṭa I and Brahmagupta in solving a quadratic equation in connection with finding out the number of terms in an arithmetical progression (A.P.) whose first term is (a) , common difference is b and the sum is s . The number of terms n is given by¹

$$n = \frac{\sqrt{2bs + (a - b/2)^2} - a + b/2}{b}$$

Two Roots of a Quadratic Equation and Brahmagupta

A quadratic equation has two roots. This must have been known to Indian algebraists even at a very early stage. Bhāskara II in his *Bījaganita* has quoted a rule ascribed to an ancient writer Padmanābha whose works are not available now:

1. *Mahāsiddhānta*. Bhāskara II, XV. 50

If (after extracting roots) the square-root of the absolute side (of the quadratic) be less than the negative absolute term on the other side, then taking it negative as well as positive two values (of the unknown) are found¹.

The term used here is *dvividhotpadyate mitih* which means that two values are obtained.

The existence of two roots of a quadratic equation appears to have been known also to Brahmagupta (628 A.D.). In illustration of his rules for the solution of a quadratic he has stated two problems involving practically the same equation :

Problem I : The square-root of the residue of the revolution of the Sun less 2 is diminished by 1, multiplied by 10 and added by 2; when will this be equal to the residue of the revolution of the Sun less 1, on Wednesday ?²

Problem II : When will the square of one-fourth the residue of the exceeding months less three, be equal to the residue of the exceeding months ?

We shall follow Prthūdaka Svāmi in solving the Problem I. In this problem the residue of the revolutions of the Sun may be supposed to be x^2+2 ; then by the question, we have

$$10(x-1)+2=x^2+1,$$

$$\text{or } x^2-10x=-9$$

Again in Problem II, if we put $4x$ for the residue of the exceeding month, then we have

$$(x-3)^2=4x$$

$$\text{or } x^2-10x=-9$$

Now by the second rule of Brahmagupta, retaining both the signs of the radical, we get :

$$x=5\pm\sqrt{25-9}=9 \text{ or } 1,$$

1. व्यक्त पक्षस्य चेन्मूलमन्यपक्षरूपतः ।

अल्पं धनर्यगं कृत्वा द्विविधोत्पद्यते मितिः ॥

—Bhāskara, *Bījaganita*

2. मण्डलशेषाद् द्रव्यनान्मूलं व्येकं दशाहतं द्वियुतमू ।

मण्डलशेषं व्येकं भानोर्द्विदिने कदा भवति ॥

—*BrSpSi*. XVIII. 49

3. अधिमासशेषपादात् व्यूनाद्वर्गोऽधिमासशेषसमः ।

अवंमोर्वर्षतो वाक्यशेषसमः कदा भवति ॥

—*BrSpSi*. XVIII. 50.

As shown by Pṛthūdaka Svāmī, the first value is taken by Brahmagupta for the Problem I and second value for the problem II. Thus it is quite clear that Brahmagupta uses sometimes the positive and at other times the negative sign with the radical. Hence we shall say that Brahmagupta knew that a quadratic equation would have two roots, and according to the requisiteness of the problem, one value out of the two would be utilised.

Simultaneous Quadratic Equations

Indian authors usually treated problems involving various forms of simultaneous quadratic equations.

$$\begin{array}{ll} \text{(i)} \quad \left. \begin{array}{l} x-y=d \\ xy=b \end{array} \right\} & \text{(ii)} \quad \left. \begin{array}{l} x+y=a \\ xy=b \end{array} \right\} \\ \text{(iii)} \quad \left. \begin{array}{l} x^2+y^2=c \\ xy=b \end{array} \right\} & \text{(iv)} \quad \left. \begin{array}{l} x^2+y^2=c \\ x+y=a \end{array} \right\} \end{array}$$

For the solution of the combination (i), Āryabhaṭa I gives the following rule in his *Āryabhaṭīya*.

The square-root of four times the product (of two quantities) added with the square of their difference, being added and diminished by their difference and halved gives the two multiplicands.¹

This means that

$$x = \frac{1}{2} \sqrt{d^2 + 4b} + d, \quad y = \frac{1}{2} (\sqrt{d^2 + 4b} - d)$$

For the solution of the same combination, Brahmagupta states as follows :

The square-root of the sum of the square of the difference of the residues and two squared times the product of the residues, being added and subtracted by the difference of the residues, and halved (gives) the desired residues severally.²

(Here by difference of the residues is meant $x-y$; and by product of the residues is meant xy .)

Brahmagupta does not seem to give the solution for simultaneous equations of the combination (ii). Mahāvīra (850 A.D.)

1. द्विकृति गुणात्संवर्गाद् द्वयन्तरवर्गेण संयुतान्मूलम् ।

अन्तरयुक्तं हीनं तद्गुण्यकारद्वयं दलितम् ॥

—Ārya. II. 24

2. शेषवधाद् द्वि कृति गुणात् शेषान्तरं कर्त्तुं संयुतान्मूलम् ।

शेषान्तरं युक्तं दलितं शेषे पृथगभीष्टे ॥

—Br SpSi. XVIII. 99

has given the solution :

Subtract four times the area (of a rectangle) from the square of the semi-perimeter then by *saṅkramaṇa* between the square-root of that (remainder) and the semi-perimeter, the base and the upright are obtained.¹ (GSS. VII. 129 $\frac{1}{2}$)

This expressed in the modern notations would be :

$$x = \frac{1}{2}(a + \sqrt{a^2 - 4b}), y = \frac{1}{2}(a - \sqrt{a^2 - 4b})$$

For the combination (iii), Mahāvīra in his *Gaṇita-Sāra-Saṃgraha* gives the following rule :

Add to and subtract twice the area (of a rectangle) from the square of the diagonal and extract the square-roots. By *saṅkramaṇa* between the greater and lesser of these (roots), the side and upright (are found).²

This put in modern notations would be :

$$x = \frac{1}{2}(\sqrt{c+2b} + \sqrt{c-2b}),$$

$$y = \frac{1}{2}(\sqrt{c+2b} - \sqrt{c-2b}).$$

For the combination (iv), Āryabhaṭa I gives the following rule :

From the square of the sum (of two quantities) subtract the sum of their squares. Half of the remainder is their product.³

The remaining operations will be similar to those for the equations (ii); so that

$$x = \frac{1}{2}(a + \sqrt{2c - a^2}), y = \frac{1}{2}(a - \sqrt{2c - a^2}).$$

Brahmagupta in this connection says :

Subtract the square of the sum from twice the sum of squares; the square-root of the remainder being added to and subtracted from the sum and halved, (gives) the desired residues.⁴

1. GSS. VII. 129 $\frac{1}{2}$

2. GSS. VII. 127 $\frac{1}{2}$

3. संपर्कस्य हि वर्गाद्विशोधयेदेव वर्गसंपर्कम् ।

यत्तस्य भवत्यर्थं विवाद् गुणकारसंपर्कम् ॥

4. कृति संयोगाद् द्विगुणाद्युति कर्त्तुं प्रोह्य शेष मूलं यत् ।

तेन युतो नो योगो दलितः शेषे पृथगभीष्टे ॥

—Ārya. II. 23

—BrSpSi. XVIII. 98

These equations have also been treated by Mahāvīra, Bhāskara II and Nārāyaṇa. Nārāyaṇa has attempted two other forms of quadratic equations :

$$(v) \begin{cases} x^2 + y^2 = c \\ x - y = d \end{cases} \quad (vi) \begin{cases} x^2 - y^2 = m \\ xy = b \end{cases}$$

For their solutions, see Datta and Singh, *Algebra*, P. 84.

Rule of Dissimilar Operations :

Datta and Singh say that the process of solving the following two particular cases of simultaneous quadratic equations was distinguished by most Indian mathematicians by the special designation *viśama-karma* or dissimilar operation :

$$(i) \begin{cases} x^2 - y^2 = m \\ x - y = n \end{cases} \quad (ii) \begin{cases} x^2 - y^2 = m \\ x + y = p \end{cases}$$

These equations have been regarded by these mathematicians as if of fundamental importance. They have given the following solutions (expressed in modern algebraic symbols) :

For the combination (i) :

$$x = \frac{1}{2} \left(\frac{m}{n} + n \right), \quad y = \frac{1}{2} \left(\frac{m}{n} - n \right),$$

For the combination (ii) :

$$x = \frac{1}{2} \left(p + \frac{m}{p} \right), \quad y = \frac{1}{2} \left(p - \frac{m}{p} \right).$$

We shall express these solutions as follows in the words of Brahmagupta :

The difference of the squares (of the unknowns) is divided by the difference of the unknowns and the quotient is increased and diminished by the difference and divided by two ; (the results will be the two unknown quantities) ; (this is) dissimilar operation.

The same rule is restated by him on a different occasion in the course of solving a problem.

If then the difference of their squares, also the difference of them (are given) ; the difference of the squares

1. योगोऽन्तरयुतहीनो दिह्यतः संक्रमणमन्तर विभक्तं वा ।
३३. व्यभिन्तरमन्तर युतहीनं दिह्यतं विषमकर्म ।

is divided by the difference of them, and this (latter) is added to and subtracted from the quotient and then divided by two ; (the results are) the residues whence the number of elapsed days (can be found).¹

This *viṣama-karma* or dissimilar operation has been described by other Indian algebraists also, as Āryabhaṭa II (*Mahāśi-
ddhānta*, XVII; 22); Śrīpati (*Siddhānta-sekhara*, XIV. 13) ; Bhā-
skara II (*Līlavatī*) and Nārāyaṇa (*Gaṇita-kaumudī*, I, 32).

Indeterminate Equations of the First Degree

Āryabhaṭa I should be given the credit of giving for the first time a treatment of the indeterminate equation of the first degree. In his *Āryabhaṭīya*, we find a method for obtaining the general solution in positive integers of the simple indeterminate equation :

$$by - ax = c$$

for integral values of a, b, c , and further indicated how to extend it to get positive integral solutions of simultaneous indeterminate equations of the first degree. His disciple Bhāskara I (522) showed that the same method might be applied to solve :

$$by - ax = -c$$

and further that the solution of this equation would follow that of $by - ax = -1$. These methods of Āryabhaṭa I and Bhāskara I have also been adopted by Brahmagupta, and in certain cases, the improvement were suggested by Āryabhaṭa II in the middle of the tenth century A.D.

The problems which were treated by ancient Indian algebraists and which led them to the investigation of the simple indeterminate equation of the first degree may be classified under three heads :

Class I : To find a number N which being divided by two given numbers (a, b) will leave two given remainders (R_1, R_2).

Thus we have :

$$N = ax + R_1 = by + R_2$$

1. तद्वर्गान्तरमाद्ये तदन्तरं चान्तरोदधृतयुतोन्म ।
वर्गान्तरं विभक्तं द्वान्यां शेषे ततोद्युगलः ।

Hence $by - ax = R_1 - R_2$

Putting $c = R_1 \cup R_2$

we get $by - ax = \pm c$

the upper or lower sign being taken according as R_1 is greater than or less than R_2 .

Class II : To find a number (x) such that its product with a given number (a) being increased or decreased by another given number (δ) and then divided by a third given number (β) will leave no remainder.

This means that in other words, we shall have to get the solution of :

$$\frac{ax \pm r}{\beta} = y$$

in positive integers.

Class III : Here we have to deal with an equation of the form :

$$by \div ax = \pm c$$

Kuṭṭaka, *Kuṭṭākāra* and *Kuṭṭa* : These are the three terms which Brahmagupta has used in regards to the subject of indeterminate analysis of the first degree. Āryabhaṭa I has also described this method in brief, but he does not use the word *kuṭṭaka*. In the *Mahābhāskarīya* of Bhāskara I we have the terms *kuṭṭākāra* and *kuṭṭa* (522 A.D.) *MBh.* I. 41.49). These words have been translated into English as *pulveriser* or *grinder*. According to Datta and Singh, the Hindu method of solving the equation $by - ax = \pm c$ is essentially based on a process of deriving from it successively other similar equations in which the values of the coefficients (a, b) become smaller and smaller. Thus the process is indeed the same as that of breaking a whole thing into smaller pieces, and this accounts for its name *kuṭṭaka* or '*pulveriser*'.

In the problems of the Class I, the quantities (a and b) are called 'divisors' (*bhāgahāra*, *bhājaka*, *cheda* etc.) and R_1 and R_2 as 'remainders' (*agra* or *śeṣa* etc.). while in a problem of the Class II ; β is ordinarily called the 'divisor' (*bhāgahāra* or *bhājaka*) and γ the 'interpolator' (*kṣepa*, *kṣepaka* etc.) ; here a is called the 'dividend' (*bhājya*), the unknown quantity to be found (x) is called the 'multiplier' or (*guṇaka* or *guṇakāra* etc) and y the

quotient or *phala*. In later years, Mahāvīra has called the unknown number (x) as *rāśi*.

Preliminary Operations in Kuṭṭaka-Karma

Usually it has been suggested that in order that an equation of the form

$$by - ax = \pm c \text{ or } by + ax = \pm c$$

may be amenable to solution, the two numbers a and b must not have a common divisor; for otherwise, the equation would be absurd, unless the number c had the same common divisor. So before the rules which we shall give hereafter, could be applied, the numbers a, b, c must be made prime (*dydha* or firm; *niccheda* or having no divisor, or *nirapavarta*, meaning irreducible to each other.

In this connection Bhāskara I writes :

The dividend and divisor will become prime to each other on being divided by the residue of their mutual division. The operation of the pulveriser should be considered in relation to them.¹

Similarly we find in the writings of Brahmagupta :

Divide the multiplier and the divisor mutually and find the last residue; those quantities being divided by the residue will be prime to each other.²

Āryabhaṭa's Rule : Āryabhaṭa I is probably the first Indian writer on this subject, but the operation given by him is rather obscure. His disciple Bhāskara I has given the solution of indeterminate equations of the first degree in more satisfactory language. We shall give here the translation of Āryabhaṭa's verse from the *Āryabhaṭaśya*, as rendered by Bibhutibhusan Datta, because other translations of this verse do very often confuse the sense :

Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remain-

1. भूदिनेष्टगणान्योन्य भक्तशेषेण भाजितौ ।

हारभाज्यौ हदौ स्यातां कुट्टाकारं तदोर्विदुः

2. हतयोः परस्परं यच्छेषं गुणकारभागहारकयोः ।

तेन हतौ निश्चेदौ तावेव परस्परं हतयोः ।

—MBh, I. 41

—BrSpSi. XVIII. 9.

Hence $by - ax = R_1 - R_2$

Putting $c = R_1 \cup R_2$

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Usually it has been suggested that in order that an equation of the form

$$by - ax = \pm c \text{ or } by + ax = \pm c$$

may be amenable to solution, the two numbers a and b must not have a common divisor; for otherwise, the equation would be absurd, unless the number c had the same common divisor. So before the rules which we shall give hereafter, could be applied, the numbers a, b, c must be made prime (*dr̥ḍha* or firm; *niccheda* or having no divisor. or *nirapavarta*, meaning irreducible to each other.

In this connection Bhāskara I writes :

The dividend and divisor will become prime to each other on being divided by the residue of their mutual division. The operation of the pulveriser should be considered in relation to them.¹

Similarly we find in the writings of Brahmagupta :

Divide the multiplier and the divisor mutually and find the last residue; those quantities being divided by the residue will be prime to each other.²

Āryabhaṭa's Rule : Āryabhaṭa I is probably the first Indian writer on this subject, but the operation given by him is rather obscure. His disciple Bhāskara I has given the solution of indeterminate equations of the first degree in more satisfactory language. We shall give here the translation of Āryabhaṭa's verse from the *Āryabhaṭīya*, as rendered by Bibhutibhusan Datta, because other translations of this verse do very often confuse the sense :

Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remain-

1. भूदिनेष्टयणान्योन्य भक्तशेषेण भाजितौ ।

हारभाज्यौ ददौ स्वातां कुट्टाकारं तबोर्विदुः

—MBh. I. 41

2. हतयोः परस्परं यच्छेषं गुणकारभागहारकयोः ।

तेन हतौ निश्छेदौ तावेव परस्परं हतयोः ।

—BrSpSi. XVIII. 9.

der. The residue (and the divisor corresponding to the smaller remainder) being mutually divided, the last residue should be multiplied by such an optional integer that the product being added (in case the number of quotients of the mutual division is even) or subtracted (in case the number of quotients is odd) by the difference of the remainders (will be exactly divisible by the last but one remainder. Place the quotients of the mutual division successively one below the other in a column; below them the optional multiplier and underneath it the quotient just obtained). Any number below the penultimate) is multiplied by the one just above it and then added by that just below it. Divide the last number (obtained by doing so repeatedly) by the divisor corresponding to the smaller remainder; then multiply the residue by the divisor corresponding to the greater remainder and add the greater remainder. (The result will be) the number corresponding to the two divisors.¹

There is an alternative rendering of this passage also as follows :

Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder. The residue (and the divisor corresponding to the smaller remainder) being mutually divided (until the remainder becomes zero), the last quotient should be multiplied by an optional integer and then added (in case the number of quotients of the mutual division is even) or subtracted (in case the number of quotients is odd) by the difference of the remainders. (Place the other quotients of mutual division successively one below the other in a column; below them the result just obtained and underneath it the optional integer). Any

1. अधिकप्रभागहारं द्विन्वादूनाप्रभागहारेण ।

शेषपरस्परभक्तं मतिगुणमग्रान्तरे क्षिप्तम् ॥

अथ उपरि गुणितमन्ययुगूनाग्रच्छेद भाजिते शेषम् ।

अधिकप्रच्छेदगुणं द्विच्छेदाप्रमधिकप्रयुतम् ।

number below (i.e. the penultimate) is multiplied by the one just above it and then added by that just below it. Divide the last number (obtained by doing so repeatedly) by the divisor corresponding to the smaller remainder; then multiply the residue by the divisor corresponding to the greater remainder and add the greater remainder. (The result will be) the number corresponding to the two divisors.

Āryabhata's problem may be enunciated thus :

To find a number (N) which being divided by two given numbers (a, b) will leave two given remainders (R_1, R_2).

This gives :

$$N = ax + R_1 = by + R_2$$

(where R_1 is a greater remainder and R_2 lesser remainder, and a is the divisor corresponding to greater remainder and b the divisor corresponding to the lesser remainder.)

Denoting as before by c the difference between R_1 and R_2 , we get

$$(i) \quad by = ax + c, \text{ if } R_1 > R_2$$

$$(ii) \quad ax = by + c, \text{ if } R_2 > R_1$$

the equation being so written as to keep c always positive.

Hence the problem now reduces to making either

$$\frac{ax+c}{b} \text{ or } \frac{by+c}{a}$$

according as $R_1 > R_2$ or $R_2 > R_1$, a positive integer. So Āryabhata says : Divide the divisor corresponding to the greater remainder etc."

Now we shall proceed with the details of the operation as proposed by Datta and Singh in his *History of Hindu Mathematics, Part II. Algebra* :

Suppose $R_1 > R_2$; then the equation to be solved will be

$$ax + c = by \quad \dots(i)$$

a, b being prime to each other.

Let

$$\begin{array}{r}
 b) \ a \ (q \\
 \underline{bq} \\
 r_1) \ b \ (q_1 \\
 \underline{r_1 q_1} \\
 r_2) \ r_1 \ (q_2 \\
 \underline{r_2 q_2} \\
 r_3 \\
 \dots \\
 \underline{r_{m-1}) \ r_{m-2} \ (q_{m-1} \\
 r_{m-1} \ q_{m-1} \\
 \underline{r_m) \ r_{m-1} \ (q_m \\
 r_m q_m \\
 r_m + 1}
 \end{array}$$

Then we get (when $a < b$, we shall have $q=0$, $r_1=a$)

$$\begin{aligned}
 a &= bq + r_1 \\
 b &= r_1 q_1 + r_2 \\
 r_1 &= r_2 q_2 + r_3 \\
 r_2 &= r_3 q_3 + r_4 \\
 &\dots \dots \dots \\
 r_{m-2} &= r_{m-1} q_{m-1} + r_m \\
 r_{m-1} &= r_m q_m + r_m + 1
 \end{aligned}$$

Now, substituting the value of a in the given equation (I), we get

$$by = (bq + r_1)x + c$$

Therefore

$$y = qx + y_1$$

where

$$by_1 = r_1 x + c$$

In other words, since $a = bq + r_1$, on putting

$$y = qx + y_1 \tag{ii}$$

the given equation (i) reduces to

$$by_1 = r_1 x + c \tag{iii}$$

Again, since $b = r_1 q_1 + r_2$

putting similarly $x=q_1y_1+x_1$

the equation (iii) can be further reduced to

$$r_1x_1=r_2y_1-c \quad (\text{iv})$$

and so on.

Writing down the successive values and reduced equations in columns, we have

(1) $y=qx+y_1$	(I.1) $by_1=r_1x+c$
(2) $x=qy_1+x_1$	(I.2) $r_1x_1=r_2y_1-c$
(3) $y_1=q_2x_1+y_2$	(I.3) $r_2y_2=r_3x_1+c$
(4) $x_1=q_3y_2+x_2$	(I.4) $r_3x_2=r_4y_2-c$
(5) $y_2=q_4x_2+y_3$	(I.5) $r_4y_3=r_5x_2+c$
(6) $x_2=q_5y_3+x_3$	(I.6) $r_5x_3=r_6y_3-c$
.....
(2n-1) $y_{n-1}=q_{2n-2}x_{n-1}+y_n$	(I. 2n-1) $r_{2n-2}y_n=r_{2n-1}x_{n-1}+c$
(2n) $x_{n-1}=q_{2n-1}y_n+x_n$	(I. 2n) $r_{2n-1}x_n=r_{2n}y_n-c$
(2n+1) $y_n=q_{2n}x_n+y_{n+1}$	(I. 2n+1) $r_{2n}y_{n+1}=r_{2n+1}x_n+c$

Now the mutual division can be continued either (i) to the finish or (ii) so as to get a certain number of quotients and then stopped. In either case the number of quotients found, neglecting the first one (q), as is usual with Āryabhaṭa, may be even or odd.

Case (i) First suppose that the mutual division is continued until the zero remainder is obtained. Since a , b are prime to each other, the last one remainder is unity.

Subcase (i.1). Let the number of quotients be even. We then have

$$r_{2n}=1, r_{2n-1}=0, q_{2n}=r_{2n-1}$$

The equations (1,2n) and (I.2n+1), therefore become

$$y_n=q_{2n}x_n+c$$

and

$$y_{n+1}=c$$

respectively. Giving an arbitrary integral value (t) to x_n we get an integral value of y_n . From that we can find the value of x_{n-1} by the equation (2n). Proceeding backwards step by step we ultimately find the values of x and y in positive integers. So that the equation (I) is solved.

Subcase (i. 2) : If the number of quotients be odd, we shall have

$$r_{2n-1}=1, r_{2n}=0, q_{2n-1}=r_{2n-2}.$$

The equations $(2n+1)$ and $(I. 2n+1)$ will then be absent and the equations $(I. 2n-1)$ and $(I. 2n)$ will be reduced respectively to

$$x_{2n-1} = q_{2n-1} y - c$$

and $x_n = -c$

Giving an arbitrary integral value (t') to y_n we get an integral value of x_{n-1} . Then proceeding backwards as before we calculate the values of x and y .

Case (ii) : Next suppose that the mutual division is stopped after having obtained an even or odd number of quotients.

Subcase (ii.1) : If the number of quotients obtained be *even* the reduced form of the original equation is

$$r_2 y + 1 = r_{2n} + 1 x_n + c$$

$$\text{or } y_n + 1 = \frac{r_{2n} + 1 x_n + c}{r_2}$$

Giving a suitable integral value (t) to x_n as will make

$$y_n + 1 = \frac{r_{2n} + 1 t + c}{r_2} = \text{an integral number,}$$

we get an integral value for y_n by $(2n+1)$. The values of x and y can then be calculated by proceeding as before.

Subcase (ii.2) : If the number of quotients be *odd* the reduced form of the quotient is

$$r_{2n-1} x_n = r_{2n} y_n - c$$

$$\text{or } x_n = \frac{r_{2n} y_n - c}{r_{2n-1}}$$

Putting $y_n = t'$, where t' is an integer, such that

$$x_n = \frac{r_{2n} t' - c}{r_{2n-1}} = \text{a whole number,}$$

we get an integral value of x_{n-1} by $(2n)$. Whence can be calculated the values of x and y in integers.

If $x = \alpha$ and $y = \beta$ be the least integral solution of $ax + c = by$, we shall have

$$a\alpha + c = b\beta$$

Therefore $a(bm + \alpha) + c = b(am + \beta)$,

m being any integer. Therefore, in general,

$$x = bm + \alpha$$

But we have calculated before that

$$x = q_1 y_1 + x_1 ;$$

$$\therefore q_1 y_1 + x_1 = bm + \alpha$$

Thus it is found that the minimum value α of x is equal to the remainder left on dividing its calculated value by b . whence we can calculate the minimum value of $N (=a\alpha + R_1)$. This will explain the *rationale* of the operations described in the latter portion of the rule of Āryabhaṭa I.

Bhāskara I and Kuṭṭaka Operation

In Chapter I of the *Mahābhāskarīya*, Bhāskara I has described the preliminary operation to be performed on the divisor and dividend of a pulveriser. We shall quote it from the edition of K.S. Shukla :

The divisor (which is "the number of civil days in a *yuga*") and the dividend (which is "the revolution number of the desired planet") become prime to each other on being divided by the (last non-zero) residue of the mutual division of the number of civil days in a *yuga* and the revolution number of the desired planet. The operations of the pulveriser should be performed on them (i. e. on the abraded divisor and abraded dividend). So has been said.¹

An indeterminate equation of the first degree of the type

$$\frac{ax - c}{a} = y$$

(with x and y unknown) is known in Hindu mathematics by the name of "pulveriser"—*kuṭṭākāra*). In this equation, a is called the "dividend" (*bhājya*), b the "divisor" (*bhāgahāra*), c the interpolator (*kṣepa*), x the "multiplier" (*guṇakāra*), and y the "quotient" (*labdha*).

In the pulveriser contemplated in the above stanza :

a = revolution number of a planet.

b = civil days in a *yuga*,

c = residue of the revolutions of the planet (*Śeṣa*)

1. भूदिनेष्टगणान्योन्य भक्तशेषेण भाजि ।
हारभाज्यैर्द्वौ स्यातां कुट्टाकारं तयोर्विदुः ।

$x = \text{ahargana}$,

and $y = \text{complete revolutions performed by the planet.}$

The text says that as a preliminary operation to the solution of this pulveriser, a and b . i.e., civil days in *yuga* and revolution-number of the planet, should be made prime to each other by dividing them out by their greatest common factor. That is to say, in solving a pulveriser, one should always make use of abraded divisor and abraded dividend.

The interpolator, i.e., the residue, should also be divided out by the same factor. (This instruction is not given in the text, but it is implied that the residue should be computed for the abraded dividend and abraded divisor).

Set down the dividend above and the divisor (*hāra*) below that. Divide them mutually and write down the quotients (*labdha*) of division one below the other (in the form of a chain). (When an even number of quotients is obtained) think out by what number the (last) remainder be multiplied so that the product being diminished by the (given) residue be exactly divisible (by the divisor corresponding to that remainder). Put down the chosen number called *mati* below the chain and then the new quotient underneath it. Then by the chosen number multiply the number which stands just above it, and to the product add the quotient (written below the chosen number). (Replace the upper number by the resulting sum and cancel the number below). Proceed afterwards also in the same way (until only two numbers remain). Divide the upper number (called the "multiplier") by the divisor by the usual process and the lower one (called the "quotient") by the dividend: the remainders (thus obtained) will respectively be the *ahargana* and the revolutions etc. or what one wants to know.¹

We shall illustrate the operation by taking a problem from the *Laghu-Bhāskariya* (VIII. 17):

The sum, the difference, and the product increased by one, of the residues of the revolution of Saturn and Mars—each is a perfect square. Taking the equations

furnished by the above and applying the method of such quadratics, obtain the (simplest) solution by the substitution of 2, 3 etc. successively in the general solution). Then calculate the *ahragana* and the revolutions performed by Saturn and Mars in that time together with the number of solar years elapsed.¹

Let x and y denote the residues of the revolution of Mars and Saturn respectively. Then we have to find out two numbers x and y such that each of the expressions $x+y$, $x-y$ and $xy+1$ may be a perfect square.

Let $x+y=4P^2$ and $x-y=4Q^2$, so that

$$x=2P^2+2Q^2$$

$$y=2P^2-2Q^2$$

and therefore $xy+1=(2P^2-1)^2+4(P^2-Q^4)$

Hence the condition that $xy+1$ be a perfect square is that $P^2=Q^4$. Substituting these values, we have

$$x=2(Q^4+Q^2)$$

$$y=2(Q^4-Q^2)$$

where Q may possess any of the values 2, 3, 4.....but not 1. (We neglect the case when x or y is zero).

1. भाज्यं न्यसेदुपरि हारमधश्च तस्य ।
खण्डयात्परस्परमधो विनिधाय लब्धम् ।
केनाऽऽहतोऽयमपनीय यथाऽस्य शेष ।
भागं ददाति परिशुद्धमिति प्रचिन्त्यम् ॥

आप्तां मतिं तां विनिधाय वल्ल्यां
नित्यं ह्यधोऽधः क्रमशश्च लब्धम् ।
मत्या हतं स्यादुपरिस्थितं य
ल्लब्धेन युक्तं पातश्च तद्वत् ॥

हारेणभाज्यो विधिनोपरिस्थो
भाज्येन नित्यं तदधः स्थितश्च ।
अर्हांगखोऽस्मिन् भगणद्वयश्चै
तद्वा भवेद्यस्य समीहितं यत् ॥

—MBh, I, 42-44

2. शेषौ मण्डलजौ यमक्षितिजयोः संयुक्तविश्लेषिता
वन्योन्महत्तविग्रहौ च पददौ रूपेण संयोजितौ ।
एवं साधु विचिन्त्य कर्माविधिना द्वित्रिकमाद्वत्सरैः ।
संगणया च गणार्कजक्षिति सुताः कालेन कालोद्भवः ॥

—LBh. VI. 17

Putting $Q=2$, we get $x=40$ and $y=24$, which is the least solution.

Assuming now that the residues of the revolution (*maṇḍa-laja-seṣa*) of Saturn and Mars are 24 and 40 respectively, we have to obtain the *ahargana* (which means the number of mean civil days elapsed since the beginning of Kaliyuga, or, in fact, any epoch).

The revolution-number of Saturn is 146564, and the number of civil days in a *yuga* is 1,577,917,500. In the present problem, these are respectively the dividend and the divisor. Their H.C.F. is 4, so that dividing them out by 4 we get 36641 and 394,479,375 as the abraded dividend and abraded divisor respectively. We have, therefore, to solve the pulveriser

$$\frac{36641x - 24}{394479375} = y$$

where x and y denote the *ahargana* and the revolutions respectively made by Saturn.

Mutually dividing 36641 and 394479375, we get

$$\begin{array}{r}
 36641 \overline{) 394479375} \quad (10766 \\
 \underline{394477006} \\
 2369 \overline{) 36641} \quad (15 \\
 \underline{35535} \\
 1106 \overline{) 2369} \quad (2 \\
 \underline{1106} \\
 157 \overline{) 1106} \quad (7 \\
 \underline{1099} \\
 7 \overline{) 157} \quad (22 \\
 \underline{154} \\
 3 \overline{) 7} \quad (2 \\
 \underline{6} \\
 1 \times 27 - 24 = 3 \overline{) 3} \quad (1 \\
 \underline{3} \\
 0
 \end{array}$$

We have chosen here the number 27 as the optional number (*mati*). In fact, *mati* may be chosen at any stage after an even number of quotients are obtained.

Writing down the quotients one below the other as prescribed in the rule, we get the chain

10766
15
2
7
22
2
(*mati*) 27
1

Reducing the chain, we successively get

10766	10766	10766	10766	10766	10766	3108044439
						(multiplier)
15	15	15	15	15	288689	288689
						(quotient)
2	2	2	2	18665	18665	
7	7	7	8714	8714		
22	22	1237	1237			
2	55	55				
(<i>mati</i>)						
27	27					
1						

(it would be seen in this reduction of chain that *mati* or 27×2 plus 1 is 55; 55×22 plus 27 is 1237; 1237×7 plus 55 is 8714; 8714×2 plus 1237 is 18665; 18665×15 plus 8714 is 288689; and finally 288689×10766 plus 18665 is 3108044439 which is the multiplier).

Dividing 3108044439 by 394479375, and 288689 by 36641, we obtain 346688814 and 32202 respectively as remainders, (This division is performed only when the multiplier and quotient are greater than the divisor and dividend respectively). These are the minimum values of x and y satisfying the above equation.

Therefore, the required *ahargana* = 346688814 and the revolutions performed by Saturn = 32202.

To obtain the *ahargana* and the revolutions of Mars, one has to solve the equation :

$$\frac{191402z - 40}{131493125} = w$$

where z and w denote the *ahargana* and the revolutions performed by Mars respectively.

The general solution of this equation is

$$z = 131493125s + 118076020$$

$$w = 191402s + 171872$$

where $s = 0, 1, 2, 3, \dots$. When $s = 0$, we have the least solution.

Brahmagupta's Rules Concerning Indeterminate Analysis of the First Degree

For the solution of Āryabhaṭa's problem, Brahmagupta gives the following rule :

What remains when the divisor corresponding to the greater remainder is divided by the divisor corresponding to the smaller remainder—that (and the latter divisor) are mutually divided and the quotients are severally set down one below the other. The last residue (of the reciprocal division after an even number of quotients has been obtained) is multiplied by such an optional integer that the product being added with the difference of the (given) remainders will be exactly divisible (by the divisor corresponding to that residue). That optional multiplier and then the (new) quotient just obtained should be set down (underneath the listed quotients). Now, proceeding from the lower-most number (in the column), the penultimate is multiplied by the number just above it and then added by the number just below it. The final value thus obtained (by repeating the above process) is divided by the divisor corresponding to the smaller remainder. The residue being multiplied by the divisor corresponding to the greater remainder and added to the greater remainder will be the number in view.¹

1. अधिकप्रमाणहारादूनाग्रच्छेद भाजिताच्छेषम् ।
यत् तत् परस्परहृतं लब्धमवोऽधः पृथक् स्थाप्यम् ॥
शेषं तदेष्टुगणितं यथाऽप्योरन्तरेण संयुक्तम् ।
शुच्यति मुख्यकः स्थाप्यो लब्धं चान्त्यादुपान्त्यगणः ॥
स्वोर्ध्वोऽन्त्य युतोऽग्रान्तो हीनाग्रच्छेदभाजितः शेषम् ।
अधिकप्रच्छेदहत मधिकप्रयुतं मकत्यग्रम् ॥

Brahmagupta further observes :

Such is the process when the quotients (of mutual division) are even in number. But if they be odd, what has been stated before as negative should be made as positive, or as positive should be made negative.¹

Regarding the direction for dividing the divisor corresponding to the greater number by the divisor corresponding to the smaller remainder, Pṛthūdaka Svāmī (860A.D.) observes that it is not absolute, rather optional; so that the process may be conducted in the same way by starting with the division of the divisor corresponding to the smaller remainder by the divisor corresponding to the greater remainder. But in this case of inversion of the process, he continues, the difference of the remainders, must be negative.

That is to say, the equation

$$by = ax + c$$

can be solved by transforming it first to the form

$$ax = by - c$$

so that we shall have to start with the division of b by a .

For the details of the "Theory of the pulveriser" as applied to the problems in Astronomy, the reader is referred to the writings of Bhaṭṭa Govind, translated by K.S. Shukla, and given as an Appendix to the edition of the *Laghu-Bhāskariya*. For the *rationale* of the rules in relation to *kuṭṭaka* or the pulveriser operation, one may also refer to the chapters by Datta and Singh in the *History of Hindu Mathematics: Algebra*.

$$\text{Solution of } by = ax \pm 1.$$

This simple indeterminate equation has a special use in astronomical calculations and therefore, Indian algebraists have paid special attention to it. In fact, this equation is solved exactly in the same way as the equation $by = ax \pm c$; it is a parti-

1. एवं समेषु विषमेष्वृणं धनं धनमृणं यदुक्तं तत् ।

अणधनयोर्व्यस्तत्वं गुण्य प्रक्षेपयोः कार्यम् ॥

cular case only of the more general latter equation. Of course, there is a little justification also for treating it separately, since both the types of equations represent two different physical conditions of the astronomical problems. In the case of $by=ax\pm c$, the conditions are such that the value of either y or x , more particularly of the latter, has to be found and the rules for solution formulated with that objective. But in the case of the equation $by=ax\pm 1$, the physical conditions require the values of both y and x .

The equation $by=ax\pm 1$ is usually known by the name *sthira-kuttaka*, literally meaning the 'constant pulveriser' Prthūdaka Svāmī also names it as *dṛḍha-kuttaka* meaning firm-pulveriser. Later on this term *dṛḍha* was confined to another sense, equivalent to *nicched* (having no divisor) or *nirapavarta* (irreducible). The origin of the name *sthira-kuttaka* or constant pulveriser has been explained by Prthūdaka Svāmī as being due to the fact that the interpolator (± 1) is here invariable.

For the solution of this equation, we shall quote Bhāskara I's rule and the rule by Brahmagupta. Bhāskara I writes in this connection as follows :

The method of the pulveriser is applied also after subtracting unity. The multiplier and quotient are respectively the numbers above and underneath. Multiplying those quantities by the desired number divide by the reduced divisor and dividend; the residues are in this case known to be the (elapsed) days and (residues of) revolutions respectively¹.

The pulveriser

$$\frac{ax-c}{b}=y \quad \dots (1)$$

may be written as

$$\frac{aX-1}{b}=Y \quad \dots (2)$$

where $x=cX$ and $y=cY$. If $X=\alpha$, $Y=\beta$ is a solution of (2), then $x=c\alpha$, $y=c\beta$ will be a solution of (i). Hence the above rule.

1. रूपमेकमपास्यापि कुदाकारः प्रसाध्यते ।
गुणकालोऽयं लब्धं च राशी स्यातामुपर्येषः ॥

Brahmagupta's Rule in this connection is as follows :

Solution of $by = ax - 1$:

Divide them (i.e., the abraded coefficient of the multiplier and the divisor) mutually and set down the quotients one below the other. The last residue (or the reciprocal division after an even number of quotients has been obtained) is multiplied by an optional integer such that the product being diminished by unity will be exactly divisible (by the divisor corresponding to that residue). The (optional) multiplier and then this quotient should be set down (underneath the listed quotients). Now proceeding from the lower most term to the uppermost, by the penultimate multiply the term just above it and then add the lowermost number. (The uppermost number thus calculated being divided by the reduced divisor, the residue (is the quantity required. This is the method of the constant pulveriser¹.

Solution of

$$by + ax = \pm c$$

Indian algebraists usually transformed this equation as $by = -ax + c$, so that it appeared as a particular case of $by = ax + c$, in which a was negative. Brahmagupta has been the first person to solve this equation, but the rule given by him is obscure :

The reversal of the negative and positive should be made of the multiplier and interpolator.²

Prthudaka Svāmī has tried to explain it, but he too is not very clear. He says :

1. हृतयोः परस्परं यच्चैवं गुणकार भागद्वययोः ।
तेन हतौ निश्छेदौ तावेव परस्परं हृतयोः ॥
लब्धमधोऽधः स्थायं तथेष्ट गुणकारसङ्गुणं शेषम् ।
शुद्ध्यति यथैकहीनं गुणकः स्थायः फलं चात्स्यात् ॥
अग्रान्तमुपान्तयेन स्वोर्ध्वं गुणितोऽन्त्य संयुतो भक्तम् ।
निःशेषभागद्वारेणैवं स्थिरकुट्टकः शेषम् ॥

—*BrSpSi*. XVIII. 9-11

2. एवं समेषु विषमेष्वर्णं धनं धनमृणं यदुक्तं तत् ।
अणधनयोर्व्यस्तत्वं गुणयप्रक्षेपयोः कार्यम् ॥

BrSpSi XVIII. 13

If the multiplier be negative, it must be made positive; and the additive must be made negative : and then the method of the pulveriser should be employed.

Prthūdaka Svāmī, however, does not indicate how to derive the solution of the equation .

$$by = -ax + c \quad \dots(1)$$

from that of the equation

$$by = ax - c \quad \dots(2)$$

The method, however, seems to have been this :

Let $x=a$, $y=\beta$ be the minimum solution of (2). Then we get

$$b\beta = a^2 - c$$

$$\text{or} \quad b(a-\beta) = -a(a-b) + c$$

Hence $x=a-b$, $y=a-\beta$ is the minimum solution of (1). This rule is very clearly indicated by Bhāskara II and others.

We shall give two examples from Bhāskara II (*Bijaganita*) to illustrate the rule :

Example I.

$$13y = -60x + 3$$

By the method described before, we find that the minimum solution of

$$13y = 60x + 3$$

is $x=11$, $y=51$. Subtracting these values from their respective abraders, namely 13 and 60, we get 2 and 90. Then by the maxim : "In the case of the dividend and divisor being of different signs, the results from the operation of division should be known to be so", making the quotient negative we get the solution of

$$13y = -60x + 3$$

as $x=2$, $y=-9$. Subtracting these values again from their respective abraders (13, 60), we get the solution of

$$13y = -60x - 3$$

as $x=11$, $y=-51$.

Example II.

$$11y = 18x + 10$$

Proceeding as before, we find the minimum solution of

$$11y=18x+10$$

to be $x=8$, $y=14$. These will also be the values of x and y in the case of the negative divisor but the quotient for the reasons stated before should be made negative. So the solution of

$$-11y=18x+10$$

is $x=8$, $y=-14$. Subtracting these (i.e., their numerical values) from their respective abraders, we get the solution of

$$-11y=18x-10$$

as $x=3$, $y=-4$,

“When the divisor is positive or negative the numerical values of the quotient and multiplier remain the same : when either the divisor or the dividend is negative, the quotient must always be known to be negative”¹.

One Linear Equation in More Than Two Unknowns

Whenever a linear equation involves more than two unknown's the Indian algebraists used to assume arbitrary values for all the unknowns except two and then to apply the method of *kut-taka* or “pulveriser”. In this connection, Brahmagupta says :

The method of the pulveriser (should be employed if there be present many unknowns (in any equation)²,

1. Bhāskara II gives the following rule :

“Those (the multiplier and quotient) obtained for a positive dividend being treated in the same manner give the results corresponding to a negative dividend.”

The treatment alluded to in this rule is that of subtraction from the respective abraders. He has further elaborated it thus :

The multiplier and quotient should be determined by taking the dividend, divisor and interpolator as positive. They will be the quantities for the additive interpolator. Subtracting them from their respective abraders, the quantities for a negative interpolator are found. If the dividend or divisor, be negative, the quotient should be stated as negative, the quotient should be stated as negative.

—*Bījaganita*

1. आद्याद्वर्णादन्यान् वर्णान् प्रोह्याद्यमानमाद्यहृतम् ।

सदृशच्छेदाक्सकृद् द्वौ व्यस्तौ कुर्वौ बहुषु ॥

—*BrSpSi. XVIII. 51*

We shall take up one of the problems posed by Brahmagupta concerning astronomy and leading to the equation :¹

$$197x - 1644y - z = 6302.$$

Hence

$$x = \frac{1644y + z + 6302}{197}$$

The commentator assumes $z=131$. Then

$$x = \frac{1644y + 6433}{197};$$

hence by the usual method of the pulveriser

$$x=41; y=1.$$

General Problem of Remainders

A certain type of simultaneous indeterminate equations of the first degree arise out of the general problem of remainders which may thus be stated : To find a number N which being severally divided by $a_1, a_2, a_3, \dots, a_n$, leaves as remainders $r_1, r_2, r_3, \dots, r_n$ respectively.

While dealing with such a case, we shall have the following series of equations :

$$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \dots = a_nx_n + r.$$

We have reasons to believe that the method of solution of these equations was known to Āryabhaṭa I. In the translation of the verse in the *Āryabhaṭīya*, II. 32-33 (the translation of which we have already given), the term *dvicchedāgram* should be translated as "the result will be the remainder corresponding to the product of the two divisors", instead of "the result will be the number corresponding to the two divisors." (the last line of the translation). This explanation is in fact given by Bhāskara I, the direct disciple and earliest commentator of Āryabhaṭa I. Such a rule is clearly stated by Brahmagupta².

1. अंशकरोषेय युतात् लिप्ताशेषात्तदन्तरादथवा ।

मानोर्द्धदिने कुर्यात् यः कथयति कुट्टकः सः ॥

—*BrSpSi*. XVIII. 55

2. स्वोर्ध्वोऽन्त्ययुतोऽग्रान्तो ह्रीनाम्यच्छेदभाजितः शेषम् ।

अधिकम्यच्छेदह्यतमधिकम्ययुतं भवत्यग्रम् ॥

—*BrSpSi*. XVIII. 5

The rationale of this method is not difficult. I shall quote it from the book of Datta and Singh: Starting with the consideration of the first two divisors, we have

$$N = a_1x_1 + r_1 = a_2x_2 + r_2.$$

By the method described before, we can find the minimum value α of x_1 satisfying this equation. Then the minimum value of N will be $a_1\alpha + r_1$. Hence the general value of N will be given by

$$\begin{aligned} N &= a_1 (a_2t + \alpha) + r_1 \\ &= a_1a_2t + a_1\alpha + r_1 \end{aligned}$$

where t is an integer. Thus $a_1\alpha + r_1$ is the remainder left on dividing N by a_1a_2 as stated by Āryabhaṭa I and Brahmagupta. Now taking into consideration the third condition, we have

$$N = a_1a_2t + a_1\alpha + r_1 = a_3x_3 + r_3$$

which can be solved in the same way as before. Proceeding in this way successively, we shall ultimately arrive at a value of N satisfying all the conditions ;

Prthūdaka Svāmī remarks :

Wherever the reduction of two divisors by a common measure is possible, there 'the product of the divisors' should be understood as equivalent to the product of the divisor corresponding to the greater remainder and quotient of the divisor corresponding to the smaller remainder as reduced (i.e. divided) by the common measure.¹ When one divisor is exactly divisible by the other, then the greater remainder is the (required) remainder and the divisor corresponding to the greater remainder is taken as 'the product of the divisors'. (The truth of) this may be investigated by an intelligent mathematician by taking several symbols.

As an illustration we shall take up a problem quoted by Bhāskara II in his *Bījaganita*, and which in its solution follows the method of Āryabhaṭa I. Prthūdaka Svāmī while commenting on several verses from Brahmagupta (*BrSpSi*. XVIII. 3-6)

1. i.e., if p be the L.C.M. of a_1 and a_2 , the general value of N satisfying the above two conditions will be

$$N = pt + a_1\alpha + r_1,$$

instead of

$$N = a_1a_2t + a_1\alpha + r_1.$$

observes that such problems were very popular amongst the ancient Indian mathematicians.

Problem : To find a number N which leaves remainders 5, 4, 3, 2 when divided by 6, 5, 4, 3 respectively.

That is to solve the equations :

$$N=6x+5=5y+4=4z+3=3w+2.$$

We have since $N=6x+5=5y+4$,

$$x = \frac{5y-1}{6}$$

But x must be integral, so $y=6t+5$, $x=5t+4$

$$\text{Hence } N=30t+29$$

Again $N=30t+29=4z+3$

$$\text{Therefore, } t = \frac{2z-13}{15}$$

Since t must be integral, we must have $z=15s+14$;
hence $t=2s+1$. Therefore

$$N=60s+59.$$

The last condition is identically satisfied. The method given here is the one followed by *Prthūdaka Svāmī*.

Thus when $N=60s+59=6x+5$

$$x = \frac{60s+54}{6} = 10s+9 \quad \dots(1)$$

Again, when $N=60s+59=5y+4$,

$$y = \frac{60s+55}{5} = 12s+11$$

Again when $N=60s+59=4z+3$

$$z = \frac{60s+56}{4} = 15s+14$$

Lastly, when $N=60s+59=3w+2$.

$$w = \frac{60s+57}{3} = 20s+19.$$

Varga Prakṛti or Kṛti Prakṛti or Square-Nature

The word *varga-prakṛti* (literally meaning 'square-nature') has been given by Indian algebraists to the indeterminate quadratic equation

$$Nx^2 \pm c = y^2$$

Here in this equation the absolute number c should be *rūpa* (or unity), which means the equation

$$Nx^2 \pm 1 = y^2$$

or it may be any absolute number. The most fundamental equation of this class has been regarded as

$$Nx^2 + 1 = y^2$$

where N is a non-square integer.

This branch of mathematics has originated from the number which is the *prakṛti* of the square of *yāvat*, etc. (the unknown x etc.), and therefore, it is called *varga- prakṛti*. The quantity N of the above equation is known as *Prakṛti*. Brahmagupta uses the term *GUṆAKA* (*multiplier*) for the same purpose¹.

This term *guṇaka* together with its variation *guṇa* appears occasionally also in the writings of later authors. For example, Śrīpati (*Siddhānta-śekhara*. XIV. 32) employs the term *guṇaka* where as Bhāskara II and Nārāyaṇa use the term *guṇa* in their *Bījagaṇitas*.

In this connection, we would now like to quote from Pṛthūdaka Svāmī (863 A.D.) from his commentary on the *Brahmasphuṭasiddhānta* :

Here are stated for ordinary use the terms which are well known to people. The number whose square, multiplied by an optional multiplier and then increased or decreased by another optional number, becomes capable of yielding a square-root, is designated by the term the "lesser root" *kaniṣṭha pada* or the "first root" *ādya-mūla*). The root which results, after those operations have been performed is called by the name the "greater root" (*jyeṣṭha pada*) or the "second root" (*anya-mūla*). If there be a number multiplying both these roots, it is called the "augmenter" (*udvartaka*); and on the contrary, if there be a number dividing the roots, it is called the "abridger" (*apavartaka*).

Thus in the equation

$$Nx^2 \pm c = y^2,$$

1. मूलं द्विवेष्ट कर्माद् गुणक गुणादिव्युत्त विद्मन्नाञ्च ।

आववधो गुणकगुणः सहान्यधारेण कृतमन्त्यम् ॥

2. *BrSpSi*. XVIII. 64 (Com.)

x is known as the lesser root, y is the greater root, N is the multiplier (*gunaka*) and c is interpolator or *kṣepaka*. Bhāskara II has used the word "*hrasvamūla*" for *kaniṣṭha pada* or *ādyamūla* literally meaning "lesser root". The earlier terms, the "first root" (*ādyamūla*) for the value of x and the "second root" or the "last root" *antya-mūla* for the value of y are quite free from ambiguity. Their use is found in the algebra of Brahmagupta. The later terms appears in the works of his commentator Pṛthūdaka Svāmī.

Brahmagupta uses the term *kṣepa*, *prakṣepa* or *prakṣepaka* in the sense of "interpolator." Again, when negative, the interpolator is sometimes distinguished as the "subtractive" or *sodhaka* and the positive interpolator is then called "the additive."

Lemmas of Brahmagupta

Prior to our giving the general solution of the Square-nature or *Varga-Prakṛti*, it would be better to give two Lemmas established by Brahmagupta. We have the following in the *Brahmasphuṭasiddhānta* :

Of the square of the optional number multiplied by the *gunaka* and increased or decreased by an other optional number, *iṣṭa*, (extract) the square root. (Proceed) twice. The product of the first roots multiplied by the *gunaka* together with the product of the second roots will give a (fresh) second root; the sum of their cross-products will be a (fresh) first root. The (corresponding) interpolator will be equal to the product of the (previous) interpolators.¹

There is a little difficulty in ascertaining the real sense of the rule given in these lines since the word *dvidha* (twice) has two implications. Firstly, it may mean that the earlier operations of finding roots are made on two optional numbers with two optional interpolators, and with the results thus obtained the subse-

1. मूलं द्विवेष्टवर्गाद् गुणकं गुणादिष्टयुतं विहीनाच्च ।

आवधयो गुणकगुणः सहान्त्यघातेन कृतमन्त्यम् ॥

वज्रवैज्यं प्रथमं प्रक्षेपः क्षेपवध तुल्यः ।

प्रक्षेपशोधकहते मूले प्रक्षेपके रूपे ॥

quent operations of their composition are performed. Secondly, it may also mean that the earlier operations are made with one optionally chosen number and one interpolator, and the subsequent ones are carried out after the repeated statement of those roots for the second time. It is also implied that in the composition of the quadratic roots, their products may be added together or subtracted from each other.

In other words, if $x=a, y=\beta$ be a solution of the equation :

$$Nx^2+k=y^2,$$

and $x=a', y=\beta'$ be a solution of

$$Nx^2+k'=y'^2,$$

then according to the above

$$x=\alpha\beta'\pm\alpha'\beta, y=\beta\beta'\pm Na\alpha'$$

is a solution of the equation

$$Nx^2+kk'=y'^2.$$

In other words, if

$$Na^2+k=3^2$$

$$Na'^2+k'=3'^2$$

then

$$N(\alpha\beta'\pm\alpha'\beta)^2+kk'=(\beta\beta'\pm Na\alpha')^2 \quad (I)$$

In particular, taking $a=a', \beta=\beta'$ and $k=k'$, Brahmagupta finds from a solution $x=a, y=\beta$ of the equation

$$Nx^2+k^2=y^2$$

a solution $x=2a\beta, y=\beta^2+Na^2$ of the equation

$$Nx^2+k=y^2$$

That is, if

$$Na^2+k=\beta^2$$

then

$$N(2a\beta)^2+k^2=(\beta^2+Na^2)^2 \quad (II)$$

This result will be hereafter called *Brahmagupta's Corollary*.

Thus *Brahmagupta's First Lemma* says that if two solutions of the equation (of the Square-nature) $Nx^2+1=y^2$ are known, then any number of other solutions can be found. For example if two solutions of the Square-nature are (a, b) and also (a', b') , then two other solutions will be :

$$x=ab'+a'b, y=bb'\pm Na a'.$$

We can compose this solution with the previous ones, and get another solution, and thus proceed on to innumerable solutions. From *Brahmagupta's Corollary to First Lemma* we get another set of solutions. If (a, b) be solution of the Square-nature, then another solution of it is

$$x=2ab, \text{ and } y=b^2+Na^2$$

Thus even if we have only one solution, we can get the other solution also (since N is known), and thus we can get any number of solutions one after the other by this Principle of Composition.

Brahmagupta's Lemmas have been described by Bhāskara II (1150 A.D.) in the following words :

Set down successively the lesser root (*hrasva*), greater root (*jyeṣṭha*) and interpolator (*kṣepaka*); and below them should be set down in order the same or another (set of similar quantities). From them by the Principle of Composition (*Bhāvanā*) can be obtained numerous roots. Therefore the Principle of Composition will be explained here. (Find) the two cross-products (*vajrābhyāsa*) of the two lesser and the two greater roots; their sum is a lesser root. Add the product of the two lesser roots multiplied by the *prakṛti* to the product of the two greater roots, the sum will be a greater root. In that (equation) the interpolator will be the product of the two previous interpolators. Again the difference of the two cross-products is a lesser root. Subtract the product of the two lesser roots multiplied by the *prakṛti* from the product of the two greater roots; (the difference) will be greater root. Here also the interpolator is the product of the two (previous) interpolators.¹

1. ह्रस्वज्येष्ठ द्वेपकान् न्यस्य तेषां तानन्यान् वाऽथो निवेश्य क्रमेण ।
साध्यान्त्येभ्यो भावनाभिर्बहुनि मूलाभ्येषां भावना प्रोच्यतेऽतः ॥
वज्राभ्यासौ ज्येष्ठलब्धोस्तदैक्यं ह्रस्वं लब्धोराहतिश्च प्रकृत्या ।
क्षुरा ज्येष्ठाभ्यास युग्ं ज्येष्ठमूलं तत्राभ्यासः क्षेपयोः क्षेपकः स्यात् ॥
ह्रस्वं वज्राभ्यासयोरन्तरं वा लब्धोर्धातो यः प्रकृत्या विनिज्जः ।
धातो यस्य ज्येष्ठयोस्तद्वियोगो ज्येष्ठं क्षेपोत्रापि च क्षेपधातः ॥

Bhāskara II, *Bijaganita, VargaPrakṛti.2-4*

Principle of Composition

The above results have been technically known amongst Indian algebraists as *Bhāvanā* (demonstrated or proved, hence theorem or lemma). The word *bhāvanā* also means "composition or combination" in algebra. *Bhāvanā* may be of two types: *Samāsa Bhāvanā* (or addition Lemma, or additive composition) and *Antara Bhāvanā* (or subtraction Lemma or subtractive composition). Whenever, again, the *bhāvanā* is made with two equal sets of roots and interpolators, it is technically named as *Tulya Bhāvanā* (or composition of equals), and when with two unequal sets of values then it is known as *Atulya Bhāvanā* (or composition of unequals).

Proof of Brahmagupta's Lemmas

It is significant to be indicated that Brahmagupta's Lemmas were rediscovered by Euler in 1764 and by Lagrange in 1768, and a considerable importance was attached to them. Kṛṣṇa, (1580 A.D.) the commentator on the *Bījaganita* of Bhāskara II gives the following proof of Brahmagupta's Lemmas:

Let (α, β) and (α', β') be the two solutions of the equation

$$nx^2 + k = y^2.$$

we have

$$N\alpha^2 + k = \beta^2$$

$$N\alpha'^2 + k = \beta'^2$$

Multiplying the first equation by β'^2 , we get

$$N\alpha^2\beta'^2 + k\beta'^2 = \beta^2\beta'^2$$

Now, substituting the value of factor β'^2 of the interpolator from the second equation, we get

$$N\alpha^2\beta'^2 + k(N\alpha'^2 + k') = \beta^2\beta'^2$$

$$\text{or } N(\alpha^2\beta'^2 + Nk\alpha'^2 + kk') = \beta^2\beta'^2$$

Again, substituting the value of k from the first equation in the second term of the left-hand side expression, we have

$$N\alpha^2\beta'^2 + N\alpha'^2(\beta^2 - N\alpha^2) + kk' = \beta^2\beta'^2$$

$$\text{or } N(\alpha^2\beta'^2 + \alpha'^2\beta^2) + kk' = \beta^2\beta'^2 + N^2\alpha^2\alpha'^2$$

Adding $\pm 2N\alpha\alpha'\beta\beta'$ to both sides, we get

$$N(\alpha\beta' \pm \alpha'\beta)^2 + kk' = (\beta\beta' \pm N\alpha\alpha')^2$$

Brahmagupta's Corollary also follows at once from the above by putting $\alpha'=\alpha$, $\beta'=\beta$ and $k'=k$.

$$N(2\alpha\beta)^2 + k^2 = (\beta^2 \pm N\alpha^2)^2$$

Thus the roots are $x=2\alpha\beta$ and $y=\beta^2 \pm N\alpha^2$ which is the Corollary.

It would be seen that modern historians of mathematics are incorrect when they say that Fermat (1657) was the first to state that the equation $Nx^2+1=y^2$, where N is a non-square integer has an unlimited number of solutions in integers. For this assertion, history takes us to the early Seventh Century A.D. when Brahmagupta wrote his classical treatise, the *Brāhmasphuṭasiddhānta*, and gave the well known two Lemmas and the Corollary to the first Lemma.

Second Lemma of Brahmagupta

In the *Brāhmasphuṭa siddhānta*, we find another important Lemma by Brahmagupta stated as follows :

On dividing the two roots (of a square- Nature) by the square-root of its additive or subtractive, the roots for interpolator unity (will be found).¹

This Lemma when expressed in the modern language of algebra would mean that if $x=\alpha, y=\beta$ be a solution of the equation.

$$Nx^2 + k^2 = y^2$$

then $x=\alpha/k, y=\beta/k$ is a solution of the equation

$$Nx^2 + 1 = y^2.$$

This rule, at another place, has been re-enunciated as follows :

If the interpolator is that divided by a square then the roots will be those multiplied by its square-root.²

1. प्रक्षेपशोधक इति मूले प्रक्षेपके रूपे ।

—BrSpSi. XVIII. 65

2. कर्माच्छिन्ने द्वे पे तत्पदगुणिते तदा मूले ।

—BrSpSi. XVIII. 70

This rule may be expressed in terms of symbols as follows.

Suppose the *Varga-prakṛti* (Square-nature) to be

$$Nx^2 \pm p^2 d = y^2,$$

so that its interpolator (*kṣepa*) $p^2 d$ is exactly divisible by the square p^2 . Then, putting therein $u = x/p$, $v = y/p$, we derive the equation

$$Nu^2 \pm d = v^2$$

whose interpolator is equal to that of the original Square-nature divided by p^2 . It is clear that the roots of the original equation are p times those of the derived equation.

Rational Solution

Indian algebraists have usually suggested the following method to obtain a first solution of $Nx^2 + 1 = y^2$:

Take an arbitrary small rational number, α , such that its square multiplied by the *gunaka* N and increased or diminished by a suitably chosen rational number k will be an exact square.

In other words, we shall have to obtain empirically a relation of the form

$$N\alpha^2 \pm k = \beta^2$$

where α , k , and β are rational numbers. Let us call this relation as the *Auxiliary Equation*. Then by Brahmagupta's Corollary, we get from it the relation

$$N(2\alpha\beta)^2 + k^2 = (\beta^2 + N\alpha^2)^2,$$

or
$$N\left(\frac{2\alpha\beta}{k}\right)^2 + 1 = \left(\frac{\beta^2 + N\alpha^2}{k}\right)^2$$

Hence, one rational solution of the equation $Nx^2 + 1 = y^2$ is given by

$$x = \frac{2\alpha\beta}{k}, \quad y = \frac{\beta^2 + N\alpha^2}{k}$$

Work on the rational solution of the Square-nature has been also done by Śrīpati. In fact, his solution, given in 1039 A.D. is of historical significance. He derives the rational solution without the aid of the "auxiliary equation." He gives the following rule :

Unity is the lesser root. Its square multiplied by the *prakṛti* is increased or decreased by the *prakṛti* combined with an (optional) number whose square-root will be the greater root. From them will be obtained two roots by the Principle of Composition¹

Thus if m^2 be the rational number optionally chosen, one shall have the identity :

$$N.1^2 + (m^2 - N) = m^2,$$

or
$$N.1^2 - (N - m^2) = m^2$$

Then by applying Brahmagupta's Corollary we get

$$N(2m)^2 + (m^2 \vee N)^2 = (m^2 + N)^2$$

$$\therefore N \left(\frac{2m}{m^2 \vee N} \right) + 1 = \left(\frac{m^2 + N}{m^2 \vee N} \right)^2$$

Hence

$$x = \frac{2m}{m \vee N} \quad y = \frac{m^2 + N}{m^2 \vee N}$$

where m is any rational number, is a solution of the equation $Nx^2 + 1 = y^2$.

This rational solution of the *varga-prakṛti* which was used by Śrīpati in 1039 A.D. was rediscovered in Europe by Brouncker in 1657.

We shall close this discussion by taking an illustration from Bhāskara II :

Problem : Tell me, O mathematician, what is that square which multiplied by 8 becomes, together with unity, a square; and what square multiplied by 11 and increased by unity, becomes a square.

This means that we have to solve the equations :

$$8x^2 + 1 = y^2 \quad \dots\dots(i)$$

$$11x^2 + 1 = y^2 \quad \dots\dots(ii)$$

In the second example, let us assume 1 as the lesser root. Following the method of Śrīpati, let us multiply its square by the *prakṛti* (here in eq. ii, *prakṛti* is 11), then let us subtract 2 (an optional number) and then extracting the square-roots we

get the greater root as 3. Hence the statement for the composition is

$$\begin{array}{llll} m=11 & l=1 & g=3 & i=-2 \\ & l=1 & g=3 & i=-2 \end{array}$$

Here m =multiplier (*gunaka* or *prakṛti*), l =lesser root (*kaniṣṭha-mūla*), g =greater root (*jyeṣṭha-mūla*) and i =interpolator (*kṣepa*).

Here we have set down successively the lesser root, greater root and interpolator, and below them again set down the same (See Brahmagupta's Lemmas described by Bhāskara II). Now proceeding as before we obtain the roots for the additive 4 :

$$l=6, g=20, (\text{for}) i=4.$$

Then by the rule :

"If the interpolator (of a *varga-prakṛti* or Square-nature) divided by the square of an optional number be the interpolator (of another Square-nature), then the two roots (of the former) divided by that optional number will be the roots (of the other). Or, if the interpolator be multiplied, their roots should be multiplied."¹

are found the roots for the additive unity

$$l=3, g=10 (\text{for}) i=1.$$

Whence by the Principle of Composition of Equals, we get the lesser and greater roots : $l=60, g=199$ (for) $i=1$. In this way an infinite number of roots can be deduced.

Alternative method:-Bhāskara II has given another method for finding the two roots for the additive unity :

Or divide twice an optional number by the difference between the square of that optional number and the *prakṛti*. This (quotient) will be the lesser root (of a Square-nature) when unity is the additive. From that (follows) the greater root.²

-
1. इष्टवर्गहतः क्षेपः क्षेपः स्यादिष्टभाजिते ।
मूले ते स्तोऽथवा क्षेपः क्षुरणः क्षुरण्ये तदा पदे ॥

Bijaganita II. 5.

2. *Siddhānta-śekhara*, XIV. 32.

Let us solve the first example $8x^2+1=y^2$. We assume the optional number to be 3. Its square is 9; the *prakṛti* of multiplier is 8, their difference is $9-8=1$. Dividing by this twice the optional number (2×3 , i.e. 6), namely 6, we get the lesser root for the additive unity as 6. Whence proceeding as before, we get the greater to be 17. Thus here $x=6$ and $y=17$.

Let us use this method for the equation $11x^2+1=y^2$. Let the optional number be 3. Its square is 9; multiplier or *prakṛti* is 11; the difference is $11-9=2$; dividing by this twice the optional number (2×3), namely 6, we get $6/2=3$, which is the lesser root. Consequently the greater root would be 10, Thus for this equation $x=3$ and $y=10$.

Solution in Positive Integers

The Indian algebraists usually aimed at obtaining solutions of the *varga-prakṛti* or Square-nature in positive integers or *abhinna*. The tentative methods of Brahmagupta and Śrīpati always did not furnish solutions in positive integers. These authors, however, discovered that if the interpolator of auxiliary equation in the tentative method be ± 1 , ± 2 or ± 4 , an integral solution of the equation $Nx^2+1=y^2$ can always be found. Thus Śrīpati says :

If 1, 2 or 4 be the additive or subtractive (of the auxiliary equation), the lesser and greater roots will be integral (*abhinna*)¹.

(i) If $k = \pm 1$, then the auxiliary equation will be

$$N\alpha^2 \pm 1 = \beta$$

where α and β are integers. Then by Brahmagupta' Corollary we get

$$x = 2\alpha\beta \text{ and } y = \beta^2 + N\alpha^2$$

as the required first solution in positive integers of the equation $Nx^2+1=y^2$

1. इष्टवर्गं प्रकृत्योर्यद्विवरं तेन वा भजेत् ।

द्विचमिष्टं कनिष्ठं तत् पदं स्यादेक संयुतौ ।

ततो ज्येष्ठमिष्टान्नयं भावनाभिस्तथेष्टतः ॥

(ii) Let $k=\pm 2$; then the auxiliary equation is

$$N\alpha^2 \pm 2 = \beta^2$$

By Brahmagupta's Corollary, we have

$$N(2\alpha\beta)^2 + 4 = (\beta^2 + N\alpha^2)^2$$

$$\text{or } N(\alpha\beta)^2 + 1 = \left(\frac{\beta^2 + N\alpha^2}{2} \right)^2$$

Hence the required first solution is

$$x = \alpha\beta, y = \frac{1}{2}(\beta^2 + N\alpha^2)$$

$$\text{Since } N\alpha^2 = \beta^2 \mp 2,$$

we have $\frac{1}{2}(\beta^2 + N\alpha^2) = \beta^2 \mp 1 = a$ whole number.

(iii) Now suppose $k = +4$: so that

$$N\alpha^2 + 4 = \beta^2$$

With an auxiliary equation like this, the first integral solution of the equation $Nx^2 + 1 = y^2$ is

$$x = \frac{1}{2}\alpha\beta$$

$$y = \frac{1}{2}(\beta^2 - 2)$$

if α is even; or

$$x = \frac{1}{2}\alpha(\beta^2 - 1)$$

$$y = \frac{1}{2}\beta(\beta^2 - 2);$$

if β is odd.

Thus we find Brahmagupta saying:

In the case of 4 as additive the square of the second root diminished by 3, then halved and multiplied by the second root will be the (required) second root: the square of the second root diminished by unity and then divided by 2 and multiplied by the first root will be the (required) first root (for the additive unity).¹

Datta and Singh has given the following *rationale* of this solution.

$$\text{Since } N\alpha^2 + 4 = \beta^2 \quad (i)$$

$$\text{we have } N(\alpha/2)^2 + 1 = (\beta/2)^2, \quad (ii)$$

Then by Brahmagupta's Corollary, we get

$$N(\alpha\beta/2)^2 + 1 = \left(\frac{\beta^2}{4} + N\frac{\alpha^2}{4} \right)^2$$

1. चतुरधिकेऽन्यपदकृतिस्त्र्यनादलिताऽन्यपद गुणाऽन्यपदम् ॥
अन्यपद कृतिर्व्येकादि हताऽऽन्यपदहताऽऽन्यपदम् ॥

Substituting the value of N in the right-hand side expression from (i), we have

$$N \cdot \left(\frac{\alpha\beta}{2} \right)^2 + 1 = \left(\frac{\beta^2 - 2}{2} \right)^2 \quad (\text{iii})$$

Composing (ii) and (iii),

$$N \left\{ \frac{\alpha}{2} (\beta^2 - 1) \right\}^2 + 1 = \left\{ \frac{\beta}{2} (\beta^2 - 3) \right\}^2$$

Hence $x = \frac{1}{2} \alpha\beta$, $y = \frac{1}{2} (\beta^2 - 2)$;

and $x = \frac{1}{2} \alpha(\beta^2 - 1)$, $y = \frac{1}{2} \beta(\beta^2 - 3)$;

are solutions of $Nx^2 + 1 = y^2$.

If β be even, the first values of (x, y) are integral. If β be odd, the second values are integral.

(iv) Finally, suppose $k = -4$; the auxiliary equation is

$$Na^2 - 4 = \beta^2$$

Then the required first solution in positive integers of

$$Nx^2 + 1 = y^2 \text{ is}$$

$$x = \frac{1}{2} \alpha\beta(\beta^2 + 3) (\beta^2 + 1)$$

$$y = (\beta^2 + 2) \left\{ \frac{1}{2} (\beta^2 + 3) (\beta^2 + 1) - 1 \right\}.$$

Brahmagupta says:

In the case of 4 as subtractive, the square of the second is increased by three and by unity; half the product of these sums and that as diminished by unity (are obtained). The latter multiplied by the first sum less unity is the (required) second root; the former multiplied by the product of the (old) roots will be the first root corresponding to the (new) second root.¹

The rationale of this solution, as given by Datta and Singh is as follows:

$$Na^2 - 4 = \beta^2 \quad (\text{i})$$

$$N(a/2)^2 - 1 = (\beta/2)^2$$

Hence by Brahmagupta's Corollary, we get

$$N \left(\frac{\alpha\beta}{2} \right)^2 + 1 = \left(\frac{\beta^2}{4} + N \frac{\alpha^2}{4} \right)^2$$

1. चतुरस्रेऽन्त्यपद कृती ज्येकयुते वधदलं पृथग्येकम् ।

ज्येकाबाह्यतन्मन्त्रं पदवध गुणमाद्यमान्त्यपदम् ॥

$$= \left\{ \frac{1}{2}(\beta^2 + 2) \right\}^2 \quad (\text{ii})$$

Again applying the Corollary, we get

$$N \left\{ \frac{1}{2} \alpha \beta (\beta^2 + 2) \right\}^2 + 1 = \left\{ \frac{1}{2} (\beta^4 + 4\beta^2 + 2) \right\}^2 \quad (\text{iii})$$

Now by the Lemma we obtain from (ii) and (iii)

$$\begin{aligned} N \left\{ \frac{1}{2} \alpha \beta (\beta^2 + 3) (\beta^2 + 1) \right\}^2 + 1 \\ = [(\beta^2 + 2) \left\{ \frac{1}{2} (\beta^2 + 3) (\beta^2 + 1) - 1 \right\}]^2 \end{aligned}$$

$$\text{Hence } x = \frac{1}{2} \alpha \beta (\beta^2 + 3) (\beta^2 + 1),$$

$$y = (\beta^2 + 2) \left\{ \frac{1}{2} (\beta^2 + 3) (\beta^2 + 1) - 1 \right\}$$

is a solution of $Nx^2 + 1 = y^2$

This can be proved without difficulty that these values of x and y are integral. Since if β is even, $\beta^2 + 2$ is also even. And hence the above values of x and y are integral. On the other hand, if β is odd, β^2 is also odd; under these conditions $\beta^2 + 1$ and $\beta^2 + 3$ are even. In this also, therefore, the above values must be integral.

Putting $p = \alpha\beta$, $q = \beta^2 + 2$, we can write the above solution in the form

$$x = \frac{1}{2} p (q^2 - 1).$$

$$y = \frac{1}{2} q (q^2 - 3).$$

This was the form in which the solution was found by Euler.

Cakravāla or Cyclic Method

We have shown in the preceding articles that the most fundamental step in Brahmagupta's method for the general solution in positive integers of the equation

$$Nx^2 + 1 = y^2$$

where N is a non-square integer, is to form an auxiliary equation of the kind

$$Na^2 + k = b^2$$

where a and b are positive integers and $k = \pm 1, \pm 2$ or ± 4 . From this auxiliary equation, by the Principle of Composition, applied repeatedly whenever necessary, one can derive, as we have already shown above, one positive integral solution of the original *Varga-prakṛti* or Square-Nature. And thence again, by means of the same principle, an infinite number of other solutions in integers can be obtained. How to form an auxiliary equation of

this type was a problem, write Datta and Singh, which could not be solved completely nor satisfactorily by Brahmagupta. In fact, Brahmagupta had to depend on trial. Success in this direction was, however, remarkably attained by Bhāskara II. He evolved a simple and elegant method which assisted in deriving an auxiliary equation having the required interpolator $\pm 1, \pm 2$, or ± 4 , simultaneously with its two integral roots, from another auxiliary equation empirically formed with any simple integral value of the interpolator, positive or negative. This method has been technically known as *Cakravāla* or the *cyclic method*. This is so called because it proceeds as in a circle, the same set of operations being applied again and again in a continuous round. For the details of this method, our reader is requested to consult the Algebra of Bhāskara II and the narrative on this method as given by Datta and Singh under the title "Cyclic Method" in their *History of Hindu Mathematics: Algebra*, 1962 Edition, pp. 161-72.

Solution of Indeterminate Quadratic Equation

It is remarkable to see that Brahmagupta was the first algebraist in the history of mathematics to find a general solution of the indeterminate quadratic equation

$$Nx^2 \pm c = y^2$$

in positive integers. We have the following verse in the *Brāhmasphuṭasiddhānta* in this connection:

From two roots (of a Square-nature or *varga-prakṛti*) with any given additive or subtractive, by making (combination) with the roots for the additive unity other first and second roots (of the equation having) the given additive or subtractive (can be found).¹

Let us take the following two equations:

$$a_1k = an + b; \text{ and } b_1k = bn + Na$$

From them we get : by eliminating n

$$a_1b - ab_1 = 1$$

1. रूप प्रद्वे पपदे पृथगिष्ट्वे व्यशोध्यमूलान्याम् ।

कृत्वान्यथापदे ये प्रद्वे पेतेने ॥

Hence $b_1 = \frac{a_1 b - 1}{a} =$ a whole number.

$$\begin{aligned} \text{Now } n^2 - N &= \frac{(a_1 k - b)^2 - N a^2}{a^2} \\ &= \frac{a_1^2 k^2 - 2b k a_1 + b^2 - N a^2}{a^2} \\ &= \frac{k(a_1^2 k - 2b a_1 + 1)}{a^2} \end{aligned}$$

Therefore $\frac{k}{a^2}(a_1^2 k - 2b a_1 + 1)$ is a whole number.

Since a, k have no common factor, it follows that

$$\frac{a_1^2 k - 2b a_1 + 1}{a^2} = \frac{n^2 - N}{k} = k_1 = \text{an integer.}$$

$$\begin{aligned} \text{Also } k_1 &= \frac{n^2 - N}{k} = \frac{a_1^2 k - 2b a_1 + 1}{a^2} \\ &= \frac{a_1^2(b^2 - N a^2) - 2b a_1 + 1}{a^2} \\ &= \left(\frac{a_1 b - 1}{a} \right)^2 - N a_1. \end{aligned}$$

Thus having known a single solution in positive integers of the equation $Nx^2 \pm c = y^2$, says, Brahmagupta, an infinite number of other integral solutions can be obtained by making use of the integral solutions of $Nx^2 + 1 = y^2$. If (p, q) be a solution of the former equation found empirically and if (α, β) be an integral solution of the latter, then by the principle of Composition

$$x = p\beta \pm q\alpha; y = q\beta \pm Np\alpha$$

will be a solution of the former. Repeating the operations, we can easily deduce as many solutions as we like.

$$\text{FORM } Mn^2x^2 \pm c = y^2:$$

In this connection, Brahmagupta says :

If the remainder is that divided by a square, the first root is that divided by its root¹.

This seems to mean that if we have the equation

$$Mn^2x^2 \pm c = y^2 \tag{i}$$

such that the multiplier (i.e. the coefficient of x^2) is divisible

1. कर्वाच्छिन्ने गुणके प्रथमं तन्मूलं भाजितं भवति ।

by n^2 , then we are justified in saying that if we put $nx=u$, the equation (i) becomes $Mu^2 \pm c = y^2$ (ii), and clearly the first root of (i) is equal to the first root of (ii) divided by n . The corresponding second root will be the same for both the equations.

$$\text{FORM } a^2x^2 \pm c = y^2 :$$

We find Brahmagupta giving the following rule in this connection : This is a solution of a particular form of a *varga-prakṛti* or Square-nature.

If the multiplier be a square, the interpolator divided by an optional number and then increased and decreased by it, is halved. The former (of these results) is the second root; and the other divided by the square-root of the multiplier is the first root.¹

Thus the solutions of the equation

$$a^2x^2 \pm c = y^2$$

are :

$$x = \frac{1}{2a} \left(\frac{\pm c}{m} - m \right)$$

$$y = \frac{1}{a} \left(\frac{\pm c}{m} + m \right)$$

where m is an arbitrary number.

Bhāskara II and Nārāyaṇa have also given the same solutions as proposed by Brahmagupta.

Rational Geometrical Figures

In the days of the *Taittiriya Samhitā* and the *Śatapatha Brāhmaṇa*, Indian mathematicians got familiarity with the solution of such equations

$$x^2 + y^2 = z^2$$

and the results were arrived geometrically on the basis of the law of rectangle as propounded by Baudhāyana in the *Śulba Sūtras* and which goes by his name. The reader is referred to the Chapter on Baudhāyana, the first Geometer in the author's "*Founders of Sciences in Ancient India*". Baudhāyana (c. 800 B.C.) gave a

1. कौ गुणके चोपः केनचिद्वृत्त्युतोन्नितो दलितः ।

प्रथमोऽन्यनूलमन्त्यो गुणकारपदोद्धृतः प्रथमः ॥

method of transforming a rectangle into a square, which is equivalent to the algebraic identity :

$$mn = \left(m - \frac{m-n}{2}\right) - \left(\frac{m-n}{2}\right)$$

where m, n , are any two arbitrary numbers.

Brahmagupta in connection with the solution of rational triangles says :

The square of the optional (*iṣṭa*) side is divided and then diminished by an optional number; half the result is the upright, and that increased by the optional number gives the hypotenuse of a rectangle.

We shall put these statements of Brahmagupta in the algebraic language thus : If m, n be any two rational numbers, then the sides of a right-angled triangle will be

$$m, \frac{1}{2} \left(\frac{m^2}{n} - n \right), \frac{1}{2} \left(\frac{m^2}{n} + n \right)$$

This Sanskrit term *iṣṭa* may either mean "given" or "optional". With the former meaning the rule would imply the method of finding rational right angles having a given leg.

Brahmagupta was the first to give a solution of the equation $x^2 + y^2 = z^2$ in integers. His solution is

$$m^2 - n^2, 2mn, m^2 + n^2.$$

m and n being two unequal integers.²

Thus if $m=7$ and $n=4$, then $m^2 - n^2 = 33$, $2mn = 56$ and $m^2 + n^2 = 65$; then the three numbers 33, 56 and 65 bear the relation $33^2 + 56^2 = 65^2$.

Mahāvira (850 A.D.) also states

The difference of the squares (of two elements) is the upright, twice their product is the base and the sum of their squares is the diagonal of a *generated* rectangle.²

Isosceles Triangles with Integral Sides : The following statement of Brahmagupta in this connection is very significant :

1. इष्टस्य मुजस्य कृतिर्मको नेष्टेन तदलं कोटिः ।

आयतचतुरस्रस्य द्वे त्रस्येष्टाधिका कर्णः ॥

—BrSpSi. XII. 35

2. GSS. VII. 90½

The sum of the squares of two unequal numbers is the side; their product multiplied by two is the altitude, and twice the difference of the squares of those two unequal numbers is the base of an isosceles triangle.¹

Thus if m, n be two integers such that m is not equal to n , the sides of all rational isosceles triangles with integral sides are given by

$$m^2 + n^2, m^2 + n^2, 2(m^2 - n^2)$$

and the altitude of the triangle is $2mn$.

This method was also followed by Mahāvīra and other Indian mathematicians. In fact, their solutions are based on the juxtaposition of two rational right triangles, equal so that they have a common leg. It is remarkably a powerful device, for every rational triangle or quadrilateral may be formed by the juxtaposition of two or four rational right triangles.

Isosceles Triangles with a Given Altitude

Here we have a rule given by Brahmagupta for finding out all rational isosceles triangles possessing the same altitude :

The (given) altitude is the producer (*karaṇi*). Its square divided by an optional number is increased and diminished by that optional number. The smaller is the base and half the greater is the side.²

Thus if m be any rational number then for a given definite altitude a , the sides of the rational isosceles triangles are $\frac{1}{2}\left(\frac{a^2}{m} + m\right)$ each and the base is $\frac{a^2 - m^2}{m}$. We shall illustrate it by an example taken from the commentary of Pṛthūdaka Svāmī. The given altitude is 8; let us take any rational number $m=4$ then the two equal sides of the isosceles are given by $\frac{1}{2}\left(\frac{8^2 + 4^2}{4}\right) = 10$ each and the base is $\frac{8^2 - 4^2}{4} = 12$. Thus the three sides of the

1. कृति युतिर सदृशराशयोर्बाहुर्धोतो द्विसंगुणो लम्बः ।

कृत्यन्तरमसदृशयोर्दिगुणं द्विसमन्विमुज्ज भूमिः ॥

—BrSpSi. XII. 33

2. करणी लम्बस्तकृतिरिच्छतेऽथेन संयुताऽल्पा भूः ।

अधिको द्विह तो बाहुः संचे प्यो यद्वधो कर्माः ॥

—BrSpSi. XII. 37.

rational isosceles triangle with altitude 8 are (10,10, 12).

Rational Scalene Triangles: Brahmagupta lays down the following rule in the case of rational scalene triangle ;

The square of an optional number is divided twice by two arbitrary numbers; the moieties of the sums of the quotients and (respective) optional numbers are the sides of a scalene triangle; the sum of the moieties of the differences is the base.¹

In other words, if m, p, q are any rational numbers, then the sides of a rational scalene triangle are :

$$\frac{1}{2} \left(\frac{m^2}{p} + p \right), \quad \frac{1}{2} \left(\frac{m^2}{q} + q \right), \\ \frac{1}{2} \left(\frac{m^2}{p} - p \right) + \frac{1}{2} \left(\frac{m^2}{q} - q \right)$$

Here the altitude (m), area and segments of the base of this triangle are all rational.

Thus putting $m=12$, $p=6$, and $q=8$ in Brahmagupta's general equation, Pṛthūdaka Svāmī derives a scalene triangle with sides (13,15) and (14) altitude (12), area (84 and the segments of the base (5) which are all integral numbers.

$$\frac{1}{2} \left(\frac{m^2}{p} + p \right) = \frac{1}{2} \left(\frac{12^2}{6} + 6 \right) = 15;$$

$$\frac{1}{2} \left(\frac{m^2}{q} + q \right) = \frac{1}{2} \left(\frac{12^2}{8} + 8 \right) = 13$$

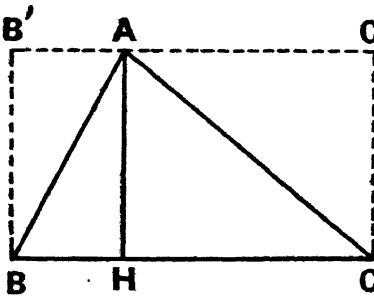


Fig. 19

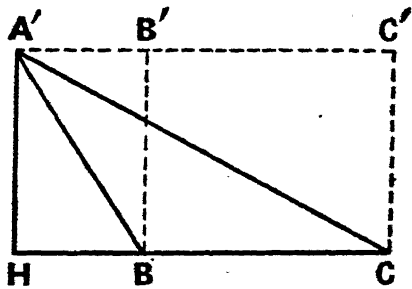


Fig. 20

1. इष्टद्वयेन भक्तो द्विषेष्ट वर्गं फलेष्टयोर्गार्धे ।
विषमत्रिभुजस्य भुजाविष्ट्येन फलार्धयोगो भूः ॥

Thus the two sides of the rational scalene triangle are 15 and 13. The base is .

$$\frac{1}{2} \left(\frac{12^2}{6} - 6 \right) + \frac{1}{2} \left(\frac{1^2}{8} - 8 \right) = 9 + 5 = 14$$

The altitude is $m=12$; area is equal to $\frac{\text{base} \times \text{altitude}}{2}$

$$= \frac{14 \times 12}{2} = \text{and the segments are } \sqrt{13^2 - 12^2} = 5 \text{ and}$$

$$\sqrt{(15^2 - 12^2)} = 9. \text{ Thus they are all integers.}$$

Rational Isosceles Trapeziums

Brahmagupta has given us a method of obtaining such isosceles trapeziums whose sides, diagonals, altitude, segments and area are all rational numbers. His rule is as follows :

The diagonals of the rectangle (generated) are the flank sides of an isosceles trapezium; the square of its side is divided by an optional number and then lessened by that optional number and divided by two; (the result) increased by the upright is the base and lessened by it is the face.¹

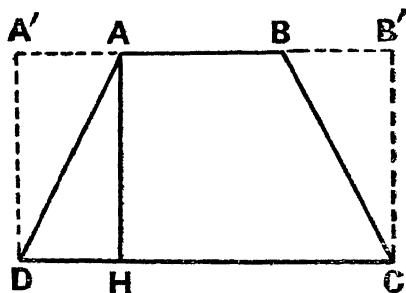


Fig. 21

Here in the figure, we have the isosceles trapezium ABCD of which CD is the base and AB is known as the face. According to Brahmagupta's rule, we have (p being the optional number).

$$CD = \frac{1}{2} \left(\frac{4m^2n^2 - p}{p} \right) + (m^2 - n^2) \quad (\text{base})$$

$$AB = \frac{1}{2} \left[\frac{4m^2n^2}{p} - p \right] - (m^2 - n^2) \quad (\text{face})$$

$$DH = (m^2 - n^2) \quad (\text{upright})$$

1. आहतकर्षो वाहू मुजकृतिरिष्टेन भावितेष्टोना ।

दिहता कोट्यधिका भूमि खमना दिसमचतुरस्रे ॥

$$AD = BC = m^2 + n^2 \quad (\text{the sides of the trapezium})$$

$$HC = \text{base-upright} = \frac{1}{2} \left[\frac{4m^2n^2}{p} - p \right] \quad (\text{segment})$$

$$AC = BD = \left[\frac{4m^2n^2}{p} + p \right] \quad (\text{diagonal})$$

$$AH = 2mn \quad (\text{altitude})$$

$$ABCD = mn \left[\frac{4m^2n^2}{p} - p \right] \quad (\text{area})$$

By choosing the values of m , n and p suitably, the values of all the dimensions of the isosceles trapezium can be made integral. Prthūdaka Svāmī starts with the rectangle (5, 12, 13) and suitably takes p as 6; then he calculates out the dimensions of the trapezium: flank sides (AD and BC) = 13, base = 14, and base = 4, altitude (AH) = 12, segments of base (DH and HC) = 5, and 9, diagonals (AC and BD) = 15, area $ABCD$ = 108. All these values are integers.

In this example, the rectangle chosen is (5, 12, 13) which is $AA'DH$, where $AD = m^2 + n^2 = 13$

$$\text{and } DH = m^2 - n^2 = 5$$

whence by adding the two we have

$$2m^2 = 18$$

This gives the value of $m = 3$, and hence $n = 2$. Prthūdaka Svāmī has taken the value of $p = 6$ by choice. Putting these values of m , n and p , the values for the dimensions of the isosceles trapezium follow from the expressions given by Brahmagupta.

$$CD = \frac{1}{2} \left(\frac{4 \cdot 3^2 \cdot 2^2}{6} - 6 \right) + \left(3^2 - 2^2 \right) = 9 + 5 = 14 \text{ (base)}$$

$$\text{Face} = 9 - 5 = 4$$

$$\text{Sides } AD = BC = 3^2 + 2^2 = 13$$

and so on for the other dimensions.

Rational Trapeziums With Three Equal Sides

This problem is very much the same as one of the rational isosceles trapezium with the only difference that in this case one of the parallel sides is also equal to the slant sides. We

have the following solution of this problem from Brahmagupta :

The square of the diagonal (of a *generated* rectangle) gives three equal sides; the fourth (is obtained) by subtracting the square of the upright from thrice the square of the side (of that rectangle). If greater, it is the base; if less, it is the face.¹

As before, the rectangle generated from m, n is given by $(m^2-n^2, 2mn, m^2+n^2)$, that is these are the three sides of the right triangle, which correspond to the two sides and the diagonal of the rectangle generated by them. Let us suppose, we have a trapezium ABCD whose sides AB, BC and AD are equal, then

$$AB = BC = AD = (m^2+n^2)^2$$

$$CD = 3(2mn)^2 - (m^2-n^2)^2 = 12m^2n^2 - m^4 - n^4$$

$$\text{or } CD = 3(m^2-n^2)^2 - (2mn)^2 = 3m^4 + 3n^4 - 10m^2n^2.$$

Prthūdaka Svāmī has taken an illustration, where $m=2, n=1$ and he deduces two rational trapeziums with three equal sides (25, 25, 25, 39) and (25, 25, 25, 11).

The segment (CH), altitude (AH), diagonals (AC, BD) and area of this trapezium are also rational, and given by :

$$CH \text{ (segment) } = 6m^2n^2 - m^4 - n^4$$

$$AH \text{ (altitude) } = 4mn(m^2-n^2)$$

$$AC = BD \text{ (diagonals) } = 4mn(m^2+n^2)$$

$$ABCD \text{ (area) } = 32m^3n^3(m^2-n^2).$$

Rational Inscribed Quadrilaterals

We find in the *Brahmasphuṭasiddhānta* a remarkable proposition formulated by Brahmagupta :

To find all quadrilaterals which will be inscribable within circles and whose sides, diagonals, perpendiculars, segments (of sides and diagonals by perpendiculars from vertices as also of diagonals by their intersection), areas, and also the diameters of the

1. कर्षकृत्तिस्त्रिसम मुखास्त्रयश्चतुर्थो विरोध्य कोटि कृत्तिम् ।

बाहुकृतेस्त्रिमुखाया वधधिको भूमुसं द्वीनः ॥

circumscribed circles will be expressible in integers. Such quadrilaterals we shall call as *Brahmagupta Quadrilaterals*.

The solution of this formidable problem has been given by Brahmagupta as follows :

The upright and bases of two right-angled triangles being reciprocally multiplied by the diagonals of the other will give the sides of a quadrilateral of unequal sides : (of these) the greatest is the base, the least is the face, and the other two sides are the two flanks.¹

Taking Brahmagupta's integral solution, the sides of the two right triangles of reference are given by :

$$(1) \quad m^2 - n^2, 2mn, m^2 + n^2;$$

$$(ii) \quad p^2 - q^2, 2pq, p^2 + q^2;$$

where m, n, p, q are integers. Then the sides of the *Brahmagupta Quadrilateral* are.

$$(m^2 - n^2)(p^2 + q^2), (p^2 - q^2)(m^2 + n^2), 2mn(p^2 + q^2), 2pq(m^2 + n^2) \quad (\text{Arrangement A})$$

Prthudaka Svāmi has illustrated the rational inscribed quadrilateral by taking an example of the right angle triangles.

$$(i) \quad (3, 4, 5) (m^2 - n^2 = 3, m^2 + n^2 = 5, \text{ whence } m = 2, n = 1)$$

$$(ii) \quad (5, 12, 13) (p^2 - q^2 = 5, p^2 + q^2 = 13, \text{ whence } p = 3, q = 2)$$

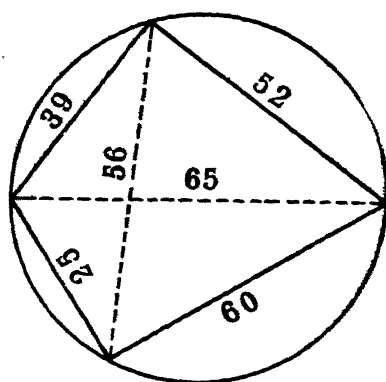


Fig. 22

Substituting these values in the above equations, we get the sides of the quadrilateral as (39, 25, 52 and 60).²

1. जाल्यद्वय कोटिमुजाः परकर्णगुणा मुजाश्चतुर्विधमे ।

अधिको भूमु खहीनो बाहुदितयं मुजावन्त्यौ ॥

—BrSpSi. XII. 38

2. The diagonals of this quadrilateral are given by Bhāskara II as 56 (=3.12+4.5) and 63 (=4.12+3.5).

(Cont. on page 268)

Put in other words, this means that one has to solve the following equations :

$$(i) \quad 5x-25 = y^2$$

$$(ii) \quad 10x-100 = y^2$$

$$(iii) \quad 83x-7635 = y^2$$

Prthūdaka Svāmī, the commentator on the *Brāhmasphuṭa-siddhānta* proceeds to solve these equations as follows :

(1.1) Suppose $y = 10$; then $x = 125$. Or put $y = 5$; then $x = 10$.

(2.1) Suppose $y = 10$; then $x = 20$.

(3.1) Assume $y = 1$; then $x = 92$.

He then remarks that by virtue of the multiplicity of suppositions there will be an infinitude of solutions in every case, But no method has been given either by Brahmagupta or his commentator to obtain the general solution.

Double Equations of the First Degree

Perhaps we have the earliest reference of the simultaneous indeterminate quadratic equations of the type

$$x \pm a = u^2$$

$$x \pm b = v^2$$

in the *Bhaskāli Manuscript* (Folio 59, recto).

Brahmagupta gives the solution of such simultaneous indeterminate quadratic equations of a general case as follows :

The difference of the two numbers by the addition or subtraction of which another number becomes a square, is divided by an optional number and then increased or decreased by it. The square of half the result diminished or increased by the greater or smaller (of the given number) is the number (required).¹

Expressed in the language of algebra, shall have :

$$= \frac{1}{2} \left\{ \frac{1}{2} \left(\frac{a-b}{m} \pm m \right) \right\}^2 \mp a$$

1. यास्यां कृतिरधिको नस्तदन्तरं हत युतो न मिष्टेन ।

तद्वल कृतिरधिकोनाऽधिकयो रधिको न यो राशिः ॥

$$\text{or } x = \left\{ \frac{1}{2} \left(\frac{a-b}{m} \mp m \right) \right\}^2 \mp b$$

where m is an arbitrary number.

Datta and Singh has given the *rationale* of this method as follows :

$$u^2 = x \pm a; \quad v^2 = x \pm b,$$

From them, we have $u^2 - v^2 = \pm a \mp b$

Therefore $u - v = m$

$$\text{and } u + v = \frac{\pm a \mp b}{m},$$

where m is arbitrary. Hence

$$u = \frac{1}{2} \left(\frac{\pm a \mp b}{m} + m \right) = \pm \frac{1}{2} \left(\frac{a-b}{m} \pm m \right)$$

Since it is obviously immaterial whether u is taken as positive or negative, we have

$$u = \frac{1}{2} \left(\frac{a-b}{m} \pm m \right)$$

$$\text{Similarly } v = \frac{1}{2} \left(\frac{a-b}{m} \mp m \right)$$

$$\text{Therefore } x = \left\{ \frac{1}{2} \left(\frac{a-b}{m} \pm m \right) \right\}^2 \mp a,$$

$$\text{or } x = \left\{ \frac{1}{2} \left(\frac{a-b}{m} \mp m \right) \right\}^2 \mp b,$$

where m is an arbitrary number.

Now we shall take up another particular case, for which Brahmagupta has given a rule :

The sum of the two numbers the addition and subtraction of which make another number (severally) a square, is divided by an optional number and then diminished by that optional number. The square of half the remainder increased by the subtractive number is the number (required)¹.

In the algebraic notations, we shall express it as follows :

1. यैरुनो यैश्च युतो रूपैर्वर्गस्तदैवयमिष्टं हृतम् ।

इष्टोनं तद्वल कृतिरुनाऽभ्यधिका भवति राशिः ॥

—BrSpSi. XVIII. 71

$$y = \frac{1}{a} \left(\frac{ad+bc}{m} + b \right)$$

if $b > c$ and $m > \frac{ad+bc}{m}$. If these conditions be reversed then x and y will have their values interchanged.

Datta and Singh have given the following *rationale* of these solutions :

$$axy = bx + cy + d,$$

$$\text{or } a^2xy - abx - acy = ad,$$

$$\text{or } (ax - c)(ay - b) = ad + bc.$$

Suppose $ax - c = m$, a rational number;

$$\text{then } ay - b = \frac{ad+bc}{m}.$$

Therefore,

$$x = \frac{1}{a}(m + c)$$

$$y = \frac{1}{a} \left(\frac{ad+bc}{m} + b \right)$$

Or, we may put $ay - b = m$;

in that case, we shall have $ax - c = \frac{ad+bc}{m}$;

$$\text{whence } x = \frac{1}{a} \left(\frac{ad+bc}{m} + c \right),$$

$$y = \frac{1}{a}(m + b).$$

Brahmagupta's own rule.

Whilst the rule given above is ascribed to an unknown author, Brahmagupta's own rule for the solution of a quadratic indeterminate equation involving a factum is as follows :

With the exception of an optional unknown, assume arbitrary values for the rest of the unknowns, the product of which forms the factum. The sum of the product of these (assumed values) and the (respective) coefficients of the unknowns will be absolute quantities. The continued products of the assumed values and of the coefficient of the factum will be the coefficient of the optionally (left out) unknown. Thus the solution

is effected without forming an equation of the factum
Why then was it done so ?¹

Datta and Singh think that the reference in the latter portion of this rule is to the method of the unknown author :

"*Kim krtam tadatah*" ? The principle underlying Brahmagupta's method is to reduce, like the Greek Diophantus (c.275 A.D.), the given indeterminate equation to a simple determinate one by assuming arbitrary values for all the unknowns except one. So undoubtedly it is inferior to the earlier method.

We now take an illustrative example from Brahmagupta :

On subtracting from the product of signs and degrees of the Sun, three and four times (respectively) those quantities, ninety is obtained. Determining the Sun within a year (one can pass as a proficient) mathematician.

If we presume x to denote the signs and y the degrees of the Sun, then the equation would be :

$$xy - 3x - 4y = 90$$

Prthūdaka Svāmī solves it in two ways :

(i) Let us assume the arbitrary number to be 17. then

$$x = \frac{1}{2} \left(\frac{90.1 + 3.4}{17} + 4 \right) = 10$$

$$y = \frac{1}{2} (17 + 3) = 10$$

(ii) Let us assume arbitrarily $y = 20$. On substituting this value of y in the above equation, we get

$$20x - 3x = 170$$

$$\text{whence } x = 10.$$

1. भावितके यद्घातो विनष्टवर्णेन तत्प्रमाणानि ।
कृत्वेष्टानि तदाहत वर्णैक्यं भवति रूपाणि ॥
वर्णं प्रमाणभाविता घातो भवतीष्ट वर्णं संख्यैवन् ।
सिध्यति विनाऽपि भावित-उमकरणात् किं कृत् तदतः ॥ —BrSpSi. XVIII. 62-63
2. भानो राश्यंशवधत् त्रिचतुर्गुणितान् विशोध्य राश्यंशान् ।
नवर्ति दृष्ट्वा सूर्यं कुर्वन्नावत्सराद् गणकः ॥ —BrSpSi. XVIII. 61.

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