

Euler and the Zeta Function Author(s): Raymond Ayoub Source: *The American Mathematical Monthly*, Vol. 81, No. 10 (Dec., 1974), pp. 1067–1086 Published by: Mathematical Association of America Stable URL: http://www.jstor.org/stable/2319041 Accessed: 10-11-2017 17:23 UTC

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EULER AND THE ZETA FUNCTION

RAYMOND AYOUB

1. Introduction. Mathematics in general appeals to the intellect; great mathematics, however, has, in addition, a kind of perceptual quality which endows it with a beauty comparable to that of great art or music. In this category belongs much of the work of the great 18th century Swiss mathematician, Leonhard Euler (1707–1783).

One of the most enchanting episodes is his work on the zeta function, to which this article is devoted. In anticipation of the later notation of G.F.B. Riemann (1826-1866), let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} ,$$

where for the moment no specification will be made on s. Euler's work on $\zeta(s)$ began about 1730 with approximations to the value of $\zeta(2)$, continued with the evaluation of $\zeta(2n)$, where n is a natural number ≥ 1 , and resulted about 1749 in the discovery of the functional equation *almost* 110 years before Riemann.

Before beginning the story, we shall give a brief sketch of Euler's life. Most of the facts are taken from a eulogy delivered by Nicholas Fuss (1755–1825), the husband of one of Euler's granddaughters.

The city of Basel in Switzerland was one of many free cities in Europe and by the 17th century had become an important center of trade and commerce. The University became a noted institution, largely through the fame of an extraordinary family — the Bernoullis. This family had come from Antwerp to Basel and the founder of the mathematical dynasty was Nicholas Bernoulli. He had 3 sons, two of whom, James (often referred to as Jacob) (1654–1705), and John (1667–1748), became noted mathematicians. Both were pupils of G. Leibniz (1646–1716) with whom John carried on an extensive correspondence and with whose work both James and John became familiar. James was a professor at Basel until his death in 1705. John, who had been a professor at Groningen, replaced him. To give an indication of the mathematical activity of this period, it is worthwhile pointing out that I. Newton (1642–1727) had started his work on the theory of fluxions about 1664, publishing his great work,

Principia Mathematica, in 1687. On the continent G. Leibniz began his studies on the calculus about 1672 and published much of his work in Ithe journal *Acta Eruditorum*. This was a monthly periodical published in Leipzig and devoted to miscellaneous articles, books and book reviews.

Paul Euler (1670–1745) was a Lutheran Pastor who was mathematically talented and who had studied mathematics with James Bernoulli at the University of Basel. Into this intellectually rich and stimulating environment, Leonhard was born in 1707. He was a precocious child who received much encouragement from his father. He entered the University of Basel and displayed such remarkable talent for mathematics that John Bernoulli gave him special instruction on Saturdays. He graduated with a kind of master's degree in 1724 at the age of 17. He had enrolled in the Faculty of Theology and had written a thesis in Latin on a comparison between Newtonian and Cartesian philosophy. Although Paul expected his son to study theology, he did not discourage Leonhard's interest in mathematics. (Still, mathematics was fine as a hobby, but surely not as a profession!)

At this period there were 3 famous centers of learning, the academies at Berlin, Paris, and St. Petersburg, and it was frequently the case that a young scholar would journey to one of these.

John Bernoulli had 3 sons. Two of them, Nicholas II (1695–1726) and Daniel (1700–1782), were mathematicians who befriended Euler. They both went to St. Petersburg in 1725 and both had a high regard for their younger colleague. After some effort, Daniel wrote to Euler that he had secured for him a stipend in the Academy. The appointment was actually in the physiology section but Euler quickly drifted into the mathematics section. He then left Basel and arrived in St. Petersburg in 1727, remaining there until 1741.

The period had its troubles. Tsarina Catherine I was committed to carrying out the policy of her husband, Peter the Great, in establishing a strong Academy. Unfortunately she died the day Euler set foot in Russia. The throne passed to Peter I's grandson, Peter II, who was only 12 and Russia was ruled by despotic regents who declared that the Academy was very costly and was of little use to the state. Euler despaired of being able to pursue his interests and decided to join the navy. Admiral Sievers saw in him a valuable asset to the navy and offered him a position as lieutenant, with promises of rapid promotion. From the sources available to the author it is not clear to what extent, if any, Euler was active in naval affairs. The death of Peter II brought to an end the despotic regency and the Academy's condition improved, but the despotism had discouraged some foreign scholars, who returned to their homeland. An opportunity arose when Bullfingr left Russia and in 1731 Euler was appointed professor of natural sciences. Two years later, when Daniel decided to return to Basel in 1733, he recommended that Euler be appointed his successor as professor of mathematics. Euler remained in this position until 1741 when he was summoned by Frederick the Great of Prussia to the Berlin Academy. He was in Berlin until 1766. Catherine II, the Great, acceded to the throne of Russia in 1762 and in 1766 summoned Euler back to the Academy in St. Petersburg where he remained until his death in 1783.

Euler did significant work in all areas of mathematics and his work in any one of these would have assured him a place in history. He was a prodigious writer whose collected works run currently to 70 quarto volumes with more to come. In editing Euler's works shortly after his death N. Fuss listed 756 articles distributed in time as follows: 1727-33:24; 1734-43:49; 1744-53:125; 1754-63:99; 1764-72:104; 1773-82:355. The most astonishing feature is the phenomenal number written in the last 10 years of his life, during which years he was blind. Since Fuss's editing activities, numerous additional manuscripts have been found and the total will run to almost 900. In addition to his articles he wrote several books, among the most noted and influential of which was his *Introductio in Analysin Infinitorum*. Some have criticized his writings as being repetitive but it is proper to ignore this kind of pedantry.

Euler's articles were mostly in Latin which is unfortunate in view of our present day ignorance of the classics. On the other hand, the Latin is comparatively simple and, with a rudimentary knowledge, together with a dictionary, the reader will be rewarded for his efforts. It is especially fortunate that the notation is familiar, and where the language is difficult, the mathematics comes to the rescue. It is customary to be surprised at how "modern" his notation is; the truth is that his influence was so profound that we still use much of the notation he helped to establish.

Reading his papers is an exhilarating experience; one is struck by the great imagination and originality. Sometimes a result familiar to the reader will take on an original and illuminating aspect, and one wishes that later writers had not tampered with it.

Euler's personal life, though relatively uneventful, was marred by several tragedies. Though apparently of a strong constitution, he developed a massive infection which resulted in the loss of one eye in 1735. The second eye developed a cataract about 1766 which rendered him blind. He could still distinguish lights and shadows and sometimes wrote mathematics in very large symbols on a blackboard. Despite this handicap, he continued unabated his mathematical activities with the help of young assistants. He once met with J. d'Alembert (1717–1783) who was utterly astonished at Euler's ability to carry out in his head the most complicated analytical calculations.

Euler married Catherine Gsell in 1733. She was the daughter of a well-known artist. She had 13 children 8 of whom tragically died in childhood. Catherine died in 1776. Euler then married her half sister.

His character was that of a kind and gentle man. He had a phenomenal memory, had studied the classics, and is said to have known the Aeneid by heart. Though the recipient of numerous honors during his lifetime, he retained his modesty and humility and it was said of him that he took as much pleasure in the discoveries of others as he did in his own.

He carried on an extensive correspondence with various mathematicians, especially Christian Goldbach (1690–1764). He also wrote a series of letters on various RAYMOND AYOUB

subjects in natural philosophy addressed to a German princess. The quality of all his letters reflects his pleasant personality.

2. Early history of the function $\zeta(s)$. In elementary courses in calculus, one of the first examples of an infinite series is that given by $\zeta(s)$.

The student quickly learns, mainly via the integral test, that

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges if s > 1 and diverges if $s \le 1$. Some enthusiastic teachers will point out that, in fact,

(1)
$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} ,$$

and perhaps remark that this relation is difficult to prove and that students who go on in mathematics will eventually learn at least one proof. More enthusiastic teachers will further point out that if k is an integer $k \ge 1$, then

(2)
$$\zeta(2k) = \frac{(-1)^{k-1}B_{2k}(2\pi)^{2k}}{2(2k)!},$$

where B_{2k} is a *rational* number, viz. a Bernoulli number, a fact first proved by Euler. The generating function for these numbers is given by

$$\frac{x}{e^{x}-1} = 1 - \frac{x}{2} + \sum_{n=2}^{\infty} \frac{B_{n}}{n!} x^{n}.$$

However, it is easily seen that $B_{2m+1} = 0$ and that $B_2 = 1/6$, $B_4 = -(1/30)$, $B_6 = 1/42, \cdots$. They might further point out that if m is odd, m = 2k + 1 ($k \ge 1$), then no such formula is known for $\zeta(m)$, and despite considerable efforts over the years, the arithmetic nature of even $\zeta(3)$ remains an unsolved problem.

Before proceeding, it is interesting to note that Euler often worked with

$$\theta(s) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s},$$

with

$$\phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},$$

and with

$$\psi(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$
.

The first two are related to $\zeta(s)$ by

$$\zeta(s) = \theta(s) + \frac{1}{2^s} \zeta(s) ;$$

hence $\theta(s) = (1 - (1/2^s))\zeta(s)$, while

$$\phi(s) = \zeta(s) - \frac{2}{2^s} \zeta(s) = (1 - 2^{1-s})\zeta(s).$$

Thus $\phi(n)$ and $\theta(n)$ can be evaluated if $\zeta(n)$ can be. One important advantage of $\phi(s)$ over $\zeta(s)$ is that the series for $\phi(s)$ converges if s > 0, while that for $\zeta(s)$ only for s > 1.

By contrast $\psi(s)$ has a superficial resemblance to $\phi(s)$ but although $\psi(2n + 1)$ has been evaluated, $\phi(2n + 1)$ has not. In fact Euler proved that

$$\psi(2n + 1) = (-1)^n \frac{E_{2n}}{2^{2n+2}(2n)!} \pi^{2n-1},$$

where

 $\sec x = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{2n!} x^{2n}$

and E_{2n} are called Euler numbers.

Let us begin the story and go back ... Infinite series have occurred sporadically in mathematics for centuries — in fact Archimedes (287–212 B.C.), when he derived his famous theorem on the quadrature of the parabola, proved in effect that the series

 \sim

$$\sum_{n=1}^{\infty} 4^{-n}$$

converges. As far as the harmonic series is concerned, however (despite Plato's interest), the earliest recorded appearance appears to be in the works of Nicholas of Oresme (1323–1382) who proved that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

The problem occurs again in 1650 in a book *Novae Quadraturae Arithmeticae* by a professor of mechanics in Bologna named Pietro Mengoli (1625–1686). He related the series to the logarithm and posed the problem of finding the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} ,$$

if it converges.

Whether through the book of Mengoli or (what seems likely) independently, the problem became known in France and England. In fact, the English mathematician John Wallis (1616–1703), professor at Oxford, commented on the problem in 1655 in his book *Arithmetica Infinitorum*. He had computed the value of $\zeta(2)$ to 3 decimal

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places but it does not appear that he recognized this value, 1.645, as being about $\pi^2/6$.

In a letter to John Bernoulli in 1673, Leibniz wrote: "let

$$dy = \frac{1}{1} + \frac{x}{2} + \frac{x^2}{3} + \cdots$$

then $dy = (-[\log(1 - x)]/x) dx$, thus

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \dots = -\int \frac{\log(1-x)}{x} \, dx.$$

As log(1 - x) is infinite when x = 1, consider instead

$$dy = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

and get $y = \int [(\log(1 + x)]/x) dx$."

He now integrates by parts and deduces that the evaluation of the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

reduces to the evaluation of integrals of the form $\int x^e (1 + x)^n dx$. He continues: "If perhaps it were possible to consider all the cases in order, some light would be shed upon the problem."

In a letter to James Bernoulli in 1691, his brother John wrote, "I see now the route for finding the sum $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$." No further work, however, was forthcoming from him until 1742 when he published a proof similar to that given by Euler in 1734.

In the St. Petersburg Academy, the members were drawn to the problem and took a great interest in the evaluation of $\zeta(2)$. That it is a tantalizing problem stems in part from the fact that the series has a superficial resemblance to the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} ,$$

whose value is easily seen to be

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1.$$

This fact was early recognized by the academicians. In 1728, Daniel Bernoulli wrote to Goldbach that he had a method for computing quickly an approximation to $\zeta(2)$ and gave as an approximate value 8/5. In reply Goldbach wrote that he could show that

$$1\frac{16}{25} = 1.64 < \zeta(2) < 1\frac{2}{3} = 1.66.$$

Neither gave indications of their computations. As noted above, Daniel Bernoulli and Euler both lived in St. Petersburg between 1727 and 1733 and it seems very probable that they discussed the problem together.

3. Euler's early contributions. There the problem lay. Euler's first contribution came in 1731 when he gave an original method for computing $\zeta(2)$. His method appeared in a paper *De summatione innumerabilium progressionum*. He deals with sums of the type

$$\sum_{k=1}^{\infty} \frac{x^k}{(ak+b)^m}$$

In the special case of $\zeta(2)$ his argument is as follows: Since

$$\frac{\log(1-x)}{x} = -\sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$$
, it follows

that $-\zeta(2) = \int_0^1 (\log(1-x)/x) dx$. Replacing 1 - x by t and splitting the integral, it follows that

$$-\zeta(2) = \int_0^1 \frac{\log t}{1-t} dt = \int_0^x \frac{\log t}{1-t} dt + \int_x^1 \frac{\log t}{1-t} dt = I_1 + I_2.$$

In I_2 , put u = 1 - t, expand in a power series and integrate termwise; then if y = 1 - x,

$$I_2 = \sum_{n=1}^{\infty} \frac{y^n}{n^2} .$$

On the other hand, in I_1 , expand $(1 - t)^{-1}$ in a series, and integrate by parts getting

$$I_1 = -\log x \log(1 - x) - \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Hence $\zeta(2) = \log x \log(1-x) + \sum_{n=1}^{\infty} x^n/n^2 + \sum_{n=1}^{\infty} (1-x)^n/n^2$. Putting $x = \frac{1}{2}$, we conclude that

$$\zeta(2) = (\log 2)^2 + \sum_{n=1}^{\infty} \frac{1}{2^n n^2}$$

What has been achieved by this next argument? The series $\sum_{n=1}^{\infty} 1/2^n n^2$ converges much more rapidly than does the series for $\zeta(2)$. Knowing that

$$\log 2 = -\log\left(1 - \frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n2^n} \sim .480453,$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \sim 1.164482,$$

Euler concludes that $\zeta(2) \sim 1.644934$.

It should be remarked that in 1730 James Stirling (1692–1770) had computed $\zeta(2)$ to 9 decimal places, of which 8 were correct, but Euler was unaware of these calculations.

Euler's next contribution came in 1732/33 in a paper entitled Methodus Generalis Summandi Progressiones. In this he states the "Euler-McLaurin" formula (Colin McLaurin (1698–1746)). In a later paper Inventio summae cuiusque seriei ex dato Termino generali, published in 1736, he gives a proof. Although the paper was published in 1736, it is reasonable to assume that the work was done before 1734. We shall give Euler's argument which we modify slightly. Moreover, we shall ignore a few technicalities. Let

$$S(x) = \sum_{n \leq x} f(n).$$

The object is to approximate S(x) by an integral. We have

(A)
$$f(x) = S(x) - S(x - 1).$$

Using the Taylor (Brooke Taylor, 1685–1731) expansion, it follows that

(B)
$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} S^{(n)}(x)}{n!};$$

(the difficulty, of course, is that in writing (A) we are assuming x to be an integer while in (B), we assume x to be any real number).

Assume now that this series can be inverted; that is, assume there exist constants b_0, b_1, b_2, \cdots such that

(C)
$$S^{(1)}(x) = \sum_{n=0}^{\infty} b_n f^{(n)}(x)$$

Differentiating (B), i nserting in (C), and equating coefficients, gives recurrence formulae for the b's, viz.,

$$b_0 = 1$$
, $b_1 = \frac{b_0}{2}$, $b_2 = \frac{b_1}{2!} - \frac{b_0}{3!}$, $b_3 = \frac{b_2}{2!} - \frac{b_1}{3!} + \frac{b_0}{4!}$, etc.

Hence $S(x) = b_0 \int f(x)dx + \sum_{n=1}^{\infty} b_n f^{(n-1)}(x)$. The *b*'s turn out to be essentially the Bernoulli numbers. This fact can be intuitively gleaned from the following argument: let *D* denote the operator d/dx, then (B) can be written as

$$f(x) = \left(1 - \frac{D}{2!} + \frac{D^2}{3!} - \cdots\right) S^{(1)}(x) = \left(\frac{e^{-D} - 1}{D}\right) S^{(1)}(x),$$

or inverting,

$$S^{(1)}(x) = \left(\frac{D}{e^{-D}-1}\right)f(x).$$

On the other hand, the generating function for the Bernoulli numbers as noted above, gives

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_2 x^2}{2!} + \frac{B_3 x^3}{3!} + \cdots$$

Hence, replacing x by -D gives the desired result. Euler is evidently excited by this discovery (as which of us would not be!) and proceeds to apply it with great enthusiasm in the paper which appeared in 1736, *Inventio summae cuiusque seriei* ex dato Termino generali, referred to above.

He derives a formula for

$$\sum_{m=1}^{n} m^{k} \qquad (k \ge 1)$$

and painstakingly computes the necessary constants B_2, \dots, B_{16} and writes out at length the results for $k = 1, \dots, 16$. Then he applies it to the harmonic series, showing that

$$\sum_{n \le x} \frac{1}{n} = \text{const} + \log x + \frac{1}{2x} - \frac{1}{12x^2} + \cdots,$$

and performs calculations for $x = 10^{l}$ for l = 1, 2, 3, 4, 5, 6. Finally among other things, he computes $\zeta(2)$ and $\zeta(3)$ with great accuracy. For $\zeta(2)$, he writes

$$\zeta(2) = \sum_{n=1}^{10} \frac{1}{n^2} + \sum_{n=11}^{\infty} \frac{1}{n^2} \, .$$

He computes the first term by hand and then estimates the remainder by the formula. His result is that approximately

$$\zeta(2) = 1.64493406684822643647.$$

Still the evaluation of $\zeta(2)$ in closed form eluded him. Needless to say, this method of approximation opened a whole new area of research.

4. First triumph. Euler's first triumph came in 1734. Having previously done work on the roots of polynomials, he conceived the idea of generalizing the factorization of polynomials to transcendental functions. Euler communicated his result to Daniel Bernoulli and, while unfortunately this letter has been lost, the reply does exist. Daniel says: "The theorem on the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{pp}{6}$$
 and $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots = \frac{p^4}{90}$

is very remarkable. You must no doubt have come upon it *a posteriori*. I should very much like to see your solution."

Here is a sketch of it as it appears in *De summis serierum reciprocarum*. Consider the expression $f(x) = 1 - (\sin x/\sin \alpha)$ with α fixed and α not a multiple of π . Leibniz

had derived the power series expansion for $\sin x$, so write

$$f(x) = 1 - \frac{x}{\sin \alpha} + \frac{x^3}{3! \sin \alpha} - \cdots$$

The right hand side is now viewed as a polynomial of infinite degree. If $a_1, a_2, \dots, a_n, \dots$ are the roots, then write

$$f(x) = \left(1 - \frac{x}{a_1}\right) \left(1 - \frac{x}{a_2}\right) \cdots \left(1 - \frac{x}{a_n}\right) \cdots = \prod_{k=1}^{\infty} \left(1 - \frac{x}{a_k}\right).$$

The roots of f(x) however, are evident from the left hand side, viz.,

$$x = \begin{cases} 2n\pi + \alpha \\ 2n\pi + \pi - \alpha \end{cases} \qquad n = 0 \pm 1, \pm 2, \cdots;$$

thus

$$f(x) = \prod_{n=-\infty}^{\infty} \left(1 - \frac{x}{2n\pi + \alpha}\right) \left(1 - \frac{x}{2n\pi + \pi - \alpha}\right)$$

(F)
$$= \left(1 - \frac{x}{\alpha}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{(2n-1)\pi - \alpha}\right) \left(1 + \frac{x}{(2n-1)\pi + \alpha}\right) \left(1 - \frac{x}{2n\pi + \alpha}\right) \left(1 + \frac{x}{2n\pi - \alpha}\right).$$

We can now expand the right hand side in a power series and equate coefficients. The expansion on the right involves the "infinite" elementary symmetric functions and Euler now derived the infinite analogues of Newton's formulae, viz., if a_1, \dots, a_n, \dots is a sequence and

$$\sigma_m = \sum_{i_1,\dots,i_m} a_{i_1}\cdots a_{i_m}$$

while $S_m = \sum_{i=1}^{\infty} a_i^m$, then in particular,

$$S_1 = \sigma_1, \quad S_2 = \sigma_1^2 - 2\sigma_2, \quad S_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

The other relations may be similarly derived.

Applying these facts to (F) we get (since $\sigma_2 = 0$),

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)\pi - \alpha} - \frac{1}{(2n-1)\pi + \alpha} + \frac{1}{2n\pi + \alpha} - \frac{1}{2n\pi - \alpha} \right) = \frac{1}{\sin \alpha} ,$$

$$\frac{1}{\alpha^2} + \sum_{n=1}^{\infty} \left(\frac{1}{((2n-1)\pi - \alpha)^2} + \frac{1}{((2n-1)\pi + \alpha)^2} + \frac{1}{(2n\pi + \alpha)^2} + \frac{1}{(2n\pi - \alpha)^2} \right)$$

$$= \frac{1}{\sin^2 \alpha} ,$$

$$\frac{1}{\alpha^3} + \sum_{n=1}^{\infty} \left(\frac{1}{((2n-1)\pi - \alpha)^3} - \frac{1}{((2n-1)\pi + \alpha)^3} + \frac{1}{(2n\pi + \alpha)^3} - \frac{1}{(2n\pi - \alpha)^3} \right)$$

$$= \frac{1}{\sin^3 \alpha} - \frac{1}{2\sin \alpha} .$$

Putting $\alpha = \pi/2$, the first gives $(4/\pi)(1 - \frac{1}{3} + \frac{1}{5} \cdots) = 1 - a$ fact already known to James Gregory (1638–1675). The second, however, leads to the long sought after objective, for it gives

$$\frac{8}{\pi^2} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) = 1.$$

However, as Euler remarks,

$$\zeta(2) = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + \frac{1}{4}\zeta(2)$$

and this, then, gives $\zeta(2) = \pi^2/6$. Similar arguments give

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots = \frac{\pi^3}{32} ,$$
$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96} ,$$

and, hence, $\zeta(4) = \pi^4/90$.

Likewise Euler computes the corresponding series with exponents 5, 6, 7, and 8. If $\alpha = \pi/4$, the first relation gives

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots$$

--- a fact he attributes to Newton.

This elegant discovery gave him one of his earliest successes and established him as a mathematician of the first rank.

One is naturally tempted to ask why, if Euler intends to use infinite products, he does not simply use sin x itself? In fact he does; as a postscript to this paper, he notes that

(G)
$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(n\pi)^2} \right)$$

and deduces more directly, $\zeta(2n)$ for n = 1, 2, 3, 4, 5, 6. (G), however, does not give the flexibility of (F) and clearly has no hope of yielding anything about $\zeta(2n + 1)$. One might surmise that he first proved (G) and then the more general result (F).

Two objections were raised to this proof by Daniel Bernoulli. In the first place, one can't compute with infinite series in the same way that one does with polynomials, and in the second place, it is not evident that all the roots of $\sin x = \sin \alpha$ are real. Euler recognizes the second objection as being valid and proceeds to prove that, in fact, all the roots are real. As to the first objection, he rightfully insisted in 1740 that the method is as well founded as any other method and, moreover, it is based upon a principle of which adequate use had not been made. Indeed, it opened up the theory

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of infinite products and partial fraction decomposition of transcendental functions and its importance goes far beyond the immediate application.

5. Connections with arithmetic. Having achieved his objective of evaluating $\zeta(2)$, Euler now turned to the arithmetic properties of $\zeta(s)$. In 1737 he communicated a paper entitled Variae Observationes circa series infinitas.

Here for the first time he proved the famous Euler product decomposition in the form

$$\zeta(s) = \frac{2^s \cdot 3^s \cdot 5^s \cdot 7^s \cdot 11^s \cdots}{(2^s - 1)(3^s - 1)(5^s - 1)(7^s - 1)(11^s - 1) \cdots} .$$

One of his theorems is the statement that

$$\sum_{p} \frac{1}{p} \sim \log \sum_{n} \frac{1}{n},$$

where the left hand side is summed over all p. Nowadays we would insist on writing that as $x \to \infty$

$$\sum_{\leq x} \frac{1}{p} \sim \log \sum_{n \leq x} \frac{1}{n} \; .$$

He also "proved" that if $n = p_1^{r_1} \cdots p_l^{r_l}$ and $\lambda(n) = (-1)^{r_1+r_2+\cdots+r_l}$, then

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$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} = 0$$

and the corresponding fact for $\mu(u)$ (what is now called the Möbius function) is stated in his "Introductio". Regretfully, we have put the word "proved" in quotation marks since the justification of this statement is as deep a result as the prime number theorem itself.

6. Return to $\zeta(s)$. He returned to $\zeta(s)$ in 1740 in a paper entitled *De Seribus Quibusdam Considerationes*. In this he developed the partial fraction decomposition of various functions. In particular, he proved that

$$\frac{\pi \cos[(b-a)/2n]\pi}{n \sin[(b+a)/2n]\pi - n \sin[(b-a)/2n]\pi} = \frac{1}{a} + \sum_{k=1}^{\infty} \frac{2b}{(2k-1)^2 n^2 - b^2} - \frac{2a}{(2kn)^2 - a^2}.$$

By specializing, once again he deduced the values of $\zeta(2), \zeta(4), \cdots$.

In the meantime what has happened to $\zeta(3)$? In this same paper he computed approximate values of $\zeta(2n + 1)$ for n = 1, 2, 3, 4, 5 to which he added the known values of $\zeta(2n)$. He wrote these in the form

$$\zeta(n) = N\pi^n.$$

He says that if n is even, then N is rational, while if N is odd then he conjectures that N is a function of log 2.

EULER AND THE ZETA FUNCTION

There is now a slight digression.

Apparently to respond to the earlier criticism concerning his first proof, Euler published a paper in an obscure journal, "Literary Journal of Germany, Switzerland and the North (The Hague)", entitled *Démonstration de la somme de la suite* $1 + \frac{1}{4} + \frac{1}{9} + \cdots$. Here he derived once again the formula for $\zeta(2)$.

Since this method is elementary, and is not generally known, and can be given in an elementary course, we present it in detail. We have

$$\frac{1}{2}(\arcsin x)^2 = \int_0^x \frac{\arcsin t}{\sqrt{1-t^2}} dt.$$

If we expand $(1 - u^2)^{-\frac{1}{2}}$ by the binomial theorem and integrate termwise, we get

$$\arcsin t = \int_0^t \frac{du}{\sqrt{1-u^2}} = t + \sum_{n=1}^\infty \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \frac{t^{2n+1}}{2n+1}$$

It follows that

$$\frac{1}{2}(\arcsin x)^2 = \int_0^x \frac{t dt}{\sqrt{1-t^2}} + \sum_{n=1}^\infty \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \frac{1}{2n+1} \int_0^x \frac{t^{2n+1}}{\sqrt{1-t^2}} dt.$$

Euler then writes, "Since the first term is integrable all the others will also be since the integration of each term reduces to that of the preceding. One can see this clearly if we reflect that in general

$$\int_0^x \frac{t^{n+2}}{\sqrt{1-t^2}} dt = \frac{n+1}{n+2} \int_0^x \frac{t^n}{\sqrt{1-t^2}} dt - \frac{x^{n+1}}{n+2} \sqrt{1-x^2}.$$

(Apparently the favorite phrases of mathematicians, "clearly etc.," are not of recent origin!) In fact, it takes a few steps to see this "clearly." Let

$$I_n(x) = \int_0^x \frac{t^{n+2}}{\sqrt{1-t^2}} = \int_0^x \frac{t^{n+1}tdt}{\sqrt{1-t^2}}$$

Integration by parts gives $I_n(x) = -x^{n+1}\sqrt{1-x^2} + (n+1)\int_0^x t^n(\sqrt{1-t^2})dt$. Multiplying the integrand by $1 = \sqrt{1-t^2}/\sqrt{1-t^2}$, and splitting into 2 parts gives

$$I_n(x) = -x^{n+1}\sqrt{1-x^2} + (n+1)I_{n-2}(x) - (n+1)I_n(x),$$

and the result follows.

Thus

(H)
$$\int_0^1 \frac{t^{2n+1}}{\sqrt{1-t^2}} = \frac{2n}{2n+1} \int_0^1 \frac{t^{2n-1}}{\sqrt{1-t^2}}$$

and as $\int_0^1 t dt / \sqrt{1 - t^2} = 1$, we conclude that

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$$\int_0^1 \frac{t^{2n+1}}{\sqrt{1-t^2}} = \frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3}$$

Therefore $\pi^2/8 = \frac{1}{2}(\arcsin 1)^2 = \sum_{n=0}^{\infty} 1/(2n+1)^2$, which as we know from above is equivalent to $\zeta(2) = \pi^2/6$. The same result may be obtained by first showing that $\frac{1}{2}(\arcsin x)^2$ satisfies the differential equation

$$(1 - x^2)y'' - xy' = 1,$$

then using undetermined coefficients to derive the series for $\frac{1}{2}(\arcsin x)^2$, and finally integrating termwise to get $\frac{1}{6}(\arcsin x)^3$, after using the above result (H). The reader will find it interesting to carry out these steps. The method gives $\zeta(2) = \pi^2/6$ directly. Euler concludes with the remark that despite repeated efforts, he was unable to use this technique to find $\zeta(2n)$ for $n \ge 2$. The reader will note that we have glossed over the mild difficulties associated with the point x = 1.

Since the time of Euler, there have been many proofs giving the value of $\zeta(2n)$. The interested reader is urged to consult K. Knopp's book on "Infinite Series."

7. The functional equation and $\zeta(3)$. In the middle of the paper De Seriebus... referred to above, Euler began a highly interesting new development. There he states that

	$1 - 3 + 5 - 7 + \dots = 0$
	$1 - 3^3 + 5^3 - 7^3 + \cdots = 0,$
etc., whereas,	
	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2,$
	$1-2+3-4+\cdots = \frac{1}{4},$
	$1 - 2^3 + 3^3 - 4^3 + \cdots = - \frac{1}{8},$
	$1 - 2^5 + 3^5 - 4^5 + \cdots = \frac{1}{4},$
	$1 - 2^7 + 3^7 - 4^7 + \cdots = - \frac{17}{16}$
On the other hand,	

 $1 - 2^2 + 3^2 - 4^2 + \cdots = 0,$ $1 - 2^4 + 3^4 - 4^4 + \dots = 0.$ $1 - 2^6 + 3^6 - 4^6 + \dots = 0.$

Where do these come from? They are derived as follows. Let

$$f(x) = 1 + x + x^{2} + \dots + x^{n} + \dots = 1/(1 - x)$$
 if $|x| < 1$.

Euler has no reluctance to put x = -1; then $1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$. To f(x) apply the operator x(d/dx). Then

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$$x \frac{d}{dx}f(x) = x + 2x^2 + 3x^3 + \dots = \frac{x}{(1-x)^2};$$

putting x = -1, gives $1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$.

Apply the operator again:

$$x + 2^2 x^2 + 3^2 x^3 + \cdots = \frac{x(1+x)}{(1-x)^2}$$

Putting x = -1, gives $1 - 2^2 + 3^2 - \cdots = 0$.

As the series converges at each stage of this process for |x| < 1, we have Euler anticipating "Abel summability" by some 75 years. Then Euler notes that

$$1 - 2 + 3 - 4 + \dots = \frac{1}{4} = \frac{2 \cdot 1}{\pi^2} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) ,$$

$$1 - 2^3 + 3^3 - 4^3 + \dots = -\frac{1}{8} = \frac{-2 \cdot 3!}{\pi^4} \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right) ,$$

$$1 - 2^5 + 3^5 - 4^5 + \dots = -\frac{1}{4} = \frac{2 \cdot 5!}{\pi^6} \left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \dots \right) ,$$

$$1 - 2^7 + 3^7 - 4^7 + \dots = -\frac{17}{16} = \frac{-2 \cdot 7!}{\pi^8} \left(1 + \frac{1}{3^8} + \frac{1}{5^8} + \dots \right) ,$$

as can be verified by an easy computation using the values of $\theta(2n)$:

As in Section 1, let $\theta(s) = \sum_{n=0}^{\infty} 1/(2n+1)^s$ and $\phi(s) = \sum_{n=1}^{\infty} (-1)^{n-1}/n^s$. These relations can be rephrased as

$$\theta(1-2n) = \frac{(-1)^{n-1} 2 \cdot (2n-1)!}{\pi^{2n}} \phi(2n) \quad (n = 1, 2, 3, 4),$$

where, of course, $\theta(m)$, $(m = 0, \pm 1, \pm 2, \cdots)$ is to be understood as

$$\lim_{x \to 1^{-}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^m} .$$

Although he does not explicitly say so, one gets the impression that Euler is trying energetically to develop a technique for evaluating $\zeta(3)$, and this impression is partially confirmed later, as we shall see.

In 1749 he gave a paper to the Berlin Academy entitled Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques.

This time he considers

$$\phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

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alone and notes the following relations:

$$\frac{1-2+3-4+5-6+\cdots}{1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\cdots} = +\frac{1\cdot(2^2-1)}{(2-1)\pi^2},$$

$$\frac{1^2-2^2+3^2-4^2+5^2-6^2+\cdots}{1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\frac{1}{6^3}+\cdots} = 0,$$

$$\frac{1^3-2^3+3^3-4^3+5^3-6^3+\cdots}{1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\cdots} = -\frac{1\cdot2\cdot3(2^4-1)}{(2^3-1)\pi^4},$$

$$\frac{1^4-2^4+3^4-4^4+5^4-6^4+\cdots}{1-\frac{1}{2^5}+\frac{1}{3^5}-\frac{1}{4^5}+\frac{1}{5^5}-\frac{1}{6^5}+\cdots} = 0,$$

or if $n \geq 2$,

(j)
$$\frac{\phi(1-n)}{\phi(n)} = \begin{cases} \frac{(-1)^{(n/2)+1}(2^n-1)(n-1)!}{(2^{n-1}-1)\pi^n} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

These relations are listed for $n = 2, 3, \dots 10$. On the other hand, if n = 1, we see that

$$\frac{1-1+1-1+\cdots}{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots} = \frac{1}{2\ln 2},$$

"whose connection with the others is entirely hidden." (J) is now rewritten in the form

$$\frac{\phi(1-n)}{\phi(n)} = \frac{-(n-1)!(2^n-1)}{(2^{n-1}-1)\pi^n} \cos \frac{\pi n}{2},$$

and Euler says "I shall hazard the following conjecture:

(K)
$$\frac{\phi(1-s)}{\phi(s)} = \frac{-\Gamma(s)(2^s-1)\cos \pi s/2}{(2^{s-1}-1)\pi^s}$$

is true for all s". Isn't this derivation beautiful!?

Now taking the limit of the right hand side as $s \to 1$ gives exactly $1/2\ln 2!$ Euler continues: "The validity of our conjecture for s = 1 (which case first appeared to deviate from the others) is already a strong justification¹ of the truth of our conjec-

¹ Although Euler uses the word "preuve," the original meaning in English (and presumably also in French) conveys the idea of testing an assumption or statement rather than proving in our sense. Compare, for example, the expression "the exception that 'proves' the rule."

ture since it appears unlikely that a false assumption could have upheld the truth of this case. We can therefore regard our conjecture as being solidly based but I shall give other justifications which are equally convincing."

He then checks the formula for $s = \frac{1}{2}, \frac{3}{2}$, and in general s = (2k + 1)/2.

We have seen in Chapter 1, that

$$\phi(s) = (1 - 2^{1-s})\zeta(s),$$

which leads at once from (K) to

$$\zeta(1-s) = \pi^{-s} 2^{1-s} \Gamma(s) \cos \frac{\pi s}{2} \zeta(s),$$

and this is the famous functional equation². It was proved by Riemann in 1859.

It should be noted that Euler could not have used $\zeta(s)$ itself since

$$\lim_{x \to 1^-} \sum_{n=1}^{\infty} n^k x^n$$

does not exist for $k = 0, 1, 2, \dots$ and therefore he could not have attached a meaning to

$$\sum_{n=1}^{\infty} n^{-(1-s)}$$

for $s = 2, 3, \cdots$.

On the other hand, it can be shown that the series

$$\phi(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1 - 2^{1-s}) \zeta(s)$$

converges for s > 0 (in fact if $s = \sigma + it$, for $\sigma > 0$), but as the pole of $\zeta(s)$ at s = 1 has been removed by the factor $(1 - 2^{1-s})$, there remains nothing in the nature of $\phi(s)$ to account for this limitation, and it turns out that

$$\sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$$

is Abel summable for every value of s.

One is naturally tempted to ask whether Riemann could have seen Euler's work. There is no evidence that he had³.

Euler continues:

"As far as the sum of the reciprocals of powers (i.e., $\sum_{n=1}^{\infty} (-1)^{n+1}/n^k$) is concerned, I have already observed that their sum can be assigned a value only when k

² Since completing this article the author has found that E. Landau has given a rigorous proof of the functional equation in the form (K). See *Bibliotheca Mathematica*, vol. 7 (1906–1907) pp. 69-79.

³ Added in proof. A. Weil remarks that the external evidence supports strongly the view that Riemann was very familiar with Euler's contributions.

is even and that when k is odd, all my efforts have been useless up to now."

Euler now observes as follows: If $s = 2\lambda + 1$, then

$$\phi(2\lambda + 1) = - \frac{(2^{2\lambda} - 1)\pi^{2\lambda+1}}{\Gamma(2\lambda + 1)(2^{2\lambda+1} - 1)} \frac{\phi(-2\lambda)}{\cos((2\lambda + 1)\pi/2)},$$

and $\phi(-2\lambda)$ as well as $\cos((2\lambda + 1)\pi/2)$ vanish if λ is an integer. Taking the limit as $\lambda \to m$ a positive integer with the help of l'Hospital's rule, we get

(L)
$$\phi(2m+1) = + \frac{2(2^{2m}-1)\pi^{2m}}{(2m)!(2^{2m+1}-1)} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} n^{2m} \log n}{\cos \pi m}$$

"It is necessary therefore to find the value of these sums

$$\sum_{n=1}^{\infty} (-1)^{n+1} n^{2m} \log n.$$

But this research is probably more difficult than the one we have in mind (meaning $\phi(2m + 1)$) and I perceive no method whatsoever which could lead us to the proposed objective."

He returned to the question for what appears to be the last time in 1772 in a paper entitled *Exercitationes Analyticae*. Through a striking and elaborate scheme, he proved that

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \dots = \frac{\pi^2}{4} \log 2 + 2 \int_0^{\pi/2} x \log \sin x dx.$$

Here is a sketch of the proof which invokes the extreme virtuosity of a master:

We know from (L) that

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \dots = \frac{\pi^2}{2} Z,$$
$$Z = \sum_{n=2}^{\infty} (-1)^n n^2 \log n.$$

where

This follows from (L) as well as the relations cited in Section 1. Of course we con-
tinue to understand that if
$$\sum_{n=1}^{\infty} a_n$$
 does not converge but $\sum_{n=1}^{\infty} a_n x^n$ converges
for $|x| < 1$, then $\sum_{n=1}^{\infty} a_n$ is defined by

$$\lim_{x \to 1^{-}} \sum_{n=1}^{\infty} a_n x^n$$
, if this limit exists.

Euler then shows that

$$Z = \sum_{n=1}^{\infty} n^2 \log \frac{(2n)^2}{(2n-1)(2n+1)} - \sum_{n=1}^{\infty} n(n+1) \log \frac{(2n+1)^2}{(2n)(2n+2)} .$$

The expansion of the logarithm is carried out and the series rearranged. Letting

 $\lambda(s) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{(n(n+1))^s}, \text{ then}$ $Z = \frac{1}{2 \cdot 2^2} + \sum_{n=2}^{\infty} \frac{1}{n 2^{2n}} (\zeta(2n-2) + (-1)^n \lambda(n-1)).$

 $\lambda(n)$ is then expressed in terms of $\zeta(2k)$ $(k = 1, 2, \dots, n)$, and if

$$S(n) = \frac{1}{n2^{2n}} + \sum_{k=1}^{\infty} \frac{(n+k-1)(n+k)\cdots(n+2k-2)}{k!(n+k)2^{2n+2k}},$$

then

$$Z = -\frac{1}{8} + S(1) + 2 \sum_{n=1}^{\infty} \zeta(2n) \left(\frac{1}{(2n+2)2^{n+2}} - S(2n+1) \right)$$

He now finds the sum S(n) by showing that

$$S_x(n) = \frac{x^n}{n} + \sum_{k=1}^{\infty} \frac{(n+k-1)(n+k)\cdots(n+2k-2)x^{n+k}}{k!(n+k)}$$

satisfies a difference differential equation and that

$$S_x(1) = \frac{1 + 2x - \sqrt{1 - 4x}}{4}$$

This is to be evaluated when $x = \frac{1}{4}$. The result of these intricate details is that

$$S(2n + 1) - \frac{1}{(2n + 2)2^{2n+2}} = \frac{1}{(2n + 1)(2n + 2)2^{2n+1}} \cdot Z = \frac{1}{2^2} - \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n + 1)(2n + 2)2^{2n}} \cdot Z$$

We know that $\zeta(2n) = \alpha_{2n} \pi^{2n}$, where α_{2n} is explicitly determined in terms of the Bernoulli numbers.

If then

$$f(x) = x^2 \sum_{n=1}^{\infty} \frac{\alpha_{2n} x^{2n}}{(2n+1)(2n+2)},$$

then by twice differentiating f(x), we see that it satisfies a differential equation which can be solved in view of the fact that we can evaluate the generating function

$$\sum_{n=1}^{\infty} \alpha_{2n} x^{2n}.$$

Is not this derivation breathtaking, especially in the light of the fact that Euler was now blind and these calculations were performed mentally!

8. Conclusion. So end the main contributions of Euler to the zeta function. He

did, however, write a brief paper on the function $\sum_{n=1}^{\infty} x^n/n^2$ toward the end of his life (1779), which was published posthumously. We have given only the highlights of his work on $\zeta(s)$. Scattered throughout his papers on analysis and in his correspondence with Goldbach and the Bernoulli's are many results which are related to the problem.

While he did not succeed in every objective he set himself, his triumphs stand like a grand fresco — a monument to his extraordinary imagination and sense of beauty and harmony.

Acknowledgements

In addition to the original papers themselves, the author has found the following sources especially helpful:

1. The preface of Vol. 16 of series 1 of Euler's Collected Works is an article entitled "Übersicht über die Bände 14, 5.16, 16," by Georg Faber. Faber gives a summary of the contents, classified by topics.

2. The article by Paul Stäckel, "Eine vergessene Abhandlung Eulers." This first appeared in the now defunct journal Bibliotheca Mathematica, 83 (1907–1908) 37–54. In this, Stäckel discusses the article "Démonstration de la somme..." and gives numerous interesting historical facts. It is reprinted in Euler's Collected Works, Vol. 14, pp. 156–176.

3. Correspondence between Euler and Goldbach published by Deutsche Akademie der Wissenschaften and edited by A. Juskevic and E. Winter. The editors' comments on the letters were very helpful.

4. The paper of Landau referred to in the footnote.

5. The referee kindly suggested stylistic changes and pointed out some errors.

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FHE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

A. P. HILLMAN

The following results of the thirty-fourth William Lowell Putnam Mathematical Competition, held on December 1, 1973, have been determined in accordance with the regulations governing the Competition. This competition is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of the **California Institute of Technology**, Pasadena, California. The members of the team were Arthur L. Rubin, James B. Shearer, and Michael F. Yoder; to each of these a prize of one hundred dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of the University of British Columbia, Vancouver, British Columbia. The