Lord Brouncker's continued fraction for π

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Lord Brouncker is remembered today chiefly for his achievement of finding a continued fraction for π. He might also have been renowned for his work on the equation $x^2 - ny^2 = 1$, but for the fact that Euler mistakenly mentioned Pell's name in connection with this equation, which has carried Pell's name ever since.

The object of this note is to establish Brouncker's continued fraction, and to show how it may be used to give a good approximation to π with little work.

Brouncker's result may be written as follows.

(1)

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ddots}}}}$$

First let us see how this result is obtained, then see how it may be used to compute approximations to π.

Let us begin by defining

$$I_n = \int_0^1 \frac{x^{2n}}{1+x^2}\,dx.$$

Then

$$I_n + I_{n+1} = \frac{1}{2n+1}.$$

We then have

$$\frac{I_{n+1} + I_n}{I_{n+2} + I_{n+1}} = \frac{2n+3}{2n+1}.$$

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If we define $r_n = I_{n+1}/I_n$, we have

$$\frac{1 + 1/r_n}{r_{n+1} + 1} = \frac{2n+3}{2n+1}$$

and it is easy to make r_n the subject:

$$r_n = \frac{2n+1}{2 + (2n+3)r_{n+1}}.$$

It follows that

$$\begin{aligned}
r_0 &= \frac{1}{2 + 3r_1} \\
&= \cfrac{1}{2 + \cfrac{3^2}{2 + 5r_2}} \\
&= \cfrac{1}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + 7r_3}}} \\
&\cdots \\
&= \cfrac{1}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{\ddots}{\cfrac{\ddots}{2 + \cfrac{(2n-1)^2}{2 + (2n+1)r_n}}}}}}
\end{aligned}$$

Now,

$$r_0 = \frac{I_1}{I_0} = \frac{1 - \frac{\pi}{4}}{\frac{\pi}{4}} = \frac{4}{\pi} - 1,$$

so

$$\frac{4}{\pi} - 1 = \cfrac{1}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{\cdot}{\cfrac{\cdot}{\cfrac{\cdot}{2 + \cfrac{(2n-1)^2}{2 + (2n+1)r_n}}}}}}}$$

It follows that

(2)

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{\cdot}{\cfrac{\cdot}{\cfrac{\cdot}{2 + \cfrac{(2n-1)^2}{2 + (2n+1)r_n}}}}}}}}$$

If we now "let $n \to \infty$", at least formally, we obtain Brouncker's continued fraction (1).

The question now arises, how can we use (1) to approximate π? We should go to (2) for some large–ish value of n, and estimate r_n. We might guess that $r_n \to 1$ as $n \to \infty$, and we shall see that is so, indeed, I shall show that

$$1 > r_n > 1 - \frac{2}{n}. \tag{3}$$

If we choose $n = 20$, we find that

$$3.140404294 < \pi < 3.142971689$$

which is not terribly good. Before going on to do much better, let us prove (3). It is clear that $r_n < 1$. We have

$$r_{n+1} = \frac{\frac{2n+1}{r_n} - 2}{2n+3} > \frac{2n-1}{2n+3} = 1 - \frac{4}{2n+3} > 1 - \frac{4}{2n+2} = 1 - \frac{2}{n+1}.$$

In order to improve on the above, let us suppose that

$$r_n = 1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots$$

for some $a_1,\ a_2,\ a_3,\ \cdots$.
Then

$$\begin{aligned} r_{n+1} &= 1 + \frac{a_1}{n+1} + \frac{a_2}{(n+1)^2} + \frac{a_3}{(n+1)^3} + \cdots \\ &= 1 + \frac{a_1}{n}\left(1 - \frac{1}{n} + \frac{1}{n^2} - + \cdots\right) + \frac{a^2}{n^2}\left(1 - \frac{2}{n} + \frac{3}{n^2} - + \cdots\right) + \cdots \\ &= 1 + \frac{a_1}{n} + \frac{-a_1 + a_2}{n^2} + \frac{a_1 - 2a_2 + a_3}{n^3} + \cdots . \end{aligned}$$

If we substitute these into the relation

$$r_{n+1} = \frac{\dfrac{2n+1}{r_n} - 2}{2n+3},$$

set $u = \dfrac{1}{n}$ and simplify, we obtain

$$1 + a_1 u + (-a_1 + a_2)u^2 + (a_1 - 2a_2 + a_3)u^3 + \cdots = \frac{\dfrac{2+u}{1 + a_1 u + a_2 u^2 + a_3 u^3 + \cdots} - 2u}{2+3u}.$$

$$= 1 + (-a_1 - 2)u + (-a_2 + a_1 + a_1^2 + 3)u^2 + (-a_3 + a_2 + 2a_1 a_2 - a_1^3 - a_1^2 - \frac{3}{2}a_1 - \frac{9}{2})u^3 + \cdots .$$

If we now compare coefficients of $u,\ u^2,\ u^3$ and so on, we find $a_1 = -1,\ a_2 = 1,\ a_3 = -\dfrac{1}{2}$ and so on, and

$$r_n \sim 1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{2n^3} - \frac{1}{4n^4} + \frac{1}{8n^5} + \frac{25}{16n^6} - \frac{25}{32n^7} - \frac{601}{64n^8} + \frac{601}{128n^9} + \frac{23089}{256n^{10}} - \frac{23089}{512n^{11}} - + \cdots .$$

(This is undoubtedly an asymptotis series rather than a convergent series.)

So the last denominator in (2) is

$$2 + (2n+1)r_n \sim 2n + 1 + \frac{1}{n} - \frac{1}{n^3} + \frac{13}{4n^5} - \frac{313}{16n^7} + \frac{11845}{64n^9} - \frac{647473}{256n^{11}} + \ \cdots \ .$$

If we now set $n = 20$, we find

$$\pi \approx 3.141592653589785$$

which is, as we know, correct to 13 decimal places.

Finally, you, dear reader, may like to guess and prove the value of the continued fraction

$$\cfrac{1}{1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1+\cfrac{4^2}{1+\cfrac{5^2}{\ddots}}}}}}$$