CHAPTER VIII

Brahmagupta and Arithmetic

Scope of Gaṇita

The word *Gaṇita* means the science of calculation. The term occurs in the *Vedāṅga Jyautiṣa* (c. 1200 B. C.):

> Just as the crest is to the peacocks, and just as the head-gem is to the snakes, so the Gaṇita among the Vedāṅga Śāstras stands at the head.[1] (VJ. 4)

In the ancient Buddhistic literature, we find mention of three classes of *gaṇita* : (i) *mudrā* (finger arithmetic). (ii) *gaṇanā* (mental arithmetic), and (iii) *saṁkhyāna* (higher arithmetic in general). In the *Brāhmasphuṭasiddhānta*, Brahmagupta uses the word *gaṇita* in the sense of entire calculations. His *gaṇitādhyāya* (Chapter XII) includes ;

> (i) *Miśraka* (mixtures). (ii) *Śreḍhī* (series). (iii) *Kṣetra* plane figures), (iv) *Vṛtta-kṣetra* (circles), (v) *Khāta* (excavations), (vi) *Citi* (piles of bricks), (vii) *Krākacika* (sawn pieces of timber), (viii) *Rāśi* (heaps or mounds of grain, and (ix) *Chāyā* (shadow).

Brahmagupta also uses the term *Dhūlikarma* (literally meaning "dustwork") for higher mathematics :

> The one learned man who knows the *dhūlikarma* or the science of mathematics as propounded by Brahmagupta would far excell them in learning who are taught the calculations according to Āryabhaṭa. Viṣṇucandra and others.[2]

> In these ten chapters of the *Brahmasiddhānta* has been given the *dhūlikarma* or the science of entire calculations

1. यथा शिखा मयूराणां नागानां मणयो यथा ।
तद् वद् वेदांग-शास्त्राणां गणितं मूर्धनि स्थितम् ॥ *VJ. 4.*

2. नावार्यो ज्ञातेरपि तन्त्रैरार्यभटविष्णुचन्द्राद्यैः ।
यो ब्रह्म धूलिकर्मविदाचार्यत्वं भवति तस्य ॥ *—BrSpSi. X. 62.*

> which is faultless.[1]
>
> This science of *dhūlikarma* has not been imparted by great teachers for blasphemy. One who would be using it for this purpose would lose all good name.[2]

Brahmagupta uses the term *gaṇita* only for those calculations which are of arithmetical in nature. The science of algebra, the foundations of which was laid by Āryabhaṭa I, was named as *kuṭṭaka* or *kuṭṭākāra* by Āryabhaṭa, and in the *Brāhmasphuṭa-siddhānta* also it is separately dealt with under *Kuṭṭādhyāya* or *kuṭṭakādhāya* (Chapter XVIII). Later on the term *bījagaṇita* was specifically given to the science of algebra.

The *Kuṭṭādhyāya* of the *Brāhmasphuṭasiddhānta* deals with the (i) concept of *kuṭṭaka* (pulveriser), addition of positive and negative as well as zero quantities, equations in one unknown (*eka-varṇa samīkaraṇa*), equations in several unknowns (*aneka-varṇa samīkaraṇa*), equations involving products of unknowns (*bhāvita*) and quadratic equations (*varga-prakṛtiḥ*) (Chapter XVIII of the *Brāhmasphuṭasiddhānta*).

Āryabhaṭa, Bhāskara and Brahmagupta use Place Value Notations.

In Europe the first definite traces of the place-value numerals are found in the tenth and eleventh centuries, but the numerals came into general use in mathematical text books only in the seventeenth century. In India, however, Āryabhaṭa I (499), Bhāskara I (522), Lalla (c. 598) and Brahmagupta (628) all use the place value numerals. There is no trace of any other system in their works. Perhaps in this country we had the place value system as early as 200 B.C. if not earlier. The use of a symbol for zero is found in Piṅgala's *Chandaḥ Sūtra* (perhaps of 200 B.C.). In literature, we have an indication of the place value from about 100 B.C. and later in the Purāṇas from the second to the fourth century A.D. The *Bakhaśālī Manuscript* (perhaps of 200 A.D.) uses the place-value notations. The earliest use of the place value principle with the letter numerals

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1. ग्रहयोगोत्र ग्रहयुतिरार्यात्रिशतियुताष्टसप्तत्या ।
अध्यायैर्दशभिर्धूलिकर्म वोच्चैर्विना ब्रह्मे ॥ —*BrSpSi*. X. 66.

2. गुरुणा न धूलिकर्म प्रतिकंचुककारिणे प्रदातव्यम् ।
दत्त सुकृतप्रणाशं कुरुते प्रतिकंचुकं यस्य ॥ —*BrSpSi*. X. 67

is found in the works of Bhāskara I about the beginning of the sixth century A.D. Thus for 3179, the expressive words are *Navādrirūpāgni*[1] (*nava* 9, *adri* 7. *rupa* 1 and *agni* 3). Similarly in the *Brāhmasphuṭasiddhānta*, for a large number like 2296828522, the expressive terms are DVIYAMAŚARĀṢṬAPAKṢAVASŪRA-SANAVADVIYAMĀḤ(*Dviyama* two twos 22, *Śara* 5, aṣṭa 8, *pakṣa* 2, *vasū* 8, *rasa* 6. *nava* 9, *dviyamāḥ* 22).[2] Such usages are to be found in all works, which clearly state the place value concept was popular as a routine. From India, this system reached Arabia. During the reign of the Khalif Al-Mansur (753-774 A.D.) there came embassies from Sindh to Baghdad, and among them were scholars, who brought along with them several works on mathematics including the *Brāhmasphuṭasiddhānta* and the *Khaṇḍa-khādyaka* of Brahmagupta: With the help of these scholars, Al-fazari, perhaps also Yakub ibn Tarik, translated them into Arabic. Both works were largely used and exercised great influence on Arab mathematics. It was on that occasion that the Arabs first became acquainted with a scientific system of astronomy. It is acceptable to all writers on the subject that it was at that time that the Hindu numerals were first definitely introduced amongst the Arabs. Arabs at first adopted the *ghobar* form of numerals which they had already obtained (but without zero) from the Alexandrians or from the Syrians. This they continued for about two centuries, but since they were not suited to their right-to-left script, they gave them up and adopted the more convenient ones. For a detailed discussion on how numerals went to the west from India and spread in Europe one is referred to this discussion in the *History of Hindu Mathematics*. Part I by Datta and Singh (1935, Single volume Edition, 1962, pp. 83-104). It is remarkable that Brahmagupta's works like the *Brāhma-sphuṭasidhānta* and the *Khaṇḍakhādyaka* became instrumental in the spread of the place-value notation in the neighbouring countries of the Middle East, and from their this system spread into Europe.

Operations and Determinations in Pāṭīgaṇita

The word *Pāṭīgaṇita* is a compound formed from the words *pāṭī*, meaning 'board', and *gaṇita*, meaning 'science of calculation',

1. *MBh. 1. 4;*
2. *BrSpSi 1-16.*

hence it means the science of calculation which requires the uses of writing material (the board). The word *pāṭi* is not Sanskrit (it originated in the non-Sanskrit literature in India); the oldest term in Sanskrit for the board is *Phalaka* or *paṭṭa*. However this term got currency in the Sanskrit literature also about the beginning of the seventh century. Brahmagupta does not use the term *pāṭīgaṇita*: he favours the use of the term *dhūlikarma* or writing figures on dust spread on a board or on the ground. The word *pāṭīgaṇita* was translated into Arabic as *ilm-hisab-al-takht* (calculation on board) and the word *dhūlikarm a* as *hisab-al-ghobār* (cal culation on dust).

Brahmagupta, in the very first verse in the Chapter XII (*Gaṇitādhyāya*) refers to twenty operations (*parikarma*) and eight determinations :

> He who distinctly and severally knows the twenty logistics, addition etc.. and the eight determinations (*vyavahāra*) inculding (measurement by) shadow is a *gaṇaka* (mathe-m atician).[1]

The commentators have given the list of these logistics (*parikarma*) and determinations (*vyavahāra*) as follows;

(A) *Parikarma* or logistics

1. Saṁkalitam (addition)
2. Vyavakalitam (subtraction)
3. Gaṇanam (multiplication)
4. Bhāgahāraḥ (division)
5. Vargaḥ (square)
6. Vargamūlam (square-root)
7. Ghanaḥ (cube)
8. Ghanamūlam (cube root)
9. 13. Five standard forms of fractions (Pañca-jāti)
14. Trairāśikam (the rule of three)
15. Vysta-trairāśikam (the inverse rule of three)
16. Pañca-rāśikam (the rule of five)
17. Sapta-rāśikam (the rule of seven)
18. Nava-rāśikam (the rule of n ine)

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1. परिकर्म विंशतिं यः संकलिताद्यां पृथक् विजानाति ।
अष्टौ च व्यवहारान् छायान्तान् भवति गणकः सः ॥ —*BrSpSi*. *XII*. *1*

19. Ekādaśa-rāśikam (the rule of eleven)
20. Bhāṇḍa-pratibhāṇḍam (barter and exchange)

(B) *Vyavahāra* or determinations

1. Miśrakaḥ (mixture)
2. Śreḍhī (progession or series)
3. Kṣetram (plane figures)
4. Khātam (excavation)
5. Citiḥ (stock)
6. Krākacikaḥ (saw)
7. Rāśiḥ (mound)
8. Chāyā (shadow)

Of the operations enlisted here, the first eight have been considered fundamental by later writers as Mahāvīra. The operations of duplation and mediation (doubling and halving) were considered fundamental by Arabs, Greeks and Egyptians; since they were not familiar with the place-value system.

Mathematics in this country developed as an aid to astronomy, and therefore, for the first time we find Āryabhṭa(499AD.) in his *Āryabhaṭīya* describing as a special section (*Gaṇitapāda*). Brahmagupta (628 A.D.) also followed Āryabhaṭa in this respect and gave the science of calculation (*gaṇita*) a special place in his treatise on astronomy. The *Siddhānta* treatises, earlier than those of Āryabhaṭa and Brahmagupta do not contain a chapter exclusively devoted to *gaṇita* (the *Sūrya-Siddhānta* and the Siddhāntas of *Vasiṣṭha,Pitāmaha* and *Romaka* are thus without *gaṇita* chapetrs). Later on Bhāskara I and Lalla also did not include *gaṇita* as a section or chapter in their treatises.It is said, however, that Lalla wrote a separate treatise on *Pāṭīgaṇita.*

It may further be remarked here that Āryabhaṭa I gives the rules for finding the square and cube-roots only whilst Brahmagupta gives the cube-root rule only (*BrSPSi.* XII. 7).

Multiplication

Undoubtedly the common Indian name 'multiplication' is '*guṇana*', this term occurs in the Vedic literature also. The other terms for this logistics are *hanana, vadha, kṣaya* etc., which all mean 'killing' or 'destroying.' The synonyms of '*hanana*' (killing) for multiplication have been used by Āryabhaṭa I (499). Brahmagupta (628), Śrīdhara (c.750) and later writers, and these terms also occur in the Bakhaśālī Manuscript.

Āryabhaṭa I does not mention the everyday methods of multiplication in his *Āryabhaṭīya* probably because they were too elementary to be included in a Siddhānta work. Brahmagupta, however, in a supplement to the section on mathematics in his *Siddhānta*, gives the names of some methods with very brief descriptions of the processes:—

> The multiplicand repeated, as in *gomūtrikā* as often as there are digits in the multiplier, is severally multiplied by them and (the results) added according to places; this gives the product. Or the multiplicand is repeated as many times as there are component parts in the multiplier.[1]

(the word *bheda* occurring in the verse has been translated as "integrant portions" by Colebrooke p. 319. Again by the term *bheda* are meant portions which added together make the whole, or aliquot parts which multiplied together make the entire quantity.

> The multiplicand is multiplied by the sum or the difference of the multiplier and an assumed quantity and, from the result the product of the assumed quantity and the multiplicand is subtracted or added.[2]

(Colebrooke thinks that this is a method to obtain the true product when the multiplier has been taken to be too great or too small by mistake.[3] Datta and Singh think, however, that this is not correct.[4]

Thus Brahmagupta mentions four methods of multiplication: (i) *gomūtrikā*, (ii) *khaṇḍa*. (iii) *bheda*, and (iv) *iṣṭa*. The common and the well known method of *kapāṭa-sandhi* has been omitted by him.

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1. गुणकारखण्डतुल्यो गुण्यौ गोमूत्रिकाकृतो गुणितः ।
सहितः प्रत्युत्पन्नो गुणकारकभेदतुल्यो वा ॥ —*BrSpSi XII. 55*

2. गुण्यो राशिर्गुणकारराशिनेष्टाधिकोनकेन गुणः ।
गुण्येष्टवधो न युतो गुणकेऽभ्यधिकोनके कार्यः ॥ —*BrSpSi. XII. 56*

3. Colebrooke, T. H., *Hindu Algebra*, p. 320.

4. Datta, B. and Singh, A. N., *History of Hindu Mathematics* Pt. I (Arithmetic), p. 135 (1962).

(i) *Gomūtrika*-method or zig-zag method. The word *gomūtrikā* means "similar to the course of cow's urine", hence "zigzag". This method in all essentials is the same as the *sthāna-khaṇḍa* method. The following illustration is based on the commentary of Pṛthūdaka Svāmī :

Example : To multiply 1223 by 235.

The numbers are written thus :

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2   1223
3    1223
5     1223
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The first line of figures is then multiplied by 2, the process beginning at units place, thus : $2\times3=6$; 3 is rubbed out and 6 substituted in its place, and so on. After all the horizontal lines have been multiplied by the corresponding numbers on the left in the vertical line, the numbers on the *pāṭī* stand thus :

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2446
 3669
  6115
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287405
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after being added together as in the present method.

The *sthāna-khaṇḍa* and the *gomūtrikā* methods resemble modern plan of multiplication most closely.

(ii) *Khaṇḍa* Method · or Parts Multiplicatian Method : Since the days of Brahmagupta, this method of multiplication also became very popular. We have two methods under this head :

(i) The multiplier is broken up into two or more parts whose sum is equal to it. The multiplicand is then multiplied severally by these and the results added.

To take an example :

$$13\times158=(6+7)\times158=(6\times158)+(7\times158)$$
$$=948+1106$$
$$=2054$$

(ii) The multiplier is broken up into two more aliquot parts. The multiplicand is then multiplied by one of these, the resulting product by the second and so on till all the parts are exhausted. The ultimate product is the result.

Thus for example :

$$96\times237=(4\times4\times6)\times237$$
$$=(4\times237)\times4\times6=948\times4\times6$$
$$=(4\times948)\times6=3792\times6$$
$$=22752$$

These methods of multiplication are found among the Arabs and the Italians, having obtained from people of India. They were known as the "Scapezzo" and "Repiego" methods respectively amongst Italians.

(iii) *Iṣṭa-guṇana* Method or the Algebraic Method.

We have already quoted the relevant verse from the *Brāhmasphuṭa-siddhānta* in this connection; (XII. 56) :

> The multiplicand is multiplied by the sum or the difference of the multiplied and an assumed quantity and from the result the product of the assvmed quantity and the multiplicand is subtracted or added.[1]

This method is of two kinds according as we (i) add or (b) subtract an assumed number. The assumed number is so chosen as to give two numbers with which multiplication will be easier than with the original multiplier. The two ways are illustrated below :

(i) $93\times13=(93+7)\times13-7\times13=1300-91=1209.$

(ii) $93\times13=(90+3)13=90\times13+3\times13=1170+39$
$=1209$

This method was in use among the Arabs and in Europe, obviously having gone out from this country.

This process has been regarded as an inverse of multiplication. The terms used for this operation are *bhāgahāra*, *bhājana*, *haraṇa*, *chedana*, etc., all these terms more or less carrying the sense "to break into parts", "to divide" etc., excepting "*haraṇa*" which denotes "to take away". This term shows the relation of division to the operation of subtraction. The dividend is termed as *bhājya*. *hārya* etc., the divisor is known as

1. गुण्यो राशिर्गुणकारराशिनेष्टाधिकोनकेन गुणः ।
गुण्येष्टवधो न युतो गुणकेऽभ्यधिकोनके कार्यः ।। —*BrSpSi. XII. 56*

bhājaka, *bhāgahāra* or simply *hara*; quotient is known as *labdhi* or *labdha* (or "what is obtained").

India never regarded this operation as a difficult one; in Europe, this operation was regarded as a tedious one till the 15th century or so. Division was such a common operation that Āryabhaṭa did not regard it as worth being included in his treatise. But since he has given the methods of extracting square-roots and cube-roots, which obviously depend on division, we conclude that the method of division was known to him. Most *Siddhānta* writers have followed Āryabhaṭa I in omitting this operation from their texts, this being regarded too elemantary to be included. Brahmagupta does not give details of this operation. The later treatises on Arithmetic as Śrīdhara's *Triśatikā* and the *Pāṭīgaṇita* (I.20) and Āryabhaṭa II (c.950 A.D.) have given the details of this operation.

Square

The Sanakrit term for square is *varga* or *kṛti* (*varga* literally means "rows" or "troops" of similar things). In mathematics, it usually means the square power and also the square figure or its area. Thus we find in the *Āryabhaṭīya* :

> A square figure of four equal sides (and the number representing its area) are called *varga*. The product of the two equal quantities is also *varga*[1].

The term *kṛti* means "doing", "making" or "action". It carries with it the idea of specific performance probably the graphical representation.

For the first time we have a definite rule for squaring in the writings of Brahmagupta. But it does not mean that prior to him it was not known. It must have been known to Āryabhaṭa I since he has given the square-root method.

Brahmagupta gives his method of squaring briefly as follows :

> Combining the product, twice the digit in the less (lowest) place into the several others (digits) with its (i.e. of the digit in the lowest place) square (repeatedly) gives the square.[2]

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1. वर्गस्समचतुरश्रः फलञ्च सदृशद्वयस्य संवर्गः । —*Ārya. II.* 3.
2. राशेरूनं द्विगुणं वहुतरगुणमूनकृतियुतं वर्गः । —*BrSpSi. XII.* 63.

The method has been more clearly enunciated by Mahāvīra (850 A.D.) in the *Gaṇitasārasaṁgraha :*

> Having squared the last (digit), multiply the rest by the digits by twice the last, (which) is moved forward (by one place). Then moving the remaining digits continue the same operation (process), This gives the square.[1]

Brahmagupta's method of squaring is shown by the following example :

To square 125.

The number is written down

125

The square of the digit in the last place, i. e., $5^2=25$ is set over it thus :

25
125

Then, $2 \times 5=10$ is placed below the other digits, and 5 is rubbed out, thus :

25
12
10

Multiplying by 10 the rest of the digits, i.e., 12 and setting the product over them (the digits), we have.

1225
12
10

Then rubbing out 10 which is not required and moving the rest of the digits, i. e. 12 we, have

1225
12

Thus one round of operations is completed.

Again as before, setting the square of 2 above it and $2 \times 2=4$ below 1. we have

1625
1
4

1. GSS. P. 12.

Multiplying the remaining digit 1 by 4, and setting the product above it, we have

5625
1

Then moving the remaining digit 1, we obtain

5625
1

Thus the second round of operations is completed.

Next setting the square of 1 above it the process is completed for there are no remaining figures, and the result stands thus :

15625

Algebraic Method of Squaring

Brahmagupta in his *Brāhmasphuṭasiddhānta* gives a minor method of squaring thus :

> The product of the sum and the difference of the number (to be squared) and an assumed number plus the square of the assumed number give square[1].

This may be represented by the following identity :

$$n^2=(n-a)\ (n+a)+a^2$$

This identity has been used for squaring by most of the Indian mathematicians. Thus

$$15^2=(15-5)\ (15+5)+5^2=225$$

We are not giving here other identities which have been used by latter mathematicians of India in getting the squares of numbers; for example, when Mahāvīra says :

> The sum of the squares of the two or more portions of the number together with their products each with the others multiplied by two gives the square[2] :

he obviously refers to the identity

$$(a+b+c\ldots\ldots\ldots\ldots)^2=a^2+b^2+c^2+\ldots\ldots+2ab+\ldots\ldots$$

Cube

The Sanskrit term for cube is *ghana*. It when used in the geometrical sense also means the solid cube. In the arithmetical sense, it means the continued product of the same number taken three times. Thus we have the definition in the *Ārya-*

1. राशेरिष्टयुतोनाद्वधः कृतिर्वेष्टकृतियुक्तः । — *BrSpSi. XII. 63*
2. *GSS*. p. 13.

bhaṭiya :

> The continued product of three equals and also the solid having twelve (equal) edges are called *ghana*.[1]

A method of cubing applicable to numbers written in the decimal place-value notation, has been in use in this country from before the 5th century A.D. Āryabhaṭa I (499 A.D.) had the familiarity with this method; he, however, does not give the method of cubing in his treatise, though he describes the inverse process of extracting the cube-root.

Brahmagupta gives the method of cubing in the following verse :

> Set down the cube of the last (*antya*); then place at the next place from it, thrice the square of the last multiplied by the succeeding; then place at the next place thrice the square of the succeeding multiplied by the last, and (at the next place) the cube of the succeeding. This gives the cube.[2]

The rule may be illustrated by an example.

Example : To cube 1357.

The given number has four places, i.e., four portions. First we take the last digit 1 and the succeeding digit 3, i.e. 13 and apply the method of cubing thus :

(i) Cube of the last (1^3)	= 1
(ii) Thrice the square of the last (3.1^2) multiplied by the succeeding (3) gives ($3.3.1^2$)	= 9 (placing at the next place)
(iii) Thrice the square of the succeeding, multiplied by the last gives ($3.3^2.1$)	= 27 (placing at the next place)
(iv) Cube of the succeeding (3^3)	= 27 (placing at the next place)
Thus 13^3 is the sum	2197

1. सदृशत्रयसंवर्गो घनस्तथा द्वादशास्रस्यात् ॥ —*Ārya, II. 3.*

2. स्थाप्योऽन्त्यघनोऽन्त्यकृतिस्त्रिगुणोत्तरसंगुणा च तत्प्रथमान् ।
उत्तरकृतिरन्त्यगुणा त्रिगुणा चोत्तर घनश्च घनः । —*BrSpSi. XII, 6.*

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1. सदृशत्रयसंवर्गो घनस्तथा द्वादशाश्रस्यात् ॥ —*Ārya*, *II*. 3.
2. स्थाप्योऽन्त्यघनोऽन्त्यकृतिस्त्रिगुणोत्तरसंगुणा च तत्प्रथमात् ।
   उत्तरकृतिरन्त्यगुणा त्रिगुणा चोत्तर घनश्च घनः । —*BrSpSi*. *XII*, 6.

After this we take the next figure, 5, i.e., the number 135, and in this consider 13 as the last and 5 as the succeeding. Then the method proceeds thus :

(i) The cube of the last (13^3) as already obtained = 2197

(ii) Thrice the square of the last multiplied by the succeeding, i.e. $3.13^2.5$ = 2535 (placing at the next place)

(iii) Thrice the square of the succeeding multiplied by the last, i.e, $3.5^2.13$ = 975 (placing at the next place)

(iv) Cube of the succeeding, i.e. 5^3 = 125 (placing at the next place)

Thus 135^3 is the sum 2460375

Now the remaining figure 7 is taken, so that the number is 1357, of which 135 is the last and 7 the succeeding. The method proceeds thus :

(i) Cube of the last, i.e. (135^3) as already obtained = 2460375

(ii) Thrice the square of the last into the succeeding, i.e, $3. (135)^2. 7$ = 382725 (placing at the next place)

(iii) Thrice the square of the succeeding into the last, i.e. $3.7^2. 135$ = 19845 (placing at the next place)

(iv) Cube of the succeeding i.e. 7^3 = 343 (placing at the next place)

Thus $(1357)^3$ is the sum 2498846293

Evidently these methods of cubing are based on the identity:

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

and keeping in mind the place values of numerals in a given number (this accounts for keeping the results of each of the four operations at the next place).

Square-Root

Indian synonyms for square-root are *vargamūla* or *pada* of a *kṛti*. The word *mūla* means the "root" of a tree, which may also mean the "foot" or the lowest part or bottom of a thing and hence "pada" or foot also became a synonym of root. Brahma-Gupta defines square-root as follows :

> The *pada* (root) of a *kṛti* (square) is that of which it is the square.[1]

While the word *mūla* for *root* is the oldest in Indian literature (it occurs in *Anuyogadvāra-sūtra*, c. 100 B.C.), the word *pada* for root probably for the first time occurs in the writings of Brahmagupta. The term *mūla* was borrowed by the Arabs who translated it by *jadhr*, meaning "basis of square". The Latin term *radix* also is a translation of the term *mūla*. In the Śulba literature and in the Prākṛta texts, we find a term *karaṇī* for square-root. In geometry, this term *karaṇī* means a "side", In later days, the term *karaṇī* was reserved for surds, i.e. a square-root which cannot be exactly evaluated, but which may be represented by a line.

We would like to quote here a rule for determining square-root of numbers from the *Āryabhaṭīya* :

> Always divide the even place by twice the square-root (upon the preceding odd place); after having subtracted from the odd place the square (of the quotient), the quotient put down at the next place (in the line of the root) gives the root[2].

As an illustration, we shall proceed to find the square-root of 18225.

The odd and even places are marked out by vertical (I) and horizontal (—) lines : The other steps are as follows :

---

1. पदं कृतिर्यत् तत् — *BrSpSi*. XVIII. 35
2. भागं हरेदवर्गान्नित्यं द्विगुणेन वर्गमूलेन ।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम् ॥ — *Ārya*, II. 4.

	1̄ – 1̄ – 1̄ 1 8 2 2 5	
Subtract square	1	root = 1
Divide by twice the root	2) 8 (3 6	placing quotient at the next place, the root=13
	22	
Subtract square of quotient	9	
Divide by twice the root	26)132(5 130	placing quotient at the next place, the root=135
Subtract square of the quotient	25 25	

The process ends. The square-root of 18225 is thus 135.

It has been stated by Kaye, that Āryabhaṭa's method of finding out the square-root is algebraic in character, and that it resembles the method given by Theon of Alexandria. Āryabhaṭa's method is purely atithmetic and not algebraic is the view of Datta and Singh who do not agree with Kaye on this point.

Cube Root

The Sanskrit term for cube-root is *ghanamūla* or *ghanapada.* The first mention of the operation of cube-root is found in the *Āryabhaṭīya* of Āryabhaṭa I (499 A.D.), though the operation is given in only a concise form:

> Divide the second *aghana* place by thrice the square of the cube-root; subtract from the first *aghana* place the square of the quotient multiplied by thrice the preceding (cube-root); and (subtract) the cube (of the quotient) from the *ghana* place; (the quotient put down) at the next place (in the line of the root) gives (the root).[1]

As has been explained by all the commentators on the *Āryabhaṭīya*, the units place is *ghana*; the tens place is first *aghana*, the hundreds place is the second aghana, the thousands place is ghana, the ten thousands place is first *aghana*, the hun-

1. अघनाद् भजेद् द्वितीयात् त्रिगुणेन घनस्य मूलवर्गेण ।
वर्गस्त्रिपूर्वं गुणितश्शोध्यः प्रथमाद् घनश्च घनात् ।। —*Ārya.* II. 5

dred-thousands place is second *aghana*, and so on. Thus to find out the cube-root, one has to mark out the *ghana*, first *aghana* and second *aghana* places, then the process of finding out the cube-root begins with the subtraction of the greatest cube number from the figures up to the last *ghana* place. Though this has not been explicitly mentioned in the rule, the commentators say that it is implied in the expression *ghanasya mūla-vargeṇa* etc. ("by the square of the cube-root etc.")

We are reproducing here an illustration given by Datta and Singh.

Example. Find the cube-root of 1953125.

The places are divided into groups of three by marking them as below [*ghana* (|) first *aghana* (—) and second *aghana* (—)]:

	| — — | — — |	
	1 9 5 3 1 2 5	
Subtract cube	1	(c) Root=1
Divide by thrice square of root, i.e. 3.1^2	3)9(2 ...	(a) Placing quotient after the root 1 gives the root 12
Subtract square of quotient multiplied by thrice the previous root, i.e. $2^2.3.1$	6 35 12 ...	(b)
Subtract cube of quotient, i.e. 2^3	233 8 ...	(c)
Divide by thrice square of the root, i.e. 3.12^2	432)2251(5 ... 2160	(a) Placing quotient after the root 12 gives the root 125
Subtract square of quotient multiplied by thrice the previous root, i.e. $5^2.3.12$	912 900 125 ...	(b)
Subtract cube of quotient, i.e. 5^3	125 ...	(c)

Thus the cube-root=125.

From the details given, it would be clear that the present method of extracting the cube-root is almost a contraction of the method first given by Āryabhaṭa I (499 A.D.)

The method of Āryabhaṭa has been invariably followed by Indian mathematicians. Brahmagupta in his *Brāhmasphuṭa-siddhānta* repeats the method in the following words :

> The divisor for the second *aghana* place is thrice the square of the cube-root; the square of the quotient multiplied by three and the preceeding (root) must be subtracted from the next (*aghana* place to the right). and the cube (of the quotient) from the *ghana* place (the procedure repeated gives) the root.[1]

Śrīdhara and Āryabhaṭa II have further improved on the method of extracting cube-root proposed by Āryabhaṭa I and followed by Brahmagupta. Rule for finding the cube-root as given by Śrīdhara in his *Pāṭīgaṇita* is as follows :

> (Divide the digits beginning with the units' place into periods of) one *ghana*-pada (one "cube" place) and two *aghana-padaṣ* (two "non-cube" places). Then subtracting the (greatest possible) cube from the (last) *ghana-pada* and placing the (cube) root underneath the third place (to the right of the last *ghana-pada*), divide out the remainder up to one place less (than that occupied by the cube root) by thrice the square of the cube-root, which, is not destroyed. Setting down the quotient (obtained from division) in the line (of the cube-root), (and designating the quotient as the 'first' (*ādima*) and the cube-root as the 'last' (*antya*), subtract the square of that quotient, as multiplied by thrice the 'last' (*antya*) from one place less than that occupied by the quotient (*uparima-rāśi*) as before, and the cube of the 'first' (*ādima*) from its own place.
>
> (The number now standing in the line of cube-root is the cube-root of the given number up to its last-but one *ghana-pada* (cube place) from the left).
>
> Again apply the rule, "(placing cube-root) under the third place" etc. (provided there be more than two *ghana-padas* (cube places) in the given number; and

1. छेदो घनाद् द्वितीयाद् घनमूलकृतिस्त्रिसंगुणाप्तकृतिः ।
शोध्या त्रिपूर्वगुणिता प्रथमाद् घनतो घनो मूलम् ।। —*BrSpSi.* XII. 7

continue the process till all *ghana-padas* (cube-places) are exhausted). This will give the (cube) root (of the given number).[1]

K.S. Shukla in his translation and commentary of this book has given the illustration of extracting cube - root as follows :

Example :- To find the cube root of 277167808.

Let us indicate *ghana-padas* or 'cube' places by "*c*" and *aghana-padas* or non-cube places as "*n*" :

n n c n n c n n c
2 7 7 1 6 7 8 0 8

Subract the greatest possible cube (i.e. 6^3 or 216) from the last 'cube' place (i.e. from 277) and place the cube-root (i.e. 6) underneath the third place to the right of the last 'cube' place; thus we have

n n c n n c n n c
6 1 1 6 7 8 0 8 (remainder)
6 (line of cube-root)

Dividing out by thrice the square of the cube-root (i.e. by 3.6^2 or 108) the remainder up to one place less than that occupied by the cube-root (i.e. 611) and setting down the quotient in the line of the cube-root (to the right of the cube-root), we have

n n c n n c n n c
7 1 6 7 8 0 8 (remainder)
6 5 (line of cube-root)

Let now quotient 5 be called the 'first' (*ādima*) and the cube-root 6 the 'last' (*antya*). Then subtracting the square of the 'first' (*ādima*) as multiplied by thrice the 'last' (*antya*) (i.e. $3 \times 6 \times 5^2$ or 450) from one place less than that occupied by the quotient (i.e. from 716), we get

1. घनपदमघनपदे द्वे घन (पद) तोऽपास्य घनमदो मूलम् ।
संयोज्य तृतीयपदस्याधस्तदनष्टवर्गेण ॥ २९ ॥
एकस्थानोनतया शेषं त्रिगुणेन (सं) भजेत्तस्मात् ।
लब्धं निवेश्य पङ्क्त्यां तद्वर्गं त्रिगुणमन्त्यहतम् ॥ ३० ॥
जह्यादुपरिमराशेः प्राग्वद् घनमादिमस्य (च) स्वपदात् ।
भूयस्तृतीय पदस्याध इत्यादिक विधिमूलम् ॥ ३१॥

—Śrīdhara, *Pāṭīgaṇita*, 29-31

n n c n n c n n c		
2 6 6 7 8 0 8	(remainder)
6 5	(line of cube-root)

And subtracting the cube of the 'first' (*ādima*) (i.e. 5^3 or 125) from its own place (i.e. from 2667), we get

n n c n n c n n c	
2 5 4 2 8 0 8	(remainder)
6 5	(line of cube-root)

One round of the operation is now over; and the number 65 standing in the line of the cube-root is the cube-root of the given number (277167808) up to its last-but-one 'cube' place (*ghana pada*) from the left (i.e. of 277167),

As there is one more 'cube' place (*ghana-pada*) on the right, the process is repeated, Thus placing the cube-root (i.e. 65) under the third place beginning with the last-but-one 'cube' place (*ghana-pada*), we have

n n c n n c n n c	
2 5 4 2 8 0 8	(remainder)
6 5	(line of cube-root)

Dividing out 25428 by 3.65^2 (=12675) as before, and placing the quotient in the line of the cube-root, we have

n n c n n c n n c	
7 8 0 8	(remainder)
6 5 2	(line of cube-root)

Subtracting $3 \times 65 \times 2^2$ (=780) we get.

n n c n n c n n c	
8	(remainder)
6 5 2	(line of cube-root)

Finally subtracting $2^3=8$ from 8, we get

n n c n n c n n c	
0	(remainder)
6 5 2	(line of cube-root)

The second round of operation is now over. There being no more of *ghana-pada* ('cube' place) on the right, the process ends, The quantity in the line of cube root, viz., 652, is the cube-root of the given number. The remainder being zero, the cube-root is exact.

Fractions

The concept of fractions in India can be traced to very early times. In the *Ṛgveda*,[1] we find such terms as one-half (*ardha*) and three-fourths (*tri-pāda*). In a passage of the *Maitrāyaṇī Saṁhitā*[2] are mentioned the fractions one-sixteenth (*kalā*), one-twelfth (*kuṣṭha*), one-eighth (*śapha*) and one-fourth (*pāda*). In the *Śūlba Sūtras*[3] we have not only a mention of fractions, but they have been used in the statement and solution of problems of geometric nature. Here in the *Śulba*, unit fractions are denoted by the use of cardinal number with the term *bhāga* or *aṁśa*; thus *pañca-daśa-bhāga* (literally "fifteen parts") is equivalent to one-fifteenth, *sapta-bhāga* (literally, "seven parts") is equivalent to one-seventh, and so on... The use of ordinal numbers with the term *bhāga* or *aṁśa* is also quite common: thus *pañcama bhāga* stands for one-fifth. The composite fractions like *tri-āṣṭama* stands for three-eighths and *dvi-saptama* for two-sevenths. In the *Bakhaśālī Manuscript*, the term *tryaṣṭa* occurs for 3/8 and $3\frac{3}{8}$ is called *trayastrayasta* (three-three-eighths).

The Sanskrit term for fraction is *bhinna* (literally meaning 'broken'). Obviously the European terms as *fractio*, *fraction*, *roupt*, *rotto* or *rocto* are translations of the same term; they are derived from the Latin *fractus* (*frangere*) or *ruptus* meaning 'broken'. The Indian term *bhinna* has a few more connotations; it stands for such numbers of the form:

$$\left(\frac{a}{b}\pm\frac{c}{d}\right), \left(\frac{a}{b} \text{ of } \frac{c}{d}\right), \left(\frac{a}{b}\pm\frac{c}{d} \text{ of } \frac{a}{b}\right) \text{or} \left(a\pm \frac{b}{c}\right).$$

These forms were termed *jāti*' i.e., 'classes', and the Indian treatises contain special rules for their reduction to proper fractions. Śrīdhara and Mahāvīra each enumerate six *jātis*. while our author, Brahmagupta, gives only five (Bhāskara II gives only four). The need for division of fractions in 'classes' arose out of the lack of proper symbolism to indicate mathematical operations. (Datta and Singh *Arithmetic*, p. 188). The only operational symbol in use was a dot, standing for the negative sign.

1. *Rv*. X. 90.4
2. *Mait* S. III. 7.7.
3. B. Datta, *Śulba*, pp. 212ff.

Reduction to lowest terms.—A non-mathematical work, *Tattvārthādhigama-Sūtra-Bhāṣya* by Umāsvāti (c.150 A.D.) casually mentions as follows in the context of a philosophic discourse:

> Or, as when the expert mathematician, for the purpose of simplifying operations, removes common factors from the numerator and denominator of a fraction, there is no change in the value of the fraction, so,...[1]

Reduction to common denominator. Whenever we have to add or subtract fractions, we follow this reduction operation to a common denominator. Brahmagupta gives the reduction along with the similar processes :

> By the multiplication of the numerator and denominator of each of the (fractional) quantities by other denominators, the quantities are reduced to a common denominator. In addition, the numerators are united, In subtraction their difference is taken.[2]

Fractions in combination :—Since there was no proper symbolism available to these early Indian mathematicians, they divided combination of fractions into four classes :

Bhāga, prabhāga, bhāgāpavāha and *bhāga-bhāga.*

(i) *Bhāga* has been mentioned by Brahmagupta (*BrSpSi.* XII, 8) thus : $\left(\frac{a}{b} \pm \frac{c}{d} \pm \frac{e}{f} \pm \ldots\ldots\ldots\right)$ usually written as

$$\begin{array}{|c|c|c|}\hline a & c & e \\ b & d & f \\ \hline\end{array} \text{ or } \begin{array}{|c|c|c|}\hline a & .c & .e \\ b & d & f \\ \hline\end{array}$$

where the dots denote subtraction.

(ii) *Prabhāga* : The form $\left(\frac{a}{b} \text{ of } \frac{c}{d} \text{ of } \frac{e}{f} \ldots\ldots\ldots\right)$

This is written as

$$\begin{array}{|c|c|c|}\hline a & c & e \\ b & d & f \\ \hline\end{array}$$

(iii) *Bhāgānubandha* :The form $\left(a + \frac{b}{c}\right)$ is written as

$$\begin{array}{|c|}\hline a \\ b \\ c \\ \hline\end{array}$$

1. II, 52.

2. विपरीतच्छेदगुणा : राश्योश्छेदांशकाः समच्छेदाः ।
संकलितेंऽशा योज्या व्यवकलितेंऽशान्तरं कार्यम् ॥ —*BrSpSi*. XII. 2.

and the form

$$\frac{p}{q}+\frac{r}{s} \text{ of } \frac{p}{q}+\frac{t}{u} \text{ of}\left(\frac{p}{q}+\frac{r}{s} \text{ of} \frac{p}{q}\right)+\ldots\ldots\ldots\ldots$$

is written as

$$\begin{array}{|c|} \hline p \\ q \\ \hline r \\ s \\ \hline t \\ u \\ \hline \end{array}$$

(iv) *Bhāgāpavāha*, i.e., the form $\left(a-\frac{b}{c}\right)$ is written as

$$\begin{array}{|c|} \hline a \\ .b \\ c \\ \hline \end{array}$$

and the form $\frac{p}{q}-\frac{r}{s} \text{ of} \frac{p}{q}-\frac{t}{u} \text{ of}\left(\frac{p}{q}-\frac{r}{s} \text{ of} \frac{p}{q}\right)-\ldots\ldots\ldots$

is written as

$$\begin{array}{|c|} \hline p \\ q \\ \hline .r \\ s \\ \hline .t \\ u \\ \hline \end{array}$$

(v) Bhāga-bhāga : The form

$$\left(a\div\frac{b}{c}\right) \text{ or} \left(\frac{p}{q}\div\frac{r}{s}\right)$$

There does not appear to have been any notation for division, such compounds being written as

$$\begin{array}{|c|} \hline a \\ b \\ c \\ \hline \end{array} \text{ or } \begin{array}{|c|} \hline p \\ q \\ \hline r \\ s \\ \hline \end{array}$$

just as for *bhāgānubandha*. That division is to be performed was known from the problem, e.g., $1\div\frac{1}{6}$ was written as ṣaḍ-bhāga-bhāga, i.e., "one-sixth *bhāga-bhāga*" or "one divided by one-sixth". It is only in the Bakhśālī Manuscript that the term *bhā* is sometimes placed before or after the quantity affected.

(vi) *Bhāga-mātṛ*, i.e., combinations of forms enumerated above. Mahāvīra, the author of the *Gaṇitasārasaṁgraha* (850

A.D.) gives twenty-six variations of this class. We shall illustrate it by the following example from Śrīdhara :

> What is the result when half, one-fourth of one-fourth, one divided by one-third, half-plus half of itself, and one-third diminished by half of itself, are added together ? (*Triśatikā*, p. 12).

A modern writer would have written it as :

$$\tfrac{1}{2}+(\tfrac{1}{4} \text{ of } \tfrac{1}{4})+(1\div\tfrac{1}{3})+(\tfrac{1}{2}+\tfrac{1}{2} \text{ of } \tfrac{1}{2})(\tfrac{1}{3}-\tfrac{1}{2} \text{ of } \tfrac{1}{3})$$

In the old Indian notation, it is written as

1	1	1	1	1	1
2	4	4	1	2	3
			3	1	.1
				2	2

The defect of the notation is obvious:

1	1
4	4

can be read also as $\tfrac{1}{4}+\tfrac{1}{4}$ and

1
1
3

can also be read as $1\tfrac{1}{3}$.

And therefore the original meaning is inferred from the context or from the enunciation of the problem.

The rules for reduction of the first two classes (*bhāga* and *prabhāga)* are those of addition or subtraction and multiplication. The rule for the reduction of the third (*bhāgānubandha)* and fourth (*bhāgāpavāha)* classes are given by Brahmagupta in the *Brāhmasphuṭa-siddhānta* thus :

> The (upper) denominator is multiplied by the denominator and the upper numerator by the same (denominator) increased or diminished by its own numerator.[1]

"Numerator" is known as "*aṁśa*" and the "denominator" as "*cheda*."

We give here from Śrīdhara's *Pāṭigaṇita* (about 900 A.D., according to K.S. Shukla, 750 A.D. according to Datta and Singh) a rule for reducing a fraction of the *bhāgānubandha* class (i.e., a whole number increased by a fraction or a fraction increased by a fraction itself) :

1. ऊर्ध्वांशाश्छेदगुणास्तृतीयजातौ द्वयोः पृथक्परयोः ।
छेदैश्छेदा गुणिताः स्वांशयुतोनैरुपरिमांशाः ॥ —*BrSpSi. XII. 9.*

> In the *bhāgānubandha* class, the whole number (*rūpagaṇa*) is multiplied by the denominator (of the fraction) should be increased by the numerator (of the fraction) or the upper denominator having been multiplied by the lower denominator, the initial numerator (i.e. the upper numerator) should be multiplied by the sum of the lower numerator and denominator.[1]

(*Pāṭīgaṇita*, 39 *cf*. *BrSpSi*. *XII*. 9 (*ii*); *GSS*. (iii) 113

This means that

(i) $a+\frac{b}{c}=\frac{ac+b}{c}$

(ii) $\frac{a}{b}+\frac{c}{d}$ of $\frac{a}{b}$ (which was written by Indians in the style

$$\boxed{\begin{array}{c} \frac{a}{b} \\ \frac{c}{d} \end{array}}$$

is equal to $\frac{a(d+c)}{bd}$

Addition and Subtraction of Fractions

In the *Brāhmasphuṭa-siddhānta*, Brahmagupta gives the rule for the addition and subtraction of fractions :

> If the denominators (*cheda*) of fractions are different then reduce these fractions to a common denominator. Now for the additions, unite the numerators and take their difference in case of subtraction.[2]

Brahmagupta and Mahāvīra give the method under *Bhāgajāti*.

Multiplication

Brahmagupta says :

The product of the numerators divided by the pro-

1. भागानुबन्धजातौ रूपगणश्छेद सङ्गुणः सांशः ।
अवरहरघ्नोर्ध्वं हरेऽधोंशाऽयुतहरघ्न आद्यंशः ॥ —*Pātīgaṇita 39.*

2. विपरीतच्छेदगुणाः राश्योश्छेदांशकाः समच्छेदाः ।
संकलितेंऽशा योज्या व्यवकलितेंऽशान्तरं कार्यम् ॥ —*BrSpSi*. XII. 2

> duct of the denominators is the (result of) multiplication of two or more fractions.[1]

While all other writers give the rule in the same way as Brahmagupta, Mahāvira in the *Gaṇitasārasaṁgraha* refers to cross reduction in order to shorten the work :

> In the multiplication of fractions, the numerators are to be multiplied by the numerators and the denominators by denominators, after carrying out the process of cross reduction, if that be possible.[2]

Division of Fractions

The *Āryabhaṭīya* does not explicitly give the rule of division, but under the Rule of Three, we have an indication of this operation. The Rule of Three states the result as $\frac{f \times i}{p}$, where f stands for *phala* i.e. "fruit", i for *icchā*, i.e., demand or requisition, and p for *pramāṇa* i.e. argument. When these quantities are fractional, we get an expression of the form

$$\frac{\frac{a}{b} \times \frac{c}{d}}{\frac{m}{n}}$$

for the evaluation of which Āryabhaṭa I states :

> The multipliers and the divisor are multiplied by the denominators of each other.

These quantities are written in the following way

$$\begin{array}{|cc|} a & m \\ b & n \\ \hline c & \\ d & \\ \end{array}$$

Transferring the denominators we have

$$\begin{array}{|cc|} a & m \\ n & b \\ c & d \\ \end{array}$$

Performing multiplication, the result is $\frac{anc}{mbd}$. The above interpretation of the obscure line in the *Āryabhaṭīya* is based

1. रूपाणिच्छेद गुणान्यंशयुतानि द्वयोर्बहूनां वा ।
प्रत्युत्पन्नो भवति च्छेदवधेनोद्धृतोऽशवधः ॥ —*BrSpSi*. XII. 3

2. GSS. p. 25. (2)

on the commentaries of Sūryadeva and Bhāskara I (the commentary of Parameśvara on this line is vague and misleading). Sūryadeva in this connection says :

> Here by the word *guṇakāra* is meant the multiplier and multiplicand, i.e., the *phala* and *icchā* quantities that are multiplied together. By *Bhāgahāra* is meant the *pramāṇa* quantity. The denominators of the *phala* and *icchā* are taken to the *pramāṇa*. The denominator of the *pramāṇa* is taken with the *phala* and *icchā*. Then multiplying these, i.e., (the numerators of) the *phala* and *icchā* and this denominator, and dividing by (the product of) the numbers standing with the *pramāṇa* the result is the quotient of the fractions.

Brahmagupta gives the method of division as follows :

> The denominator and numerator of the divisor having been interchanged, the denominator of the dividend is multiplied by the (new) numerator. Thus division of *proper fractions* is performed.[1]

Square and Square-Root of Fractions

Brahmagupta says as follows in this connection:—

> The square of the numerator of a proper fraction divided by the square of the denominator gives the square[2].

This rule of Brahmagupta has been followed by other authors also. The rule regarding the square-root as given by Brahmagupta is as follows :

> The square-root of the numerator of a proper fraction divided by the square-root of the denominator gives the square-root.[3]

The Rule of Three :

The Indian term in Sanskrit for the Rule of Three is *Trairāśika* (literally, "three terms"). The term occurs in the *Bakhśāli Manuscript* also, and also in the *Aryabhaṭīya*, indicating the

1. परिवर्त्य भागहारं छेदांशौ छेद संगुणच्छेदः ।
अंशोऽशगुणो भाज्यस्य भागहारः सवर्णितयोः ॥ —*BrSpSi*. XII. 4
2. संवर्णितांशवर्गश्छेदकृतिविभाजितो भवति वर्गः । —*BrSpSi*. XII. 5 (1)
3. संवर्णितांशमूलं छेदपदेनोद्धृतं मूलम् । —*BrSpSi*. XII. 5 (2)

antiquity of the term. Bhāskara in his commentary of the *Āryabhaṭīya* gives a justification of the use of this term for the Rule of Three thus :

> Here three quantities are needed (in the statement and calculation) so the method is called *trairāśika* (meaning thereby the "rule of three terms").

The problem of the Rule of Three has the form :

> If *p* (*pramāṇa*) yields *f* (*phala*), what will *i* (*icchā*) yield ?

Āryabhaṭa II (the author of the *Mahāsiddhānta*, 950 A.D.) uses the terms *māna*, *vinimaya* and *icchā*, instead of *pramāṇa*, *phala* and *icchā* respectively. It has also been pointed out by several authors that the first and third terms are similar, i.e., of the same denomination.

We shall give here the Rule of Three as given by Āryabhaṭa I and Brahmagupta :

> In the Rule of Three, the *phala* ("fruit"), being multiplied by the *icchā* ("requisition') is divided by the *pramāṇa* ("argument"). The quotient is the fruit corresponding to the *icchā* The denominators of one being multiplied with the other give the multiplier (i.e. numerator) and the divisor (i.e. denominator).[1]
> In the Rule of Three *pramāṇa* ("argument"), *phala* ("fruit") and *icchā* ("requisition") are the (given) terms; the first and the last terms must be similar. The *icchā* multiplied by the *phala* and divided by the *pramāṇa* gives the fruit (of the demand).[2]

Śrīdhara also gives the Rule of Three almost in the same words. Bhāskara II, Nārāyaṇa and others follow Brahmagupta and Śrīdhara in the *Trairāśika* operation. Śrīdhara in his *Pāṭīgaṇita* says :

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1. त्रैराशिकफलराशिं तमथेच्छाराशिनाहतं कृत्वा ।
लब्धं प्रमाणभजितं तस्मादिच्छाफलमिदं स्यात् ॥
छेदाः परस्परं हता भवन्ति गुणकार भागहाराणां ।
छेदगुणं सच्छेदं परस्परं तत्सवर्णत्वम् ॥ —*Arya*: II 26-27.

2. त्रैराशिके प्रमाणं फलमिच्छाद्यन्तयोः सदृशराशी ।
इच्छाफलेन गुणिता प्रमाणभक्ता फलं भवति ॥ —*BrSpSi* XII. 10

> In (solving problems on) the Rule of Three, the argument (*pramāṇa*) and the requisition (*icchā*), which are of the same denomination, should be set down in the first and last places; the fruit (*phala*), which is of a different denomination, should be set down in the middle, (this having been done) that (middle quantity multiplied by the last quantity should be divided by the first quantity.[1]

We shall illustrate the Rule of Three by an example from the *Pāṭīgaṇita* (Example 25) :

> Example, If 1 *pala* and 1 *karṣa* of sandalwood are obtained for ten and a half *paṇas*, then for how much will nine *palas* and one *karṣa* (of sandalwood) be obtained ?[2]

Here in this Example.

argument=1 *pala* and 1 *karṣa*=$1\frac{1}{4}$ or 5/4 *palas*; fruit= $10\frac{1}{2}$ or 21/2 *paṇas*;

and requisition=9 palas and 1 karṣa=$9\frac{1}{4}$ or 37/4 palas. According to the Rule we shall write them as :

$$\left|\begin{array}{ccc} 1 & 10 & 9 \\ 1 & 1 & 1 \\ 4 & 2 & 4 \end{array}\right|$$

Converting these into proper fractions we have

$$\left|\begin{array}{ccc} 5 & 21 & 37 \\ 4 & 2 & 4 \end{array}\right|$$

Then applying the rule, (i.e. multiplying the second and the last and dividing by the first), we have

$$\left|\begin{array}{cc} 21 & 5 \\ 2 & 4 \\ \hline 37 & \\ 4 & \end{array}\right| \qquad = \frac{\frac{21}{2} \times \frac{37}{4}}{\frac{5}{4}}$$

Or transferring denominators $\left|\begin{array}{cc} 21 & 5 \\ 4 & 2 \\ 37 & 4 \end{array}\right| = \frac{21.4.37}{5.2.\ 4}$ *pala*

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1. आद्यन्तयोस्त्रिराशावभिन्नजाती प्रमाणमिच्छा च ।
फलमन्यजातिमध्ये तदन्त्यगुणमादिना विभजेत् ॥ —*Pāṭīgaṇita* 43.

2. चन्दनपलं सकर्षं सार्धैर्यदि लभ्यते पणैर्दशभिः ।
[illegible] लभ्यन्ते पलानि नव कर्षयुक्तानि ॥ —*Pāṭīgaṇita*. Ex. 25.

=4 purāṇa, 13 paṇas, 2 kākiṇīs and 16 varāṭakas. (One purāṇa is equivalent to 16 paṇas; one paṇa is equivalent to 4 kākiṇīs, and one kākiṇī is equivalent to 20 varāṭakas or cowries.

Inverse Rule of Three

This is known as *vyasta-trairāśika* (literally meaning "inverse rule of three terms)". After having described the rule of three, Brahmagupta proceeds to give an account of this inverse rule of three :

> Divide the *phala* with *icchā* and multiply by *pramāṇa*; this gives the *vyasta-trairāśika* inverse rule of three[1].

Here *pramāṇa* is the argument also known as the first term and, and *phala* is the fruit also known as the middle term and *iccha* is known as requisition or the last term. As Bhāskara II clearly states, this rule is applied where with the increase of the *icchā*, the *phala* decreases or with its decrease the *phala* increases (*Līlāvatī*).

Rule of Compound Proportion

Brahmagupta and other writers call the rule of compound proportions as *pañca-rāśika*, *sapta-rāśika* etc., meaning the rule of five terms, rule of seven terms etc. depending on the number of terms involved the problems. These are sometimes grouped under the general application of the "Rule of Odd Terms". Āryabhaṭa I (499 A.D.) though actually gives the rule of three appears to have been quite familiar with the rule of compound proportion also. In fact the difference between the rule of three and compound proportion is more or less artificial. This view was expressed by Bhāskara I (525 A.D.) in his commentary on the *Āryabhaṭīya* :

> Here Ācarya Āryabhaṭa has described the Rule of Three only. How the well-known Rules of Five etc. are to be obtained ? I say thus : The Ācarya has described only the fundamentals of *anupāta* (proportion). All others such as the Rule of Five etc. follow from that fundamental rule of proportion. How ? The Rule of Five etc. consist of combinations of the Rule of Three. ......In the Rule of Five, there are two Rules of

1. व्यस्त त्रैराशिक फलमिच्छा भक्तः प्रमाण फलघातः ।
त्रैराशिकादिषु फलं विषमेष्वेकादशान्तेषु ॥ —*BrSpSi* XII. 11

> Three, in the Rule of Seven three Rules of Three, and so on. This I shall point out in the examples.

Brahmagupta gives the following rule relating to the solution of problems in compound proportion :

> In the case of odd terms beginning with three terms up to eleven, the result is obtained by transposing the fruits of both sides, from one side to the other, and then dividing the product of the larger set of terms by the product of the smaller set. In all the fractions, the transposition of denominators, in like manner, takes place on both sides.[1]

This may be illustrated by taking an example from the commentary of Pṛthūdaka Svāmī on the *Brāhmasphuṭasiddhānta* :

Example —If there is an increase of 10 in 3 months on 100 (*niṣkas*), what would be the increase on 60 (*niṣkas*) in 5 months.

Here the *Pramāṇa pakṣa* (the first set of terms) is
100 niṣkas, 3 months, 10 niṣkas (phala)

The second set or the *icchā pakṣa* is
60 *niṣkas*, 5 months, x *niṣkas*

The terms are written in compartments as below :

100	60
3	5
10	0

In the above 10 (written lowest) is the *fruit* of the first side (*pramāṇa pakṣa*), and there is no *fruit* on the second side or the *iccā pakṣa*. Interchanging the *fruits* we get

100	60
3	5
0	10

The larger set of terms is on the second side (*icchā pakṣa*). The product of the numbers is 3,000. The product of the

1. व्यस्त त्रैराशिक फलमिच्छा भक्तः प्रमाणफलघातः ।
त्रैराशिकादिषु फलं विषमेष्वेकादशान्तेषु ॥
फलसंक्रमणमुभयतो बहुराशि वधोऽल्पवधहृतो ज्ञेयम् ।
सकलेष्वेवं भिन्नेषूभयतश्छेदसंक्रमणम् ॥ —*BrSpSi.* XII. 11-12.

number on the side of the smaller set of terms is 300. Therefore the required result is $\frac{3000}{300}=10$.

Rule of Three as a Particular Case

According to Brahmagupta, the above method of "compound proportion" may be applied to the Rule of Three. Taking the example solved under the Rule of Three:

> If one *pala* and one *karṣa* of sandal wood are obtained for ten and a half *paṇas*, for how much will be obtained nine *palas* and one *karṣa*?
>
> (4 karṣas=1 pala).

We shall represent them according to the Rule of Compound Proportion as

Pramāṇa pakṣa : 1 *pala*, 1 *karṣa*, $10\frac{1}{2}$ *paṇa*
or $\frac{5}{4}$ *pala* , $\frac{21}{2}$ *paṇa*

Icchā pakṣa : 9 *pāla*, 1 *karṣa*, *x paṇa*
or 37/4 *pala* , *x paṇa*

This we shall represent as

5	37
4	4
21	0
2	

Transposing the fruits, we have

5	37
4	4
0	21
	2

Transposing denominators

5	37
4	4
0	21
2	

The product of numbers on the side of the larger set is divided by the product of the numbers on the side of the smaller set, 0 in this case is not a number. It is the symbol for the unknown or absence, Hence the result is :

$$\frac{37.4.21}{5.4.2} \textit{ paṇas}$$

> The above method of working Rule of Three is found among Arabs, although it does not seem to have been used in India after Brahmagupta.

Problem Containing Quadratic Equation

Perhaps Āryabhaṭa I is the first man in the history of mathematics to give a solution of a quadratic equation (499 A.D.). In his *Āryabhaṭīya*, he gives a rule for the solution of the following problem (I am reproducing it as described by Datta and Singh):

> The principal sum p (=100) is lent for one month (interest unknown=x). This unknown interest is then lent out for t(=six) months. After this period, the original interest (x) plus the interest on this interest amounts to A(=16). The rate-interest (x) on the principal (p) is required.

This problem requires the solution of the quadratic equation :—

$$tx^2+px-AP=0$$

which gives $x=\dfrac{-p/2\pm\sqrt{(p/2)^2+Apt}}{t}$

The negative value of the radical does not give a solution of the problem; so that the result is

$$x=\frac{\sqrt{Apt+(p/2)^2}-p/2}{t}$$

This solution is stated by Āryabhaṭa I in the following words:

> Multiply the sum of the interest on the principal and the interest (A) by the time (t) and by the principal (p). Add to this result the square of half the principal $(p/2)^2$. Take square-root this. Subtract half the principal ($p/2$) and divide the remainder by the time (t). The result will be the (unknown) interest (x) on the principal.[1]

Here the Sanskrit terms are *mūla* for principal and *phala* for interest.

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1. मूलफलं सफलं कालमूलगुणमर्धमूलकृतियुक्तं ।
मूलं मूलार्धोनं कालहृतं स्यात्स्वमूलफलम् ॥ *Ārya. II. 25,*

Brahmagupta (628 A.D.) gives a more general rule: He enunciates his problem thus:

> The principal (p) is lent out for t_1 months and the unknown interest on this ($=x$) is lent out for t_2 months at the same rate and becomes A. To find x.

This evidently gives the quadratic:

$$x^2+\frac{pt_1}{t_2}\ x-\frac{Apt_1}{t_2}=0$$

whose solution is

$$x=\pm\sqrt{\frac{Apt_1}{t_2}+\left(\frac{pt_1}{2t_2}\right)^2}-\frac{pt_1}{2t_2}$$

The negative value of the radical does not give a solution of the problem, so it is discarded.

Brahmagupta states the formula thus:

> Multiply the principal (p) by its time (t_1) and divide by the other time (t_2) (placing the result) at two places. Multiply the first of these by the mixture (A). Add to this the square of half the other. Take the square-root of this (sum). From the result subtract half the other. This will be the interest (x) on the principal.[1]

A Problem on Interest

Brahmagupta gives a solution of a problem on interest:

> In what time will a given sum s, the interest on which for t months is r, become k times itself?

The rule for the solution of this problem as given by Brahmagupta is:

> The given sum multiplied by its time and divided by the interest (*phala*), being multiplied by the factor (*guṇa*) less one, is the time (*required*).[2]

Miscellaneous Problems

Brahmagupta in his *Gaṇitādhyāya* of the *Brāhmasphuṭa-siddhānta* gives numerous solutions in relation to miscellaneous problems. Here I shall be quoting a few of the problems which

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1. कालप्रमाणघातः परकालहृतो द्विधाऽऽद्यमिश्रवधात् ।
अन्यार्धकृतियुतात् पदमन्यार्धोनं प्रमाणफलम् ॥ —*BrSpSi*. XII. 15.

2. कालगुणितं प्रमाणं फलभक्तं व्येकगुणहतं कालः ।
स्वफलयुतरूपभक्तं मूलफलैक्यं भवति मूलम् ॥ —*BrSpSi*. XII. 14.

have been quoted by his commentator Pṛthūdaka Svāmī in connection with one of his karaṇa-sūtra.[1]

1. A horse was purchased by (nine) dealers in partnership, whose contributions were one, etc. up to nine; and was sold by them for five less than five hundred. Tell me what was each man's share of the sale proceed[2]
2. Four colleges (mathas), containing an equal number of pupils, were invited to partake of a sacrificial feast. A fifth, a half, a third and a quarter (of the total number of pupils in the college) came from the respective colleges to the feast; and added to one, two, three and four, they were found to amount to eighty-seven; or, with those deducted, they were sixty seven. Find the actual number of the pupils that came from each college.[3]

3. Three (unequal) jars of liquid butter, of water and of honey, contained thirty-two, sixty and twenty-four *Pala* respectively; the whole was mixed together and the jars filled again. Tell me the quantity of butter, of water and of honey in each jar[4].

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1. प्रक्षेपयोगहृतया लब्ध्वा प्रक्षेपका गुणा लाभाः ।
ऊनाधिकोद्धरास्तद्युतोनया स्वफलमूनयुत ॥ *BrSpSi*. XII. *16*.

2. एकाद्यैर्नव पर्यन्तैर्वणिग्भिर्मूलराशिभिः ।
क्रीतो हयोऽसौ विक्रीतः पञ्चोनैः पञ्चोनैः पंचभिः शतैः ।
किमेकैकस्य तत्रासीद् ब्रूहि त्वं मिश्रकान् मम ॥

3. मठस्थानानि चत्वारि छात्राणां समसंख्यया ।
भोक्तुं संमन्त्रितान्यासन् दीक्षायां किल यज्वना ॥
पंचार्धत्रिचतुर्थांशास्तेभ्यो भोक्तुं समागताः ।
एकद्वित्रिचतुर्युक्ता ह्यशीतिः ससप्तका ॥
एवोत्तरैरथवा हीना सप्तषष्टिश्चतेंऽशकाः ।
मठेभ्यश्छात्रसंख्यां मे ब्रूहि ये चागता यतः

4. घृतोदक मधूनां ये त्रयः कलसकाः पलैः ।
रदषष्टिजिनैः पूर्णा एकीभूतास्ततः पुनः ॥
मिश्रेण पूरिता यावत् तावत् संख्यां न वेद्म्यहम् ।
घृतोदकमधूनां तामेकैकत्र गतां वद ॥

—:o:—

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