DEVELOPMENT OF CALCULUS IN INDIA

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Abstract

In this article we shall present an overview of the development of calculus in Indian mathematical tradition. The article is divided naturally into two parts. In the first part we shall discuss the developments during what may be called the classical period, starting with the work of Āryabhaṭa (c. 499 CE) and extending up to the work Nārāyaṇa Paṇḍita (c. 1350). The work of the Kerala School starting with Mādhava of Saṅgamagrāma (c. 1350), which has a more direct bearing on calculus, will be dealt with in the second part. Here we shall discuss some of the contributions of the Kerala School during the period 1350–1500 as outlined in the seminal Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva (c. 1530).

PART I : THE CLASSICAL PERIOD

Āryabhaṭa to Nārāyaṇa Paṇḍita (c. 500–1350 CE)

1 Introduction

In his pioneering history of calculus written sixty years ago, Carl Boyer was totally dismissive of the Indian contributions to the conceptual development of the subject.[1] Boyer's historical overview was written around the same time when (i) Ramavarma Maru Thampuran and Akhileswarayyar brought out the first edition of the Mathematics part of the seminal text *Gaṇita-yukti-bhāṣā*, and (ii) C.T. Rajagopal and his collaborators, in a series of pioneering studies, drew attention to the significance of the results and techniques outlined in *Yuktibhāṣā* (and the work of the Kerala School of Mathematics in general), which seem to have been forgotten after the initial notice by Charles Whish in early nineteenth century. These and the subsequent studies have led to a somewhat different perception of the Indian contribution to the development of calculus as may be gleaned from the following quotation from a recent work on the history of mathematics:[2]

> We have here a prime example of two traditions whose aims were completely different. The Euclidean ideology of proof which was so influential in the Islamic world had no apparent influence in India (as al-Biruni

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[1] C.B.Boyer, *The History of the Calculus and its Conceptual Development*, Dover, New York 1949, pp. 61–62.

[2] L. H. Hodgekin, *A Histoty of Mathematics: From Mesopotamia to Modernity*, Oxford 2005, p. 168.

> had complained long before), even if there is a possibility that the Greek tables of 'trigonometric functions' had been transmitted and refined. To suppose that some version of 'calculus' underlay the derivation of the series must be a matter of conjecture.
>
> The single exception to this generalization is a long work, much admired in Kerala, which was known as *Yuktibhāṣā*, by Jyeṣṭhadeva; this contains something more like proofs—but again, given the different paradigm, we should be cautious about assuming that they are meant to serve the same functions. Both the authorship and date of this work are hard to establish exactly (the date usually claimed is the sixteenth century), but it does give explanations of how the formulae are arrived at which could be taken as a version of the calculus.

The Malayalam work *Gaṇita-yukti-bhāṣā* (c. 1530) of Jyeṣṭhadeva indeed presents an overview of the work of Kerala School of mathematicians during the period 1350–1500 CE. The Kerala School was founded by Mādhava (c. 1340–1420), who was followed by the illustrious mathematician-astronomers Parameśvara (c. 1380–1460), his son Dāmodara and the latter's student Nīlakaṇṭha Somayājī (c. 1444–1550). While the achievements of the Kerala School are indeed spectacular, it has now been generally recognised that these are in fact very much in continuation with the earlier work of Indian mathematicians, especially of the *Āryabhaṭan* school, during the period 500–1350 CE.

In the first part of this article, we shall consider some of the ideas and methods developed in Indian mathematics, during the period 500–1350, which have a bearing on the later work of the Kerala School. In particular, we shall focus on the following topics: Mathematics of zero and infinity; iterative approximations for irrational numbers; summation (and repeated summations) of powers of natural numbers; use of second-order differences and interpolation in the calculation of *jyā* or Rsines; the emergence of the notion of instantaneous velocity of a planet in astronomy; and the calculation of the surface area and volume of a sphere.

2 Zero and Infinity

2.1 Background

The *śānti-mantra* of *Īśāvāsyopaniṣad* (of *Śukla-yajurveda*), a text of *Brahmavidyā*, refers to the ultimate absolute reality, the Brahman, as *pūrṇa*, the perfect, complete or full. Talking of how the universe emanates from the Brahman, it states:

> पूर्णमदः पूर्णमिदं पूर्णात्पूर्णमुदच्यते।
> पूर्णस्य पूर्णमादाय पूर्णमेवावशिष्यते॥
>
> That (Brahman) is *pūrṇa*; this (the universe) is *pūrṇa*; [this] *pūrṇa* emanates from [that] *pūrṇa*; even when *pūrṇa* is drawn out of *pūrṇa*, what remains is also *pūrṇa*.

Pāṇini's *Aṣṭādhyāyī* (c. 500 BCE) has the notion of *lopa* which functions as a null-morpheme. *Lopa* appears in seven *sūtras* of Chapters 1, 3, 7, starting with

> अदर्शनं लोपः। (1.1.60).

Śūnya appears as a symbol in Piṅgala's *Chandaḥ-sūtra* (c. 300 BCE). In Chapter VIII, while enunciating an algorithm for evaluating any positive integral power of 2 in terms of an optional number of squaring and multiplication (duplication) operations, *śūnya* is used as a marker:

> रूपे शून्यम्। द्विः शून्ये। (8.29-30).

Different schools of Indian philosophy have related notions such as the notion of *abhāva* in Nyāya School, and the *śūnyavāda* of the Bauddhas.

2.2 Mathematics of zero in *Brāhmasphuṭa-siddhānta* (c. 628 CE) of Brahmagupta

The *Brāhmasphuṭa-siddhānta* (c. 628 CE) of Brahmagupta seems to be the first available text that discusses the mathematics of zero. *Śūnya-parikarma* or the six operations with zero are discussed in the chapter XVIII on algebra (*kuṭṭakādhyāya*), in the same six verses in which the six operations with positives and negatives (*dhanarṇa-ṣaḍvidha*) are also discussed. Zero divided by zero is stated to be zero. Any other quantity divided by zero is said to be *taccheda* (that with zero-denominator):[3]

> धनयोर्धनमृणमृणयोर्धनर्णयोरन्तरं समैक्यं खम्।
> ऋणमैक्यं च धनमृणधनशून्ययोः शून्यम्॥
> ऊनमधिकाद्धिशोध्यं धनं धनादृणमृणादधिकमूनात्।
> व्यस्तं तदन्तरं स्यादृणं धनं धनमृणं भवति॥
> शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम्।
> शोध्यं यदा धनमृणादृणं धनाद्वा तदा क्षेप्यम्॥
> ऋणमृणधनयोर्घातो धनमृणयोर्धनवधो धनं भवति।
> शून्यर्णयोः खधनयोः खशून्ययोर्वा वधः शून्यम्॥
> धनभक्तं धनमृणहृतमृणं धनं भवति खं खभक्तं खम्।
> भक्तमृणेन धनमृणं धनेन हृतमृणमृणं भवति॥
> खोद्धृतमृणं धनं वा तच्छेदं खमृणधनविभक्तं वा।
> ऋणधनयोर्वर्गः स्वं खं खस्य पदं कृतिर्यत् तत्॥
>
> …[The sum of] positive (*dhana*) and negative (*ṛṇa*), if they are equal, is zero (*kham*). The sum of a negative and zero is negative, of a positive and zero is positive and of two zeros, zero (*śūnya*).

[3] *Brāhmasphuṭasiddhānta* of Brahmagupta, Ed. with his own commentary by Sudhakara Dvivedi, Benaras 1902, verses 18.30–35, pp. 309–310.

> ...Negative subracted from zero is positive, and positive from zero is negative. Zero subtracted from negative is negative, from positive is positive, and from zero is zero (*ākāśa*).
>
> ...The product of zero and a negative, of zero and a positive, or of two zeroes is zero.
>
> ...A zero divided by zero is zero.
>
> ...A positive or a negative divided by zero is that with zero-denominator.

2.3 Bhāskarācārya on *Khahara*

Bhāskarācārya II (c. 1150), while discussing the mathematics of zero in *Bījagaṇita*, explains that infinity (*ananta-rāśi*) which results when some number is divided by zero is called *khahara*. He also mentions the characteristic property of infinity that it is unaltered even if 'many' are added to or taken away from it, in terms similar to the invocatory verse of *Īśāvāsyopaniṣad* mentioned above:[4]

> खहरो भवेत् खेन भक्तश्च राशिः ॥
>
> द्विघ्नं त्रिहृत् खं खहृतं त्रयं च शून्यस्य वर्गं वद मे पदं च॥
>
> ...अयमनन्तो ३/० राशिः खहरः इत्युच्यते।
>
> अस्मिन्विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु।
> बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत्॥
>
> A quantity divided by zero will be (called) *khahara* (an entity with zero as divisor).
>
> Tell me ...three divided by zero ...This infinite (*ananta* or that without end) quantity $\frac{3}{0}$ is called *khahara*.
>
> In this quantity, *khahara*, there is no alteration even if many are added or taken out, just as there is no alteration in the Infinite (*ananta*), Infallible (*acyuta*) [Brahman] even though many groups of beings enter in or emanate from [It] at times of dissolution and creation.

2.4 Bhāskarācārya on multiplication and division by zero

Bhāskarācārya while discussing the mathematics of zero in *Līlāvatī*, notes that when further operations are contemplated, the quantity being multiplied by zero should not be changed to zero, but kept as is. Further he states that when the quantity which is multiplied by zero is also divided by zero, then it remains unchanged. He follows this up with an example and declares that this kind of calculation has great relevance in astronomy:[5]

[4] *Bījagaṇita* of Bhāskarācārya, Ed. by Muralidhara Jha, Benaras 1927, *Vāsanā* on *Khaṣaḍvidham* 3, p. 6.

[5] *Līlāvatī* of Bhāskarācārya, Ed. by H. C. Bannerjee, Calcutta 1927, *Vāsanā* on verses 45–46, pp. 14–15.

> योगे खं क्षेपसमं वर्गादौ खं खभाजितो राशिः।
> खहरः स्यात् खगुणः खं खगुणश्चिन्त्यश्च शेषविधौ॥
> शून्ये गुणके जाते खं हारश्चेत्पुनस्तदा राशिः।
> अविकृत एव ज्ञेयस्तथैव खेनोनितश्च युतः॥
> खं पञ्चयुग्भवति किं वद खस्य वर्गं मूलं घनं घनपदं खगुणाश्च पञ्च।
> खेनोद्धृता दश च कः खगुणो निजार्धयुक्तस्त्रिभिश्चगुणितः खहृतस्त्रिषष्टिः॥
>
> ...अज्ञातो राशिः तस्य गुणः ०। सार्धं क्षेपः १/२। गुणः ३। हरः ०।
> दृश्यं ६३ ।
> ततो वक्ष्यमाणेन विलोमविधिना इष्टकर्मणा वा लब्धो राशिः १४।
> अस्य गणितस्य ग्रहगणिते महानुपयोगः।
>
> ...A quantity multiplied by zero is zero. But it must be retained as such when further operations [involving zero] are contemplated. When zero is the multiplier of a quantity, if zero also happens to be a divisor, then that quantity remains unaltered ...
>
> ...What is the number which when multiplied by zero, being added to half of itself multiplied by 3 and divided by zero, amounts to sixty-three?
>
> ...Either following the inverse process or by choosing a desired number for the unknown ('rule of false position'), the quantity is obtained to be 14. This kind of calculation is of great use in mathematical astronomy.

Bhāskara works out his example as follows:

$$\begin{aligned} 0\left[\left(x+\frac{x}{2}\right)\times\frac{3}{0}\right] &= 63 \\ \frac{3x}{2}\times 3 &= 63. \\ \text{Therefore,} \qquad x &= 14. \end{aligned} \tag{1}$$

Bhāskara, it seems, had not fully mastered this kind of "calculation with infinitesimals" as is clear from the following example that he presents in *Bījagaṇita* while solving quadratic equations by eliminating the middle term:[6]

> कः स्वार्धसहितो राशिः खगुणो वर्गितो युतः।
> स्वपदाभ्यां खभक्तश्च जाताः पञ्चदशोच्यताम्॥
>
> Say what is the number which when added to half of itself, multiplied by zero, squared and the square being augmented by twice its root and divided by zero, becomes fifteen?

Clearly the problem as stated is

$$\frac{\left[0(x+\frac{x}{2})\right]^2 + 2\times\left[0(x+\frac{x}{2})\right]}{0} = 15. \tag{2}$$

[6] *Bījagaṇita*, cited above, *Vāsanā* on *avyaktavargādi-samīkaraṇam* 5, pp. 63–64.

Bhāskara in his $V\bar{a}san\bar{a}$ seems to just cancel out the zeros without paying any heed to the different powers of zero involved. He converts the problem into the equation

$$\left[x+\frac{x}{2}\right]^2+2\times\left[x+\frac{x}{2}\right]=15. \tag{3}$$

Solving this, by the method of elimination of the middle term, Bhāskara obtains the solution $x=2$. The other solution $(-\frac{10}{3})$ is not noted.

3 Irrationals and iterative approximations

3.1 Background

$Baudh\bar{a}yana$-$\acute{s}ulva$-$s\bar{u}tra$ gives the following approximation for $\sqrt{2}$:[7]

> प्रमाणं तृतीयेन वर्धयेत्तच्च चतुर्थेनात्मचतुस्त्रिंशोनेन। सविशेषः।
>
> The measure [of the side] is to be increased by its third and this [third] again by its own fourth less the thirty-fourth part [of the fourth]. That is the approximate diagonal ($savi\acute{s}e\d{s}a$).

$$\begin{aligned}\sqrt{2} &\approx 1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}\\ &= \frac{577}{408}\\ &= 1.4142156.\end{aligned} \tag{4}$$

The above approximation is accurate to 5 decimal places.

$Baudh\bar{a}yana$-$\acute{s}ulva$-$s\bar{u}tra$ also gives an approximation for π: [8]

> चतुरश्रं मण्डलं चिकीर्षन्नक्ष्णयार्धं मध्यात्प्राचीमभ्यापातयेत्।
> यदतिशिष्यते तस्य सहतृतीयेन मण्डलं परिलिखेत्।
>
> If it is desired to transform a square into a circle, [a cord of length] half the diagonal of the square is stretched from the centre to the east; with one-third [of the part lying outside] added to the remainder [of the half-diagonal] the [required] circle is drawn.

If a is half-the side of the square, then the radius r of the circle is given by

$$r\approx\left(\frac{a}{3}\right)(2+\sqrt{2}). \tag{5}$$

This corresponds to $\pi\approx 3.0883$.

[7] $Baudh\bar{a}yana\acute{s}ulvas\bar{u}tram$ (1.61-2), in *The Śulvasūtrās*, Ed. by S. N. Sen and A. K. Bag, New Delhi 1983, p. 19.

[8] $Baudh\bar{a}yana\acute{s}ulvas\bar{u}tram$ (1.58), ibid., p. 19.

3.2 Algorithm for square-roots in *Āryabhaṭīya*

The *Āryabhaṭīya* of Āryabhaṭa (c. 499 CE) gives a general algorithm for computing the successive digits of the square root of a number. The procedure given in the following verse is also elucidated by an example:[9]

भागं हरेदवर्गान्नित्यं द्विगुणेन वर्गमूलेन।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम्॥

Always divide the non-square (even) place by twice the square-root [already found]. Having subtracted the square [of the quotient] from the square (odd) place, the quotient gives the [digit in the] next place in the square-root.

	7	5	
5	6	2	5
4	9		
14)	7	2	(5
	7	0	
		2	5
		2	5
		0	0

3.3 Approximating the square-root of a non-square number

The method for obtaining approximate square-root (*āsanna-mūla*) of a non-square number (*amūlada-rāśi*) is stated explicitly in *Triśatikā* of Śrīdhara (c. 750):[10]

राशेरमूलदस्याहतस्य वर्गेण केनचिन्महता।
मूलं शेषेण विना विभजेद्गुणवर्गमूलेन॥

Multiply the non-square number by some large square number, take the square-root [of the product] neglecting the remainder, and divide by the square-root of the multiplier.

Nārāyaṇa Paṇḍita (c. 1356) has noted that the solutions of *varga-prakṛti* (the so called Pell's equation) can be used to compute successive approximations to the square-root of a non-square number:[11]

मूलं ग्राह्यं यस्य च तद्रूपक्षेपजे पदे तत्र।
ज्येष्ठं ह्रस्वपदेन च समुद्धरेत् मूलमासन्नम्॥

[With the number] whose square-root is to be found as the *prakṛti* and unity as the *kṣepa*, [obtain the greater and smaller] roots. The greater root divided by the lesser root is an approximate value of the square-root.

Nārāyaṇa considers the example

$$10x^2 + 1 = y^2, \tag{6}$$

[9] *Āryabhaṭīya* of Āryabhaṭa, Ed. by K. S. Shukla and K. V. Sarma, New Delhi 1976, *Gaṇitapāda* 4, p. 36.

[10] *Triśatikā* of Śrīdhara, Ed. by Sudhakara Dvivedi, Varanasi 1899, verse 46, p. 34.

[11] *Gaṇitakaumudī* of Nārāyaṇa Paṇḍita, Ed. by Padmakara Dvivedi, Part II, Benaras 1942, verse 10.17, p. 244.

and gives the approximate values:

$$\sqrt{10} \approx \frac{19}{6},\ \frac{721}{228},\ \frac{27379}{8658}, \tag{7}$$

which are obtained by successive compositions ($bh\bar{a}van\bar{a}$) of the basic solutions 6, 19:[12]

$$228 = (2)(6)(19), \qquad 721 = (10)(6)^2 + (19)^2, \qquad \text{and so on.}$$

3.4 Approximate value of π in *Āryabhaṭīya*

Āryabhaṭa (c. 499) gives the following approximate value for π:[13]

> चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम्।
> अयुतद्वयविष्कम्भस्यासन्नो वृत्तपरिणाहः॥
>
> One hundred plus four multiplied by eight and added to sixty-two thousand: This is the approximate measure of the circumference of a circle whose diameter is twenty-thousand.

Thus as per the above verse $\pi \approx \frac{62832}{20000} = 3.1416$.

3.5 Successive doubling of the sides of the circumscribing polygon

It appears that Indian mathematicians (at least in the Āryabhaṭan tradition) employed the method of successive doubling of the sides of a circumscribing polygon—starting from the circumscribing square leading to an octagon, etc.—to find successive approximations to the circumference of a circle. This method has been described in the later Kerala texts *Yuktibhāṣā* (c. 1530) of *Jyeṣṭhadeva* and *Kriyākramakarī* commentary (c. 1535) of Śaṅkara Vāriyar on *Līlāvatī*, of Bhāskara II. The latter cites the verses of Mādhava (c. 1340–1420) in this connection and notes at the end that:[14]

> एवं यावदभीष्टं सूक्ष्मतामापादयितुं शक्यम्।
>
> Thus, one can obtain [an approximation to the circumference of the circle] to any desired level of accuracy.

We now outline this method as described in *Yuktibhāṣā*.[15] In Figure 1, $EOSA_1$ is the first quadrant of the square circumscribing the given circle. EA_1 is half the side of the

[12] *Bhāvanā* or the rule of composition enunciated by Brahmagupta is the tranformation $(X, Y) \rightarrow (X^2 + DY^2,\ \ 2XY)$ which tranforms a solution $x = X$, $y = Y$ of the equation $x^2 - Dy^2 = 1$, into another solution with larger values for x, y, which correspond to higher convergents in the continued fraction expansion of $\sqrt{D}$ and thus give better approximations to it.

[13] *Āryabhaṭīya*, cited above, *Gaṇitapāda* 10, p. 45.

[14] *Līlāvatī* of Bhāskara II, Ed. with commentary *Kriyākramakarī* of Śaṅkara Vāriyar by K. V. Sarma, Hoshiarpur 1975, comm. on verse 199, p. 379.

[15] *Gaṇita-yukti-bhāṣā* of Jyeṣṭhadeva, Ed. and Tr. by K. V. Sarma, with Exp. Notes by K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, 2 Vols, Hindustan Book Agency, New Delhi 2008. Reprint Springer 2009, Vol. I Section 6.2, pp. 46–49, 180–83, 366–69.

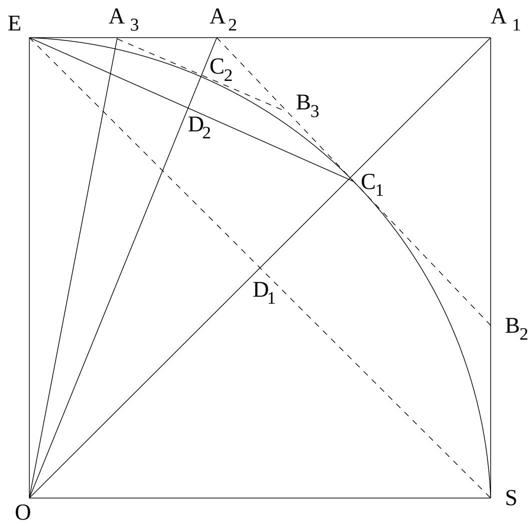

Figure 1: Finding the circumference of a square from cirumscribing polygons.

circumscribing square. Let OA_1 meet the circle at C_1. Draw $A_2C_1B_2$ parallel to ES. EA_2 is half the side of the circumscribing octagon.

Similarly, let OA_2 meet the circle at C_2. Draw $A_3C_2B_3$ parallel to EC_1. EA_3 is now half the side of a circumscribing regular polygon of 16 sides. And so on. Let half the sides of the circumscribing square, octagon etc., be denoted

$$l_1 = EA_1,\ l_2 = EA_2,\ l_3 = EA_3,\ \ldots \tag{8}$$

The corresponding $karṇas$ (diagonals) are

$$k_1 = OA_1,\ k_2 = OA_2,\ k_3 = OA_3,\ \ldots \tag{9}$$

And the $\bar{a}bh\bar{a}dhas$ (intercepts) are

$$a_1 = D_1A_1,\ a_2 = D_2A_2,\ a_3 = D_3A_3,\ \ldots \tag{10}$$

Now

$$l_1 = r,\ k_1 = \sqrt{2}r \qquad \text{and} \qquad a_1 = \frac{r}{\sqrt{2}}. \tag{11}$$

Using the *bhuja-koṭi-karṇa-nyāya* (Pythagoras theorem) and *trairāśika-nyāya* (rule of three for similar triangles), it can be shown that

$$l_2 = l_1 - (k_1 - r)\,\frac{l_1}{a_1} \tag{12}$$

$$k_2^2 = r^2 + l_2^2 \tag{13}$$

$$\text{and} \qquad a_2 = \frac{[k_2^2 - (r^2 - l_2^2)]}{2k_2}. \tag{14}$$

In the same way l_{n+1}, k_{n+1} and a_{n+1} are to be obtained in terms of l_n, k_n and a_n. These can be shown equivalent to the recursion relation:[16]

[16]If we set $r = 1$ and $l_n = \tan\theta_n$, then equation (15) gives $l_{n+1} = \tan\left(\frac{\theta_n}{2}\right)$. Actually, $\theta_n = \frac{\pi}{2^{n+1}}$ and the above method is based on the fact that for large n, $2^n \tan\frac{\pi}{2^{n+2}} \approx 2^n \frac{\pi}{2^{n+2}} = \frac{\pi}{4}$.

$$l_{n+1} = \frac{r}{l_n}\,[\sqrt{(r^2 + l_n^2)} - r]. \tag{15}$$

4 Summation (and repeated summations) of powers of natural numbers (*saṅkalita*)

4.1 Sum of squares and cubes of natural numbers in *Āryabhaṭīya*

The ancient text *Bṛhaddevatā* (c. 5^{th} century BCE) has the result

$$2 + 3 + \ldots + 1000 = 500,499. \tag{16}$$

Āryabhaṭa (c. 499 CE), in the *Gaṇitapāda* of *Āryabhaṭīya*, deals with a general arithmetic progression in verses 19–20. He gives the sum of the squares and cubes of natural numbers in verse 22:[17]

> सैकसगच्छपदानां क्रमात् त्रिसंवर्गितस्य षष्ठोंऽशः।
> वर्गचितिघनः स भवेत् चितिवर्गो घनचितिघनश्च॥
>
> The product of the three quantities, the number of terms plus one, the same increased by the number of terms, and the number of terms, when divided by six, gives the sum of squares of natural numbers (*varga-citi-ghana*). The square of the sum of natural numbers gives the sum of the cubes of natural numbers (*ghana-citi-ghana*).

In other words,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{17}$$

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + n^3 &= [1 + 2 + 3 + \ldots + n]^2 \\ &= \left[\frac{n(n+1)}{2}\right]^2. \end{aligned} \tag{18}$$

4.2 Repeated sum of natural numbers in *Āryabhaṭīya*

Āryabhaṭa also gives the repeated sum of the sum of the natural numbers (*saṅkalita-saṅkalita* or *vāra-saṅkalita*):[18]

> एकोत्तराद्युपचितेर्गच्छाद्येकोत्तरत्रिसंवर्गः।
> षड्भक्तः स चितिघनः सैकपदघनो विमूलो वा॥
>
> Of the series (*upaciti*) 1, 2, …, n, take three terms in continuation of which the first is the given number of terms (*gaccha*), and find their product; that [product], or the number of terms plus one subtracted from its own cube divided by six, gives the repeated sum (*citi-ghana*).

[17] *Āryabhaṭīya*, cited above, *Gaṇitapāda* 22, p. 65.
[18] *Āryabhaṭīya*, cited above, *Gaṇitapāda* 21, p. 64.

We have

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \qquad (19)$$

$\bar{\mathrm{A}}\mathrm{ryabha\d{t}a}$'s result expresses the sum of these triangular numbers in two forms:

$$\begin{aligned} 1\frac{(1+1)}{2} + 2\frac{(2+1)}{2} + \ldots + n\frac{(n+1)}{2} &= \frac{[n(n+1)(n+2)]}{6} \\ &= \frac{[(n+1)^3 - (n+1)]}{6}. \qquad (20) \end{aligned}$$

4.3 Nārāyaṇa Paṇḍita's general formula for *Vārasaṅkalita*

In his *Gaṇita-kaumudī*, Nārāyaṇa Paṇḍita (c. 1356) gives the formula for the r^{th}-order repeated sum of the sequence of numbers 1, 2, 3, …, n:[19]

> एकाधिकवारमिताः पदादिरूपोत्तरा पृथक् तेंऽशाः।
> एकाद्येकचयहरास्तद्घातो वारसङ्कलितम्॥
>
> The *pada* (number of terms in the sequence) is the first term [of an arithmetic progression] and 1 is the common difference. Take as numerators [the terms in the AP] numbering one more than *vāra* (the number of times the repeated summation is to be made). The denominators are [terms of an AP of the same length] starting with one and with common difference one. The resultant product is *vāra-saṅkalita*.

Let

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = V_n^{(1)}. \qquad (21)$$

Then, Nārāyaṇa's result is

$$\begin{aligned} V_n^{(r)} &= V_1^{(r-1)} + V_2^{(r-1)} + \ldots + V_n^{(r-1)} \qquad (22) \\ &= \frac{[n(n+1)\ldots(n+r)]}{[1.2\ldots(r+1)]}\,. \qquad (23) \end{aligned}$$

Nārāyaṇa's result can also be expressed in the form of a sum of polygonal numbers:

$$\sum_{m=1}^{n} \frac{[m(m+1)\ldots(m+r-1)]}{[1.2\ldots r]} = \frac{[n(n+1)\ldots(n+r)]}{[1.2\ldots(r+1)]}\,. \qquad (24)$$

This result can be used to evaluate the sums $\sum_{k=1}^{n} k^2,\ \sum_{k=1}^{n} k^3,\ \ldots$ by induction. It can also be used to estimate the behaviour of these sums for large n.

4.4 Summation of geometric series

The geometric series $1 + 2 + 2^2 + \ldots 2^n$ is summed in Chapter VIII of Piṅgala's *Chandaḥ-sūtra* (c. 300 BCE). As we mentioned earlier, Piṅgala also gives an algorithm for evaluating any positive integral power of a number (2 in this context) in terms of an optimal number of squaring and multiplication operations.

[19] *Gaṇitakaumudī* of Nārāyaṇa Paṇḍita, Ed. by Padmakara Dvivedi, Part I, Benaras 1936, verse 3.19–20, p. 123.

Mahāvīrācārya (c. 850), in his *Gaṇita-sāra-saṅgraha* gives the sum of a geometric series and also explains the Piṅgala algorithm for finding the required power of the common ratio between the terms of the series:[20]

> पदमितगुणहतिगुणितप्रभवः स्याद्गुणधनं तदाद्यूनम्।
> एकोनगुणविभक्तं गुणसङ्कलितं विजानीयात्॥
> समदलविषमस्वरूपो गुणगुणितो वर्गताडितो गच्छः।
> रूपोनः प्रभवघ्नो व्येकोत्तरभाजितः सारम्॥
>
> The first term when multiplied by the product of the common ratio (*guṇa*) taken as many times as the number of terms (*pada*), gives rise to the *guṇadhana*. This *guṇadhana*, when diminished by the first term and divided by the common ratio less one, is to be understood as the sum of the geometrical series (*guṇa-saṅkalita*).

That is

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{(r - 1)}. \tag{25}$$

Vīrasena (c. 816), in his commentary *Dhavalā* on the *Ṣaṭkhaṇḍāgama*, has made use of the sum of the following infinite geometric series in his evaluation of the volume of the frustum of a right circular cone:[21]

$$1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{4}{3}. \tag{26}$$

The proof of the above result is discussed in the *Āryabhaṭīya-bhāṣya* (c. 1502) of Nīlakaṇṭha Somayājī. As we shall see later (section 10.1), Nīlakaṇṭha makes use of this series for deriving an approximate expression for a small arc in terms of the corresponding chord in a circle.

5 Use of Second-order differences and interpolation in computation of Rsines (*Jyānayana*)

Jyā, Koṭi and *Śara*

The *jyā* or *bhujā-jyā* of an arc of a circle is actually the half-chord (*ardha-jyā* or *jyārdha*) of double the arc. In the Figure 2, if R is the radius of the circle, *jyā* (Rsine), *koṭi* or *koṭi-jyā* (Rcosine) and *śara* (Rversine) of the *cāpa* (arc) EC are given by:

$$\begin{aligned}
jy\bar{a}\ (arc\ EC) = CD &= R\sin(\angle COE) && (27)\\
ko\d{t}i\ (arc\ EC) = OD &= R\cos(\angle COE) && (28)\\
\acute{s}ara\ (arc\ EC) = ED &= R\text{vers}(\angle COE) \\
&= R - R\cos(\angle COE). && (29)
\end{aligned}$$

For computing standard Rsine-tables (*paṭhita-jyā*), the circumference of a circle is

[20] *Gaṇitasārasaṅgraha* of Mahāvīrācārya, Ed. by Lakshmi Chanda Jain, Sholapur 1963, verses 2.93–94, pp. 28–29.

[21] See, for instance, T. A. Sarasvati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarsidass, Delhi 1979, Rep. 2007, pp. 203–05.

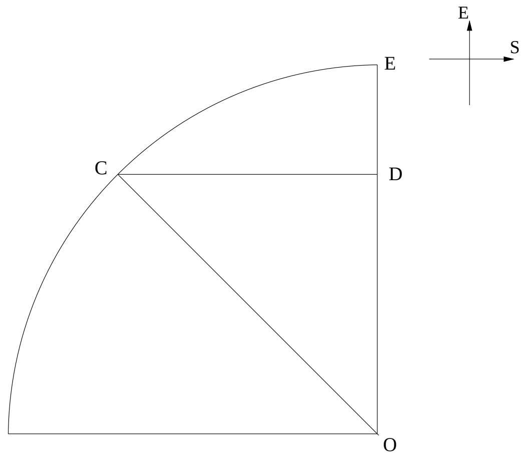

Figure 2: *Jyā*, *Koṭi* and *Śara*.

divided into $21600'$ and usually the Rsines are tabulated for every multiple of $225'$, thus giving 24 tabulated Rsines in a quadrant. Using the value of $\pi \approx \frac{62832}{20000} = 3.1416$, given by Āryabhaṭa, the value of the radius then turns out to be $3437'\,44''\,19'''$. This is accurate up to the seconds, but is usually approximated to $3438'$. Using a more accurate value of π, Mādhava (c. 1340–1420) gave the value of the radius correct to the thirds as $3437'\,44''\,48'''$ which is also known by the *Kaṭapayādi* formula *devo-viśvasthalī-bhṛguḥ*.

5.1 Computation of Rsines

Once the value of the radius R is fixed (in units of minutes, seconds etc.) the 24 Rsines can be computed (in the same units) using standard relations of *jyotpatti* (trigonometry). For instance, Varāhamihira has given the following Rsine values and relations in his *Pañcasiddhāntikā* (c. 505):[22]

$$
\begin{aligned}
R\sin(30^\circ) &= \frac{R}{2} && (30a)\\
R\sin(45^\circ) &= \frac{R}{\sqrt{2}} && (30b)\\
R\sin(60^\circ) &= \frac{\sqrt{3}}{2}R && (30c)\\
R\sin(90^\circ) &= R && (30d)
\end{aligned}
$$

$$
\begin{aligned}
R\sin(A) &= R\cos(90-A) && (31)\\
R\sin^2(A) + R\cos^2(A) &= R^2 && (32)\\
R\sin\left(\frac{A}{2}\right) &= \left(\frac{1}{2}\right)\left[R\sin^2(A) + R\,\text{vers}^2(A)\right]^{\frac{1}{2}}
\end{aligned}
$$

[22] *Pañcasiddhāntikā* of Varāhamihira, Ed. by T. S. Kuppanna Sastry and K. V. Sarma, Madras 1993, verses 4.1–5, pp. 76–80.

$$= \left(\frac{R}{2}\right)^{\frac{1}{2}} [R - R\cos A]^{\frac{1}{2}}. \tag{33}$$

The above Rsine values (30) and relations (31)–(33) can be derived using the *bhujā-koṭi-karṇa-nyāya* (Pythagoras theorem) and *trairāśika* (rule of three for similar triangles), as is done for instance in the *Vāsanā-bhāṣya* of Pṛthūdakasvāmin (c. 860) on *Brāhmasphuṭasiddhānta* (c. 628) of Brahmagupta. Equations (30)–(33) can be used to compute all 24 tabular Rsine values.

5.2 Āryabhaṭa's computation of Rsine-differences

The computation of tabular Rsine values was made much simpler by Āryabhaṭa who gave an ingenious method of computing the Rsine-differences, making use of the important property that the second-order differences of Rsines are proportional to the Rsines themselves:[23]

> प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम्।
> तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि॥
>
> The first Rsine divided by itself and then diminished by the quotient will give the second Rsine-difference. The same first Rsine, diminished by the quotients obtained by dividing each of the preceding Rsines by the first Rsine, gives the remaining Rsine-differences.

Let $B_1 = R\sin\,(225'),\ B_2 = R\sin\,(450'), ...,\ B_{24} = R\sin\,(90°)$, be the twenty-four Rsines, and let $\Delta_1 = B_1,\ \Delta_2 = B_2 - B_1, ...,\ \Delta_k = B_k - B_{k-1}, ...$ be the Rsine-differences. Then, the above rule may be expressed as[24]

$$\Delta_2 = B_1 - \frac{B_1}{B_1} \tag{34}$$

$$\Delta_{k+1} = B_1 - \frac{(B_1 + B_2 + \ldots + B_k)}{B_1} \qquad (k = 1, 2, \ldots, 23). \tag{35}$$

This second relation is also sometimes expressed in the equivalent form

$$\Delta_{k+1} = \Delta_k - \frac{(\Delta_1 + \Delta_2 + \ldots + \Delta_k)}{B_1} \qquad (k = 1, 2, \ldots, 23). \tag{36}$$

From the above it follows that

$$\Delta_{k+1} - \Delta_k = \frac{-B_k}{B_1} \qquad (k = 1, 2, \ldots, 23). \tag{37}$$

Since Āryabhaṭa also takes $\Delta_1 = B_1 = R\sin(225') \approx 225'$, the above relations reduce to

$$\Delta_1 = 225' \tag{38}$$

$$\Delta_{k+1} - \Delta_k = \frac{-B_k}{225'} \qquad (k = 1, 2, \ldots, 23). \tag{39}$$

[23] *Āryabhaṭīya*, cited above, *Gaṇitapāda* 12, p. 51.

[24] Āryabhaṭa is using the approximation $\Delta_2 - \Delta_1 \approx 1'$ and the second terms in the RHS of (34)–(36) and the RHS of (37) and (39) have an implicit factor of $(\Delta_2 - \Delta_1)$. See (45) below which is exact.

5.3 Derivation of the Āryabhaṭa-relation for the second-order Rsine-differences

Āryabhaṭa's relation for the second-order Rsine-differences is derived and made more exact in the *Āryabhaṭīya-bhāṣya* (c. 1502) of Nīlakaṇṭha Somayājī and *Yuktibhāṣā* (c. 1530) of Jyeṣṭhadeva. We shall present a detailed account of the first and second-order Rsine-differences as given in *Yuktibhāṣā*[25] later in Section 16. Here we shall only summarize the argument.

In Figure 3, the arcs EC_j and EC_{j+1} are successive multiples of $225'$. The Rsine and Rcosine of the arcs EC_j and EC_{j+1} are given by

$$\begin{aligned} B_j &= C_jP_j,\ B_{j+1} = C_{j+1}P_{j+1} && (40)\\ \text{and}\qquad K_j &= C_jT_j,\ K_{j+1} = C_{j+1}T_{j+1}, && (41)\end{aligned}$$

respectively. Let M_{j+1} and M_j be the mid-points of the arcs $C_jC_{j+1},\ C_{j-1}C_j$ and the Rsine and Rcosine of the arcs EM_j and EM_{j+1} be denoted respectively by $B_{j-\frac{1}{2}},\ B_{j+\frac{1}{2}},\ K_{j-\frac{1}{2}}, K_{j+\frac{1}{2}}$.

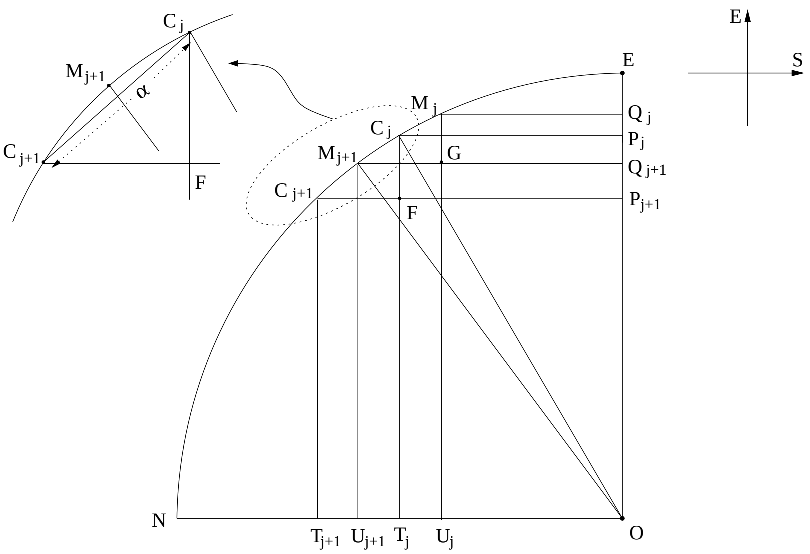

Figure 3: Derivation of *Āryabhaṭa* relation.

Let the chord of the arc C_jC_{j+1}, be denoted by α and let R be the radius. Then a simple argument based on *trairāśika* (similar triangles) leads to the relations:[26]

$$\begin{aligned} B_{j+1} - B_j &= \left(\frac{\alpha}{R}\right) K_{j+\frac{1}{2}} && (42)\\ K_{j-\frac{1}{2}} - K_{j+\frac{1}{2}} &= \left(\frac{\alpha}{R}\right) B_j. && (43)\end{aligned}$$

[25] *Gaṇita-yukti-bhāṣā*, cited above, Section 7.5.1, pp. 94–96, 221–24, 417–20.

[26] Equations (42) and (43) are essentially the relations:

$$\begin{aligned} R\sin(x+h) - R\sin x &= \left(\frac{\alpha}{R}\right) R\cos\left(x+\frac{h}{2}\right)\\ R\cos\left(x-\frac{h}{2}\right) - R\cos\left(x+\frac{h}{2}\right) &= \left(\frac{\alpha}{R}\right) R\sin x, \end{aligned}$$

Thus we get

$$\begin{aligned}\Delta_{j+1}-\Delta_j &= (B_{j+1}-B_j)-(B_j-B_{j-1}) \\ &= -\left(\frac{\alpha}{R}\right)^2 B_j. \end{aligned} \tag{44}$$

We can also express this relation in the form

$$\Delta_{j+1}-\Delta_j = \frac{-B_j(\Delta_1-\Delta_2)}{B_1}. \tag{45}$$

The above relations are exact. $\bar{\mathrm{A}}\mathrm{ryabha\d{t}a}$'s relation (39) corresponds to the approximations, $B_1 \approx 225'$ and $\Delta_1-\Delta_2 \approx 1'$ so that

$$\left(\frac{\alpha}{R}\right)^2 = \frac{(\Delta_1-\Delta_2)}{B_1} \approx \left(\frac{1}{225'}\right). \tag{46}$$

In *Tantrasaṅgraha*, Nīlakaṇṭha Somayājī has given the finer approximation:[27]

$$\left(\frac{\alpha}{R}\right)^2 = \frac{(\Delta_1-\Delta_2)}{B_1} \approx \left(\frac{1}{233\frac{1}{2}'}\right). \tag{47}$$

This is further refined by Śaṅkara Vāriyar in his commentary *Laghu-vivṛti* in the form:[28]

$$\left(\frac{\alpha}{R}\right)^2 = \frac{(\Delta_1-\Delta_2)}{B_1} \approx \left(\frac{1}{233'32''}\right). \tag{48}$$

Since $\alpha = 2R\sin 112'30''$, we find that the above relation is correct up to seconds.

Commenting on Āryabhaṭa's method of computing Rsines, Delambre had remarked:[29]

> The method is curious: it indicates a method of calculating the table of sines by means of their second-differences... This differential process has not up to now been employed except by Briggs, who himself did not know that the constant factor was the square of the chord ΔA $(= 3°45')$ or of the interval, and who could not obtain it except by comparing the second differences obtained in a different manner. The Indians also have probably done the same; they obtained the method of differences only from a table calculated previously by a geometric process. Here then is a method which the Indians possessed and which is found neither amongst the Greeks, nor amongst the Arabs.

with $\alpha = 2R\sin\frac{h}{2}$. These lead to (44) in the form:

$$(R\sin(x+h)-R\sin x)-(R\sin x - R\sin(x-h)) = -\left(\frac{\alpha}{R}\right)^2 R\sin x.$$

[27] *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī, Ed. with *Laghu-vivṛti* of Śaṅkara Vāriyar by S. K. Pillai, Trivandrum 1958, verse 2.4, p. 17.

[28] Ibid., comm. on verse 2.4.

[29] Delambre, *Historie de l' Astronomie Ancienne*, t 1, Paris 1817, pp. 457, 459f, cited from B. B. Datta and A. N. Singh, 'Hindu Trigonometry', Ind. Jour. Hist. Sc. **18**, 39–108, 1983, p. 77.

5.4 The Rsine-table of Āryabhaṭa

In the *Gītikā-pāda* of *Āryabhaṭīya*, Āryabhaṭa has given a table of Rsine-differences:[30]

> मखि भखि फखि धखि णखि ञखि
> ङखि हस्झ स्ककि किष्ग श्घकि किघ्व ।
> घ्लकि किग्र हक्य धकि किच
> स्ग श्झ ङ्व क्ल प्त फ छ कलार्धज्याः ॥
>
> 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, and 7—these are the Rsine-differences [at intervals of 225’ of arc] in terms of the minutes of arc.

The above values follow directly from Āryabhaṭa’s relation (39) for the second order Rsine-differences. To start with, $\Delta_1 = B_1 = R\sin(225') \approx 225'$. Then we get, $\Delta_2 = B_1 - \frac{B_1}{B_1} = 224'$ and so on.

The Rsine-table of Āryabhaṭa[31] (see Table 1), obtained this way, is accurate up to minutes. In this table, we also give the Rsine values given by Govindasvāmin (c. 825) in his commentary on *Mahābhāskarīya* of Bhāskara I, and by Mādhava (c. 1340–1420) as recorded in the *Āryabhaṭīya-bhāṣya* (c. 1502) of Nīlakaṇṭha Somayājī. Though Govindasvāmin gives the Rsine values up to the thirds, his values are accurate only up to the seconds; those of Mādhava are accurate up to the thirds.

5.5 Brahmagupta’s second-order interpolation formula

The Rsine table of Āryabhaṭa gives only the Rsine values for the twenty-four multiples of $225'$. The Rsines for arbitrary arc-lengths have to be found by interpolation only. In his *Khaṇḍakhādyaka* (c. 665), Brahmagupta gives a second-order interpolation formula for the computation of Rsines for arbitrary arcs. In this work, which is in the form of a manual (*karaṇa*) for astronomical calculations, Brahmagupta uses a simpler Rsine-table which gives Rsines only at intervals of $15°$ or $900'$:[32]

> गतभोग्यखण्डकान्तरदलविकलवधात् शतैर्नवभिराप्त्या ।
> तद्युतिदलं युतोनं भोग्यादूनाधिकं भोग्यम् ॥
>
> Multiply the residual arc after division by $900'$ by half the difference of the tabular Rsine difference passed over (*gata-khaṇḍa*) and to be passed over (*bhogya-khaṇḍa*) and divide by $900'$. The result is to be added to or subtracted from half the sum of the same tabular sine differences according as this [half-sum] is less than or equal to the Rsine tabular difference to be passed. What results is the true Rsine-difference to be passed over.

[30] *Āryabhaṭīya*, cited above, *Gītikāpāda* 12, p. 29.
[31] See, for instance, A. K. Bag, *Mathematics in Ancient and Medieval India*, Varanasi 1979, pp. 247–48.
[32] *Khaṇḍakhādyaka* of Brahmagupta, Ed. by P. C. Sengupta, Calcutta 1941, p. 151.

Table 1: Rsine-table of Āryabhaṭa, Govindasvāmin and Mādhava.

	Āryabhaṭa (c. 499)	Govindasvāmin (c. 825)	Mādhava (c. 1375)
$3^\circ 45'$	$225'$	$224'\ 50''\ 23'''$	$224'\ 50''\ 22'''$
$7^\circ 30'$	$449'$	$448'\ 42''\ 53'''$	$448'\ 42''\ 58'''$
$11^\circ 15'$	$671'$	$670'\ 40''\ 11'''$	$670'\ 40''\ 16'''$
$15^\circ 00'$	$890'$	$889'\ 45''\ 08'''$	$889'\ 45''\ 15'''$
$18^\circ 45'$	$1105'$	$1105'\ 01''\ 30'''$	$1105'\ 01''\ 39'''$
$22^\circ 30'$	$1315'$	$1315'\ 33''\ 56''$	$1315'\ 34''\ 07'''$
$26^\circ 15'$	$1520'$	$1520'\ 28''\ 22'''$	$1520'\ 28''\ 35'''$
$30^\circ 00'$	$1719'$	$1718'\ 52''\ 10'''$	$1718'\ 52''\ 24'''$
$33^\circ 45'$	$1910'$	$1909'\ 54''\ 19'''$	$1909'\ 54''\ 35'''$
$37^\circ 30'$	$2093'$	$2092'\ 45''\ 46'''$	$2092'\ 46''\ 03'''$
$41^\circ 15'$	$2267'$	$2266'\ 38''\ 44'''$	$2266'\ 39''\ 50'''$
$45^\circ 00'$	$2431'$	$2430'\ 50''\ 54'''$	$2430'\ 51''\ 15'''$
$48^\circ 45'$	$2585'$	$2584'\ 37''\ 43'''$	$2584'\ 38''\ 06'''$
$52^\circ 30'$	$2728'$	$2727'\ 20''\ 29'''$	$2727'\ 20''\ 52'''$
$56^\circ 15'$	$2859'$	$2858'\ 22''\ 31'''$	$2858'\ 22''\ 55'''$
$60^\circ 00'$	$2978'$	$2977'\ 10''\ 09'''$	$2977'\ 10''\ 34'''$
$63^\circ 45'$	$3084'$	$3083'\ 12''\ 51'''$	$3083'\ 13''\ 17'''$
$67^\circ 30'$	$3177'$	$3176'\ 03''\ 23'''$	$3176'\ 03''\ 50'''$
$71^\circ 15'$	$3256'$	$3255'\ 17''\ 54'''$	$3255'\ 18''\ 22'''$
$75^\circ 00'$	$3321'$	$3320'\ 36''\ 02'''$	$3320'\ 36''\ 30'''$
$78^\circ 45'$	$3372'$	$3371'\ 41''\ 01'''$	$3371'\ 41''\ 29'''$
$82^\circ 30'$	$3409'$	$3408'\ 19''\ 42'''$	$3408'\ 20''\ 11'''$
$86^\circ 15'$	$3431'$	$3430'\ 22''\ 42'''$	$3430'\ 23''\ 11'''$
$90^\circ 00'$	$3438'$	$3437'\ 44''\ 19'''$	$3437'\ 44''\ 48'''$

Let h be the basic unit of arc in terms of which the Rsine-table is constructed, which happens to be $225'$ in the case of *Āryabhaṭīya*, and $900'$ in the case of *Khaṇḍakhādyaka*. Let the arc for which Rsine is to be found be given by

$$s = jh + \varepsilon \qquad \text{for some } j = 0, 1, \ldots \tag{49}$$

Now $R\sin(jh) = B_j$ are the tabulated Rsines. Then, a simple interpolation (*trairāśika*) would yield

$$\begin{aligned} R\sin(jh+\epsilon) &= B_j + \left(\frac{\varepsilon}{h}\right)(B_{j+1} - B_j) \\ &= R\sin(jh) + \frac{\varepsilon}{h}\Delta_{j+1}. \end{aligned} \tag{50}$$

Instead of the above simple interpolation, Brahmagupta prescribes

$$R\sin(jh+\epsilon) = B_j + \left(\frac{\varepsilon}{h}\right)\left[\left(\frac{1}{2}\right)(\Delta_j + \Delta_{j+1}) \pm \left(\frac{\varepsilon}{h}\right)\frac{(\Delta_j \sim \Delta_{j+1})}{2}\right]. \tag{51}$$

Here, the sign is chosen to be positive if $\Delta_j < \Delta_{j+1}$, and negative if $\Delta_j > \Delta_{j+1}$ (as in the case of Rsine). So Brahmagupta's rule is actually the second-order interpolation formula

$$R\sin(jh+\varepsilon) \quad = \quad R\sin(jh) + \left(\frac{\varepsilon}{h}\right)\left[\left(\frac{1}{2}\right)(\Delta_j + \Delta_{j+1}) - \left(\frac{\varepsilon}{h}\right)\frac{(\Delta_j - \Delta_{j+1})}{2}\right]$$

$$
\begin{aligned}
&= R\sin(jh) + \left(\frac{\varepsilon}{h}\right)\frac{(\Delta_{j+1}+\Delta_j)}{2} + \left(\frac{\varepsilon}{h}\right)^2 \frac{(\Delta_{j+1}-\Delta_j)}{2} \\
&= R\sin(jh) + \left(\frac{\varepsilon}{h}\right)\Delta_{j+1} + \left(\frac{\varepsilon}{h}\right)\left[\frac{\varepsilon}{h}-1\right]\frac{(\Delta_{j+1}-\Delta_j)}{2}. \qquad (52)
\end{aligned}
$$

6 Instantaneous velocity of a planet (*tātkālika-gati*)

6.1 True daily motion of a planet

In Indian Astronomy, the motion of a planet is computed by making use of two corrections: the *manda-saṃskāra* which essentially corresponds to the equation of centre and the *śīghra-saṃskāra* which corresponds to the conversion of the heliocentric longitudes to geocentric longitudes. The *manda* correction for planets is given in terms of an epicycle of variable radius r, which varies in such a way that

$$\frac{r}{K} = \frac{r_0}{R}, \qquad (53)$$

where K is the *karṇa* (hypotenuse) or the (variable) distance of the planet from the centre of the concentric and r_0 is the tabulated (or mean) radius of the epicycle in the measure of the concentric circle of radius R.

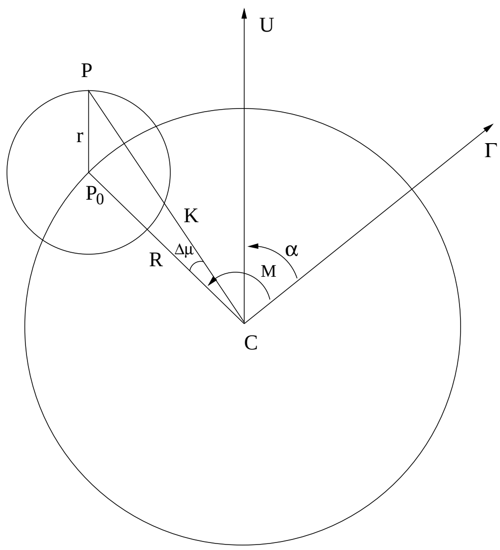

Figure 4: *Manda* correction.

In Figure 4, C is the centre of concentric on which the mean planet P_0 is located. CU is the direction of the *ucca* (aphelion or apogee as the case may be). P is the true planet which lies on the epicycle of (variable) radius r centered at P_0, such that P_0P is parallel to CU. If M is the mean longitude of a planet, α the longitude of the *ucca*, then the correction (*manda-phala*) $\Delta\mu$ is given by

$$
\begin{aligned}
R\sin(\Delta\mu) &= \left(\frac{r}{K}\right) R\sin(M-\alpha) \\
&= \left(\frac{r_0}{R}\right) R\sin(M-\alpha). \qquad (54)
\end{aligned}
$$

For small r, the left hand side is usually approximated by the arc itself. The *manda*-correction is to be applied to the mean longitude M, to obtain the true or *manda*-corrected longitude μ given by

$$\mu = M - \left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right) R\sin(M-\alpha). \tag{55}$$

If n_m and n_u are the mean daily motions of the planet and the *ucca*, then the true longitude on the next day is given by

$$\mu + n = (M + n_m) - \left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right)\ R\sin\,(M + n_m - \alpha - n_u). \tag{56}$$

The true daily motion is thus given by

$$n = n_m - \left(\frac{r_0}{R}\right)\ \left(\frac{1}{R}\right)\ [R\sin\{(M-\alpha) + (n_m - n_u)\} - R\sin\,(M-\alpha)]. \tag{57}$$

The second term in the above is the correction to mean daily motion (*gati-phala*). An expression for this was given by Bhāskara I (c. 629) in *Mahābhāskarīya*, where he makes use of the approximation:[33]

$$\left.\begin{array}{r} R\sin\{(M-\alpha)+(n_m-n_u)\} \\ -\ R\sin(M-\alpha) \end{array}\right\} \approx \left\{\begin{array}{l} (n_m - n_u)\ \times \\ \left(\frac{1}{225}\right)\ \text{Rsine-difference at } (M-\alpha). \end{array}\right. \tag{58}$$

In the above approximation, $(n_m - n_u)$ is multiplied by tabular Rsine-difference at the $225'$ arc-bit in which (the tip of the arc) $(M-\alpha)$ is located. Therefore, under this approximation, as long as the anomaly (*kendra*), $(M-\alpha)$, is in the same multiple of $225'$, there will be no change in the *gati-phala* or the correction to the mean velocity. This defect was noticed by Bhāskara also in his later work *Laghubhāskarīya*:[34]

> अभिन्नरूपता भुक्तेश्चापभागविचारिणः।
> रवेरिन्दोश्च जीवानामूनभावाद्यसम्भवात्॥
> एवमालोच्यमानेयं जीवाभुक्तिर्विशीयर्ते।
>
> Whilst the Sun or the Moon moves in the [same] element of arc, there is no change in the rate of motion (*bhukti*), because the Rsine-difference does not increase or decrease; viewed thus, the rate of motion [as given above] is defective.

The correct formula for the true daily motion of a planet, employing the Rcosine as the 'rate of change' of Rsine, seems to have been first given by Muñjāla (c. 932) in his short manual *Laghumānasa* [35] and also by Āryabhaṭa II (c. 950) in his *Mahāsiddhānta*:[36]

> कोटिफलघ्नी भुक्तिर्गज्याभक्ता कलादिफलम्॥
>
> The *koṭiphala* multiplied by the [mean] daily motion and divided by the radius gives the minutes of the correction [to the rate of the motion].

[33] *Mahābhāskarīya* of Bhāskara I, Ed. by K. S. Shukla, Lucknow 1960, verse 4.14, p. 120.
[34] *Laghubhāskarīya* of Bhāskara I, Ed. by K. S. Shukla, Lucknow 1963, verses 2.14-5, p. 6.
[35] *Laghumānasa* of Muñjāla, Ed. by K. S. Shukla, New Delhi 1990, verse 3.4, p. 125.
[36] *Mahāsiddhānta* of Āryabhaṭa II, Ed. by Sudhakara Dvivedi, Varanasi 1910, verse 3.15, p. 58.

This gives the true daily motion in the form

$$n = n_m - (n_m - n_u)\left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right)\ R\cos(M - \alpha). \tag{59}$$

6.2 The notion of instantaneous velocity (*tātkālikagati*) according to Bhāskarācārya II

Bhāskarācārya II (c. 1150) in his *Siddhāntaśiromaṇi* clearly distinguishes the true daily motion from the instantaneous rate of motion. And he gives the Rcosine correction to the mean rate of motion as the instantaneous rate of motion. He further emphasizes the fact that the velocity is changing every instant and this is particularly important in the case of Moon because of its rapid motion.[37]

> दिनान्तरस्पष्टखगान्तरं स्याद् गतिः स्फुटा तत्समयान्तराले ॥
> कोटीफलघ्नी मृदुकेन्द्रभुक्तिस्त्रिज्योद्धृता कर्किमृगादिकेन्द्रे ।
> तया युतोना ग्रहमध्यभुक्तिस्तात्कालिकी मन्दपरिस्फुटा स्यात्॥
> समीपतिथ्यन्तसमीपचालनं विधोस्तु तत्कालजयैव युज्यते।
> सुदूरसञ्चालनमाद्यया यतः प्रतिक्षणं सा न समा महत्यतः॥

> The true daily motion of a planet is the difference between the true planets on successive days. And it is accurate (*sphuṭa*) over that period. The *koṭiphala* (Rcosine of anomaly) is multiplied by the rate of motion of the *manda*-anomaly (*mṛdu-kendra-bhukti*) and divided by the radius. The result added or subtracted from the mean rate of motion of the planet, depending on whether the anomaly is in *Karkyādi* or *Mṛgādi*, gives the true instantaneous rate of motion (*tātkālikī manda-sphuṭagati*) of the planet.
>
> In the case of the Moon, the ending moment of a *tithi*[38] which is about to end or the beginning time of a *tithi* which is about to begin, are to be computed with the instantaneous rate of motion at the given instant of time. The beginning moment of a *tithi* which is far away can be calculated with the earlier [daily] rate of motion. This is because Moon's rate of motion is large and varies from moment to moment.

Here, Bhāskara explains the distinction between the true daily rate of motion and the true instantaneous rate of motion. The former is the difference between the true longitudes on successive days and it is accurate as the rate of motion, on the average, for the entire period. The true instantaneous rate of motion is to be calculated from the Rcosine of the anomaly (*koṭiphala*) for each relevant moment.

Thus if ω_m and ω_u are the rates of the motion of the mean planet and the *ucca*, then $\omega_m - \omega_u$ is the rate of motion of the anomaly, and the true instantaneous rate of motion

[37] *Siddhāntaśiromaṇi* of Bhāskarācārya, Ed. by Muralidhara Chaturvedi, Varanasi 1981, verses 2.36–8, p. 119.

[38] *Tithi* is the time taken by the Moon to lead the Sun exactly by 12° in longitude.

of the planet at any instant is given by Bhāskara to be

$$\omega = \omega_m + (\omega_m - \omega_u) \left(\frac{r_0}{R}\right) \left(\frac{1}{R}\right) R \cos(M - \alpha), \tag{60}$$

where $(M - \alpha)$ is the anomaly of the planet at that instant.

Bhāskara explains the idea of the instantaneous velocity even more clearly in his *Vāsanā*:[39]

> अद्यतनश्वस्तनस्फुटग्रहयोः औदयिकयोर्दिनार्धजयोर्वा अस्तकालिकयोर्वा यदन्तरं कलादिकं सा स्फुटा गतिः। अद्यतनाच्छ्वस्तने न्यूने वक्रागतिर्ज्ञेया। तत्समयान्तराल इति। तस्य कालस्य मध्येऽनया गत्या ग्रहश्चालयितुं युज्यत इति। इयं किल स्थूला गतिः। अथ सूक्ष्मा तात्कालिकी कथ्यते। तुङ्गगत्यूना चन्द्रगतिः केन्द्रगतिः। अन्येषां ग्रहाणां ग्रहगतिरेव केन्द्रगतिः। मृदुकेन्द्रकोटिफलं कृत्वा तेन केन्द्रगतिर्गुण्या त्रिज्यया भाज्या लब्धेन कर्क्यादिकेन्द्रे ग्रहगतिर्युक्ता कार्या। मृगादौ तु रहिता कार्या। एवं तात्कालिकी मन्दपरिस्फुटा स्यात्। तात्कालिक्या भुक्त्या चन्द्रस्य विशिष्टं प्रयोजनम्। तदाह 'समीपतिथ्यन्तसमीपचालनम्' इति। यत्कालिकश्चन्द्रस्तस्मात् कालाद्गतो वा गम्यो वा यदासन्नस्तितिथ्यन्तस्तदा तात्कालिक्या गत्या तिथिसाधनं कर्तुं युज्यते। तथा समीपचालनं च। यदा तु दूरतरस्तिथ्यन्तो दूरचालनं वा चन्द्रस्य तदाद्यया स्थूलया कर्तुं युज्यते। स्थूलकालत्वात्। यतश्चन्द्रगतिर्महत्वात् प्रतिक्षणं समा न भवति अतस्तदर्थमयं विशेषोऽ-भिहितः।
>
> अथ गतिफलवासना। अद्यतनश्वस्तनग्रहयोरन्तरं गतिः। अत एव ग्रहफलयोरन्तरं गतिफलं भवितुमर्हति। अथ तत्साधनम्। अद्यतनश्वस्तनकेन्द्रयोरन्तरं केन्द्रगतिः। भुजज्याकरणे यद्भोग्यखण्डं तेन सा गुण्या शरद्विदस्रैः (२२५) भाज्या। तत्र तावत् तात्कालिकभोग्यखण्डकरणायानुपातः। यदि त्रिज्यातुल्यया कोटिज्ययाद्यं भोग्यखण्डं शरद्विदस्रतुल्यं लभ्यते तदेष्टया किमित्यत्र कोटिज्यायाः शरद्विदस्रा गुणस्त्रिज्या हरः। फलं तात्कालिकं स्फुटभोग्यखण्डं तेन केन्द्रगतिर्गुणनीया शरद्विदस्रैर्भाज्या। अत्र शरद्विदस्रमितयोर्गुणकभाजकयोस्तुल्यत्वान्नाशे कृते केन्द्रगतेः कोटिज्यागुणस्त्रिज्याहरः स्यात्। फलमद्यतनश्वस्तनकेन्द्रदोर्ज्ययोरन्तरं भवति। तत्फलकरणार्थं स्वपरिधिना गुण्यं भांशैः (३६०) भाज्यम्। पूर्वं किल गुणकः कोटिज्या सा यावत् परिधिना गुण्यते भांशैः ह्रियते तावत्कोटिफलं जायत इत्युपपन्नं 'कोटीफलघ्नी मृदुकेन्द्रभुक्ति'रित्यादि। एवमद्यतनश्वस्तनग्रहफलयोरन्तरं तद्गतेः फलं कर्क्यादिकेन्द्रे ग्रहर्णफलस्यापचीयमानत्वात् तुलादौ धनफलस्यापचीयमानत्वात् धनम्। मकरादौ तु धनफलस्यापचीयमानत्वात् मेषादावृणफलस्योपचीयमानत्वादृणम् इत्युपपन्नम्।
>
> The true daily velocity is the difference in minutes etc., between the true planets of today and tomorrow, either at the time of sunrise, or mid-day

[39] *Siddhāntaśiromaṇi*, cited above, *Vāsanā* on 2.36–38, p. 119–20.

or sunset. If tomorrow's longitude is smaller than that of today, then we should understand the motion to be retrograde. It is said "over that period". This only means that, during that intervening period, the planet is to move with this rate [on the average]. This is only a rough or approximate rate of motion. Now we shall discuss the instantaneous rate of motion... In this way, the *manda*-corrected true instantaneous rate of motion (*tātkālikī manda-parisphuṭagati*) is calculated. In the case of Moon, this instantaneous rate of motion is especially useful...Because of its largeness, the rate of motion of Moon is not the same every instant. Hence, in the case of Moon, the special [instantaneous] rate of motion is stated.

Then, the justification for the correction to the rate of motion (*gati-phala-vāsanā*)...The rate of motion of the anomaly is the difference in the anomalies of today and tomorrow. That should be multiplied by the [current] Rsine-difference used in the computation of Rsines and divided by 225. Now, the following rule of three to obtain the instantaneous Rsine-difference: If the first Rsine-difference 225 results when the Rcosine is equal to the radius, then how much is it for the given Rcosine. In this way, the Rcosine is to be multiplied by 225 and divided by the radius. The result is the instantaneous Rsine-difference and that should be multiplied by the rate of motion in the anomaly and divided by 225...

Thus, Bhāskara is here conceiving also of an instantaneous Rsine-difference, though his derivation of the instantaneous velocity is somewhat obscure. These ideas are more clearly set forth in the *Āryabhaṭīya-bhāṣya* (c. 1502) of Nīlakaṇṭha Somayājī and other works of the Kerala School.

6.3 The *śīghra* correction to the velocity and the condition for retrograde motion

Bhāskara then goes on to derive the correct expression for the true rate of motion as corrected by the *śīghra*-correction. In the language of modern astronomy, the *śīghra*-correction converts the heliocentric longitude of the planets to the geocentric longitudes. Here also, the Indian astronomers employ an epicycle, but with a fixed radius, unlike in the case of the *manda*-correction.

If μ is the *manda*-corrected (*manda-sphuṭa*) longitude of the planet, ζ is the longitude of the *śīghrocca*, and r_s, the radius of the *śīghra*-epicycle, then the correction (*śīghra-phala*) $\Delta\sigma$ is given by

$$R\sin(\Delta\sigma) = \left(\frac{r_s}{K}\right)\ R\sin\,(\mu - \zeta), \tag{61}$$

where $(\mu - \zeta)$ is the *śīghrakendra* and K is the hypotenuse (*śīghrakarṇa)* given by

$$K^2 = R^2 + r_s^2 - 2Rr_s\cos\,(\mu - \zeta). \tag{62}$$

The calculation of the *śīghra*-correction to the velocity is indeed much more difficult as the denominator in (61), which is the hypotenuse which depends on the anomaly, also varies with time in a complex way. This has been noted by Bhāskara who was able to obtain the correct form of the *śīghra*-correction to the velocity (*śīghra-gati-phala*) in an ingenious way.[40]

> फलांशखाङ्कान्तरशिञ्जिनीघ्नी द्राक्केन्द्रभुक्तिः श्रुतिहृद्विशोध्या।
> स्वशीघ्रभुक्तेः स्फुटखेटभुक्तिः शेषं च वक्रा विपरीतशुद्धौ ॥
>
> The Rsine of ninety degrees, less the degrees of *śīghra*-correction for the longitude (*śīghra-phala*), should be multiplied by the rate of motion of the *śīghra*-anomaly (*drāk-kendra-bhukti*) and divided by the hypotenuse (*śīghra-karṇa*). This, subtracted from the rate of motion of the *śīghrocca*, gives the true velocity of the planet. If this is negative, the planet's motion is retrograde.

If ω is the rate of motion of the *manda*-corrected planet and ω_s is the rate of motion of the *śīghrocca*, then the rate of motion of the *śīghra*-anomaly is $(\omega - \omega_s)$, and the true velocity of the planet ω_t is given by

$$\omega_t = \omega_s - \left[\frac{(\omega_s - \omega) R \cos(\Delta\sigma)}{K} \right]. \tag{63}$$

The details of the ingenious argument given by Bhāskara for deriving the correct form (63) of the *śīghra*-correction to the velocity has been outlined by D. Arkasomayaji in his translation of *Sīddhāntaśiromaṇi*.[41]

Since Bhāskara's derivation is somewhat long-winded, here we shall present a modern derivation of the result just to demonstrate that the expression given by Bhāskara is indeed exact.

In Figure 5a, S, E and P represent the positions of the Sun, Earth and an exterior planet respectively. Let v and v_s be the linear velocities of the planet and the Earth with respect to the Sun. PP' and EE' are lines perpendicular to the line EP joining the Earth to the planet. Let R, r represent the radii of the orbits of the planet and the Earth (assumed to be cicular) around the Sun respectively and K, the distance of the planet from the Earth. For an exterior planet, the *śīghra*-correction $\Delta\sigma$ is given by the angle $S\hat{P}E$.

If v_t be the linear velocity of the planet as seen from the Earth, then the angular velocity is given by

$$\omega_t = \frac{d\theta}{dt} = \frac{v_t}{K}. \tag{64}$$

The magnitude of v_t in terms of v and v_s (for the situation depicted in the figure) is

$$v_t = v \cos\Delta\sigma + v_s \cos\theta. \tag{65}$$

Also from the triangle SEP, the distance of the planet from the Earth—known as *karṇa*, and denoted K in the figure—may be expressed as

$$\begin{aligned} K &= R\cos\Delta\sigma + r\cos\theta, \\ \text{or} \qquad \cos\theta &= \frac{K - R\cos\Delta\sigma}{r}. \end{aligned} \tag{66}$$

[40] *Siddhāntaśiromaṇi*, cited above, verse 2.39, p. 121.
[41] D. Arkasomayaji, *Siddhāntaśiromaṇi* of Bhāskarācārya, Tirupati 1980, pp. 157–161.

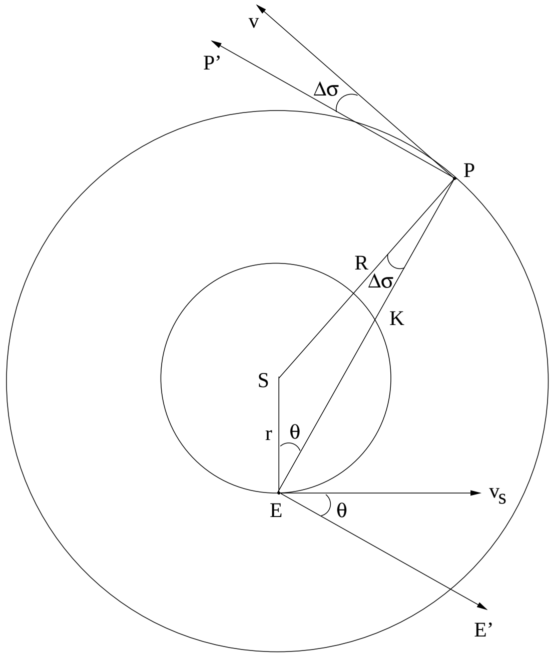

Figure 5*a*: Velocity of a planet as seen from the Earth.

Using (66) in (65) we have

$$\begin{aligned} v_t &= v\cos\Delta\sigma + \frac{v_s}{r}(K - R\cos\Delta\sigma) \\ &= \frac{v_s K}{r} + \cos\Delta\sigma\left(v - v_s\frac{R}{r}\right) \\ \text{or} \qquad \frac{v_t}{K} &= \frac{v_s}{r} + \frac{\cos\Delta\sigma\left(v - v_s\frac{R}{r}\right)}{K}. \qquad (67) \end{aligned}$$

Making use of (64) and the fact that $v = R\omega$ and $v_s = r\omega_s$, the above equation reduces to

$$\omega_t = \omega_s - \left[\frac{(\omega_s - \omega)R\cos\Delta\sigma}{K}\right],$$

which is same as the expression given by Bhāskara (63).

Bhāskara in his *Vāsanā*:[42] justifies as to why in the *śīghra* process a different procedure for finding the rate of motion of the planet has to be employed than the one used in the *manda* process:

अत्रोपपत्तिः। अद्यतनश्वस्तनशीघ्रफलयोरन्तरं गतेः शीघ्रफलं स्यात्। तच्च यथा मान्दं गतिफलं ग्रहफलवदानीतं तथा यद्यानीयते कृतेऽपि कर्णानुपाते सान्तरमेव स्यात्। यथा धीवृद्धिदे। नहि केन्द्रगतिजमेव फलयोरन्तरं स्यात् किन्त्वन्यदपि अद्यतनभुजफलश्वस्तनभुजफलान्तरे त्रिज्यागुणेऽद्यतन-कर्णहृते यादृशं फलं न तादृशं श्वस्तनकर्णहृते। स्वल्पान्तरेऽपि कर्णे

[42] Ibid., *Vāsanā* on 2.39.

भाज्यस्य बहुत्वाद् बह्वन्तरं स्यादित्येतदानयनं हित्वान्यत् महामतिमद्भिः कल्पितम्। तद्यथा...

Here is the justification. The *śīghra*-correction to the rate of motion is the difference between the *śīghra-phalas* of today and tomorrow. If that is derived in the same way as the *manda*-correction to the rate of motion, the result will be incorrect even if it were to be divided by the hypotenuse (*śīghra-karṇa*)... The difference is not just due to the change in the anomaly [which is the argument of the Rsine] but also otherwise... The result of dividing by today's hypotenuse is different from that of dividing by that of tomorrow. Even if the hypotenuses turn out to differ by small amount, since the quantities they divide are large and thus a large difference could result. Hence, this way of approach [which was adopted in the case of *manda*-correction to the rate of the motion] has been forsaken and another has been devised by the great intellects. That is as follows...

6.4 The equation of centre is extremum when the velocity correction vanishes

Later, in the *Golādhyāya* of *Siddhāntaśiromaṇi*, Bhāskara considers the situation when the correction to the velocity (*gati-phala*) vanishes:[43]

कक्ष्यामध्यगतिर्यग्रेखाप्रतिवृत्तसंपाते।
मध्यैव गतिः स्पष्टा परं फलं तत्र खेटस्य॥

Where the [North-South] line perpendicular to the [East-West] line of apsides through the centre of the concentric meets the eccentric, there the mean velocity itself is true and the equation of centre is extremum.

In his *Vāsanā*, Bhāskara explains this relation between vanishing of the velocity correction and the extrema of the correction to the planetary longitude:[44]

कक्ष्यावृत्तमध्ये या तिर्यग्रेखा तस्याः प्रतिवृत्तस्य च यः संपातस्तत्र मध्यैव गतिः स्पष्टा। गतिफलाभावात्। किंच तत्र ग्रहस्य परमं फलं स्यात्। यत्र ग्रहस्य परमं फलं तत्रैव गतिफलाभावेन भवितव्यम्। यतोऽद्यतनश्वस्तन-ग्रहयोरन्तरं गतिः। फलयोरन्तरं गतिफलम्। ग्रह्स्य गतेर्वा फलाभावस्था-नमेव धनर्णसन्धिः। यत् पुनर्लल्लोक्तं 'मध्यैव गतिः स्पष्टा वृत्तद्वययोगगे द्युचरे' इति तदसत्। न हि वृत्तद्वययोगे ग्रहस्य परमं फलम्।

The mean rate of motion itself is exact at the points where the line perpendicular [to the line of apsides], at the middle of the concentric circle, meets the eccentric circle. Because, there is no correction to the rate of motion [at those points]. Also, because there the equation of centre [or

[43] *Siddhāntaśiromaṇi*, cited above, *Golādhyāya* 4.39, p. 393.
[44] Ibid., *Vāsanā* on *Golādhyāya* 4.39.

correction to the planetary longitudes] is extreme. Wherever the equation of centre is maximum, there the correction to the velocity should be absent. Because, the rate of motion is the difference between the planetary longitudes of today and tomorrow. The correction to the velocity is the difference between the equations of centre. The place where the correction to the velocity vanishes, there is a change over from positive to the negative. And, what has been stated by Lalla, "the mean rate of motion is itself true when the planet is on the intersection of the two circles [concentric and eccentric]", that is incorrect. The planet does not have maximum equation of centre at the confluence of the two circles.

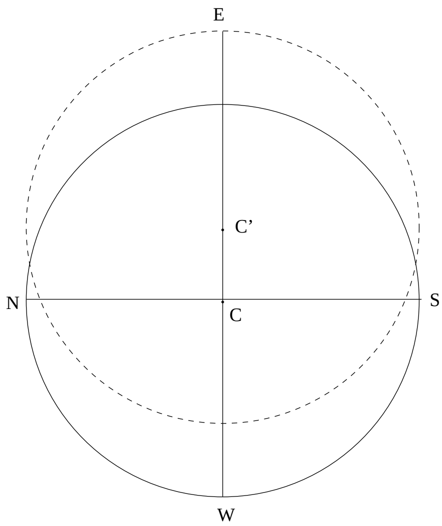

Figure 5*b*: Equation of centre is extremum where the correction to velocity vanishes.

Bhāskara explains that when the anomaly is ninety degrees, or the mean planet is at N along the line CN perpendicular to the line of apsides CE (see Figure 5*b*), the equation of centre is maximum. It is precisely then that the correction to the velocity vanishes, as it changes sign from positive to negative. It is incorrect to state (as Lalla did in his *Śiṣyadhīvṛddhida-tantra*) that the correction to the velocity is zero at the point where the concentric and eccentric meet.

7 Surface area and volume of a sphere

In *Āryabhaṭīya* (*Golapāda* 7), the volume of a sphere has been incorrectly estimated as the product of the area of a great circle by its square-root. Śrīdhara (c. 750) seems to have given the correct expression for the volume of a sphere (*Triśatikā* 56), though his estimate of π is fairly off the mark. Bhāskarācārya (c. 1150) has given the correct relation between the diameter, the surface area and the volume of a sphere in his *Līlāvatī*:[45]

[45] *Līlāvatī*, cited above (fn. 5), verse 203, p. 79–80.

वृत्तक्षेत्रे परिधिगुणितव्यासपादः फलं यत्
क्षुण्णं वेदैरुपरि परितः कन्दुकस्येव जालम्।
गोलस्यैवं तदपि च फलं पृष्ठजं व्यासनिघ्नं
षड्भिर्भक्तं भवति नियतं गोलगर्भे घनाख्यम्॥

In a circle, the circumference multiplied by one-fourth the diameter is the area, which, multiplied by four, is its surface area going around it like a net around a ball. This [surface area] multiplied by the diameter and divided by six is the volume of the sphere.

The surface area and volume of a sphere have been discussed in greater detail in the *Siddhāntaśiromaṇi* (*Golādhyāya* 2.53-61), where Bhāskara has also presented the *upapatti* or justification for the results in his commentary *Vāsanā*. As regards the surface area of the sphere, Bhāskara argues as follows:[46]

अथ बालावबोधार्थं गोलस्योपरि दर्शयेत्। भूगोलं मृण्मयं दारुमयं वा कृत्वा तं चक्रकलापरिधिं (२१६००) प्रकल्प्य तस्य मस्तके बिन्दुं कृत्वा तस्माद्बिन्दोर्गोलषण्णवतिभागेन शरद्विदस्रसङ्ख्येन (२२५) धनूरूपेणैव वृत्तरेखामुत्पादयेत्। पुनस्तस्मादेव बिन्दोः तेनैव द्विगुणसूत्रेणान्यां त्रिगुणेनान्यामेवं चतुर्विंशतिगुणं यावच्चतुर्विंशतिर्वृत्तानि भवन्ति। एषां वृत्तानां शरनेत्रबाहवः (२२५) इत्यादीनि ज्यार्धानि व्यासार्धानि स्युः। तेभ्योऽनुपाताद्वृत्तप्रमाणानि। तत्र तावदन्त्यवृत्तस्य मानं चक्रकलाः (२१६००)। तस्य व्यासार्धं त्रिज्या ३४३८। ज्यार्धानि चक्रकलागुणानि त्रिज्याभक्तानि वृत्तमानानि जायन्ते। द्वयोर्द्वयोर्वृत्तयोर्मध्य एकैकं वलयाकारं क्षेत्रम्। तानि चतुर्विंशतिः। बहुज्यापक्षे बहूनि स्युः। तत्र महदधोवृत्तं भूमिमुपरितनं लघुमुखं शरद्विदस्रमितं लम्बं प्रकल्प्य लम्बगुणं कुमुखयोगार्धमित्येवं पृथक् पृथक् फलानि। तेषां फलानां योगो गोलार्धपृष्ठफलम्। तद्द्विगुणं सकलगोलपृष्ठफलम्। तद्व्यासपरिधिघाततुल्यमेव स्यात्।

In order to make the point clear to a beginner, the teacher should demonstrate it on the surface of a sphere. Make a model of the earth in clay or wood and let its circumference be $21,600$ minutes. From the point at the top of the sphere at an arc-distance of $1/96^{th}$ of the circumference, i.e., $225'$, draw a circle. Similarly draw circles with twice, thrice,... twenty-four times $225'$ [as the arc-distances] so that there will be twenty-four circles. These circles will have as there radii Rsines starting from $225'$. The measure [circumference] of the circle will be in proportion to these radii. Here, the last circle has a circumference $21,600'$ and its radius is $3,438'$. The Rsines multiplied by $21,600$ and divided by the radius $[3,438]$ will give the measure of the circles. Between any two circles, there is an annular region and there are twenty-four of them. If more [than 24] Rsines are used, then there will be as many regions. In each figure [if it is cut and spread across as a trapezium] the larger lower circle may be taken as the base and the smaller upper circle as the face and $225'$

[46] *Siddhāntaśiromaṇi*, cited above, *Vāsanā* on *Golādhyāya* 2.57, p. 362.

as the altitude and the area calculated by the usual rule: [Area is] altitude multiplied by half the sum of the base and face. The sum of all these areas is the area of half the sphere. Twice that will be the surface area of the entire sphere. That will always be equal to the product of the diameter and the circumference.

Here $\mathrm{Bh\bar{a}skara}$ is taking the circumference to be $C = 21600'$, and the corresponding radius is approximated as $R \approx 3438'$. As shown in Figure 6, circles are drawn parallel to the equator of the sphere, each separated in latitudes by $225'$. This divides the northern hemisphere into 24 strips, each of which can be cut and spread across as a trapezium. If we denote the 24 tabulated Rsines by B_1, B_2,... B_{24}, then the area A_j of j-th trapezium will be

$$A_j = \left(\frac{C}{R}\right)\frac{(B_j + B_{j+1})}{2}225.$$

Therefore, the surface area S of the sphere is estimated to be

$$S = 2\left(\frac{C}{R}\right)\left[B_1 + B_2 + ...B_{23} + \left(\frac{B_{24}}{2}\right)\right](225). \tag{68}$$

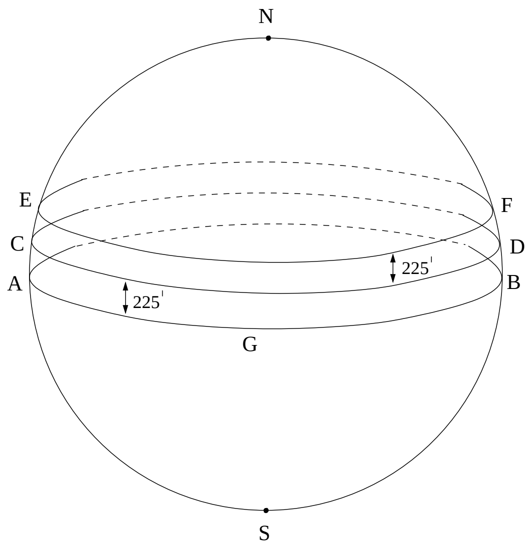

Figure 6: Surface area of a sphere.

Now, $\mathrm{Bh\bar{a}skara}$ states that the right hand side of the above equation reduces to $2CR$. This can be checked by using $\mathrm{Bh\bar{a}skara}$'s Rsine-table. $\mathrm{Bh\bar{a}skara}$ himself has done the summation of the Rsines in his $V\bar{a}san\bar{a}$ on the succeeding verses,[47] where he gives another method of derivation of the area of the sphere, by cutting the surface of the sphere into lunes. In that context, he computes the sum

$$\begin{aligned} B_1 + B_2 .... + B_{23} + \left(\frac{B_{24}}{2}\right) &= B_1 + B_2 .... + B_{23} + B_{24} - \left(\frac{R}{2}\right) \\ &\approx 54233 - 1719 = 52514. \end{aligned} \tag{69}$$

[47] $Siddh\bar{a}nta\acute{s}iroma\d{n}i$, cited above, $V\bar{a}san\bar{a}$ on $Gol\bar{a}dhy\bar{a}ya$ 2.58–61, p. 364.

Thus, according to Bhāskara's Rsine table

$$\begin{aligned}\left[B_1 + B_2 + ... + B_{23} + \left(\frac{B_{24}}{2}\right)\right](225) &= 52514 \times (225) \\ &= 11815650 \\ &\approx (3437.39)^2. \qquad (70)\end{aligned}$$

Taking this as $R^2 = (3438)^2$, we obtain the surface area of the sphere to be [48]

$$S = 2\left(\frac{C}{R}\right) R^2 = 2CR. \qquad (71)$$

Of course, the grossness of the result (70) is due to the fact that the quadrant of the circumference was divided into only 24 bits. Bhāskara also mentions that we may consider dividing the circumference into many more arc-bits, instead of the usual 24 divisions which are made for computing Rsine-tables. This is the approach taken in *Yuktibhāṣā*, where the circumference of the circle is divided into a large number, n, of equal arc-bits. If Δ is the Rsine of each arc-bit, the surface area is estimated to be

$$S = 2\left(\frac{C}{R}\right)(B_1 + B_2 + ...B_n)(\Delta). \qquad (72)$$

Then it is shown that in the limit of large n,

$$(B_1 + B_2 + ...B_n)(\Delta) \approx R^2, \qquad (73)$$

which leads to the result $2CR$ for the surface area.[49]

As regards the volume of a sphere, Bhāskara's justification is much simpler:[50]

> गोलपृष्ठफलस्य व्यासगुणितस्य षडंशो घनफलं स्यात्। अत्रोपपत्तिः। पृष्ठफलसङ्ख्यानि रूपबाहूनि व्यासार्धतुल्यवेधानि सूचीखातानि गोलपृष्ठे प्रकल्प्यानि। सूच्यग्राणां गोलगर्भे संपातः। एवं सूचीफलानां योगो घनफलमित्युपपन्नम्। यत् पुनः क्षेत्रफलमूलेन क्षेत्रफलं गुणितं घनफलं स्यादिति तत् प्रायः चतुर्वेदाचार्यः परमतमुपन्यस्तवान्।
>
> The surface area of a sphere multiplied by its diameter and divided by six is its volume. Here is the justification. As many pyramids as there are units in the surface area with bases of unit side and altitude equal to the semi-diameter should be imagined on the surface of the sphere. The apices of the pyramids meet at the centre of the sphere. Then the volume of the sphere is the sum of the volumes of the pyramids and thus our result is justified. The view that the volume is the product of the area times its own root, is perhaps an alien view (*paramata*) that has been presented by Caturavedācārya [Pṛthūdakasvāmin].

[48] As has been remarked by one of the reviewers, it is indeed intriguing the Bhāskara chose to sum the tabular Rsines numerically, instead of making use of the relation between Rsines and Rcosine-differences which was well known since the time of Āryabhaṭa. In fact, the proof given in *Yuktibhāṣā* (cited below in fn. 49) makes use of the relation between the Rsines and the second order Rsine-diffferences to estimate this sum.

[49] *Gaṇita-yukti-bhāṣā*, cited above, Section 7.18, pp. 140–42, 261–63, 465–67. In modern terminology, this amounts to the evaluation of the integral $\int_0^{\frac{\pi}{2}} R\sin\theta R d\theta = R^2$.

[50] *Siddhāntaśiromaṇi*, cited above, *Vāsanā* on *Golādhyāya* 2.61, p. 364.

We may note that it is the *Āryabhaṭīya* rule which is referred to as *paramata* in the above passage. Bhāskara's derivation of the volume of a sphere is similar to that of the area of a circle by approximating it as the sum of the areas of a large numbers of triangles with their vertices at the centre, which is actually the proof given in *Yuktibhāṣā*. In the case of the volume of a sphere, *Yuktibhāṣā*, however, gives the more "standard" derivation, where the sphere is divided into a large number of slices and the volume is found as the sum of the volumes of the slices—which ultimately involves estimating the sum of squares of natural numbers (*varga-saṅkalita*), $1^2 + 2^2 + 3^2 + ... + n^2$, for large n.[51]

[51] *Gaṇita-yukti-bhāṣā*, cited above, Section 7.19, pp. 142–45, 263–66, 468–70.

PART II : WORK OF THE KERALA SCHOOL

Mādhava to Śaṅkara Vāriyar (c. 1350–1550 CE)

8 Kerala School of Astronomy

The Kerala School of Astronomy in the medieval period, pioneered by Mādhava (c. 1340–1420) of *Saṅgamagrāma*, extended well into the 19th century as exemplified in the work of Śaṅkaravarman (c. 1830), *Rājā* of Kaḍattanaḍu. Only a couple of astronomical works of Mādhava (*Veṇvāroha* and *Sphuṭacandrāpti*) seem to be extant now. Most of his celebrated mathematical discoveries—such as the infinite series for π and the sine and cosine functions—are available only in the form of citations in later works.

Mādhava's disciple *Parameśvara* (c. 1380–1460) of Vaṭasseri, is reputed to have carried out detailed observations for over 50 years. A large number of original works and commentaries written by him have been published. However, his most important work on mathematics, the commentary *Vivaraṇa* on *Līlāvatī* of Bhāskara II, is yet to be published.

Nīlakaṇṭha Somayājī (c. 1444–1550) of Kuṇḍagrāma, disciple of Parameśvara's son Dāmodara (c. 1410–1520), is the most celebrated member of Kerala School after Mādhava. Nīlakaṇṭha has cited several important results of Mādhava in his various works, the most prominent of them being *Tantrasaṅgraha* (c. 1500) and *Āryabhaṭīya-bhāṣya*. In the latter work, while commenting on *Gaṇitapāda* of *Āryabhaṭīya*, Nīlakaṇṭha has also dealt extensively with many important mathematical issues.

However, the most detailed exposition of the work of the Kerala School, starting from Mādhava, and including the seminal contributions of Parameśvara, Dāmodara and Nīlakaṇṭha, is to be found in the famous Malayalam work *Yuktibhāṣā* (c. 1530) of Jyeṣṭhadeva (c. 1500–1610). Jyeṣṭhadeva was also a disciple of Dāmodara but junior to Nīlakaṇṭha. The direct lineage from Mādhava continued at least till Acyuta Piśāraṭi (c. 1550–1621), a disciple of Jyeṣṭhadeva, who wrote many important works and a couple of commentaries in Malayalam also.

At the very beginning of *Yuktibhāṣā*, Jyeṣṭhadeva states that he intends to present the rationale of the mathematical and astronomical results and procedures which are to be found in *Tantrasaṅgraha* of Nīlakaṇṭha. *Yuktibhāṣā*, comprising 15 chapters, is naturally divided into two parts, Mathematics and Astronomy. Topics in astronomy proper, so to say, are taken up for consideration only from the eighth chapter onwards, starting with a discussion on mean and true planets.

The first seven chapters of *Yuktibhāṣā* are in fact in the nature of an independent treatise on mathematics and deal with various topics which are of relevance to astronomy. It is here that one finds detailed demonstrations of the results of Mādhava such as the infinite series for π, the arc-tangent, sine and the cosine functions, the estimation of correction terms and their use in the generation of faster convergent series. Demonstrations are also provided for the classical results of Āryabhaṭa (c. 499) on *kuṭṭākāra* (linear indeterminate equations), of Brahmagupta (c. 628) on the diagonals and the area of a cyclic quadrilateral, and of Bhāskara II (c. 1150) on the surface area and volume of a sphere. Many of these rationales have also been presented mostly in the form of Sanskrit verses by Śaṅkara Vāriyar (c. 1500–1560) of Tṛ·ikkuṭaveli in his commentaries *Kriyākramakarī* (c. 1535) on *Līlāvatī* of Bhāskara II and *Yukti-dīpikā* on *Tantrasaṅgraha* of Nīlakaṇṭha. In fact, Śaṅkara Vāriyar ends his commenatary on the first chapter of *Tantrasaṅgraha* with the acknowledgement:[52]

> इत्येषा परक्रोडावासद्विजवरसमीरितो योऽर्थः।
> स तु तन्त्रसङ्ग्रहस्य प्रथमेऽध्याये मया कथितः॥
>
> Whatever has been the meaning as expounded by the noble *dvija* of *Parakroḍa* [Jyeṣṭhadeva] the same has now been stated by me for the first chapter of *Tantrasaṅgraha*.

In the following sections we shall present an overview of the contribution of the Kerala School to the development of calculus (during the period 1350–1500), following essentially the exposition given in *Yuktibhāṣā*. In order to indicate some of the concepts and methods developed by the Kerala astronomers, we first take up the issue of irrationality of π and the summation of infinite geometric series as discussed by Nīlakaṇṭha Somayājī in his *Āryabhaṭīya-bhāṣya*. We then cosider the derivation of binomial series expansion and the estimation of the sum of integral powers of integers, $1^k + 2^k + \ldots + n^k$ for large n, as presented in *Yuktibhāṣā*. These results constitute the basis for the derivation of the infinite series for $\frac{\pi}{4}$ due to Mādhava. We shall outline this as also the very interesting work of Mādhava on the estimation of the end-correction terms and the transformation of the π-series to achieve faster convergence. Finally we shall summarize the derivation of the infinite series for Rsine and Rcosine due to Mādhava.

In the final section, we shall deal with another topic which has a bearing on calculus, but is not dealt with in *Yuktibhāṣā*, namely the evaluation of the instantaneous velocity of a planet. Here, we shall present the result of Dāmodara, as cited by Nīlakaṇṭha, on the instantaneous velocity of a planet which involves the derivative of the arc-sine function. There are indeed many works and commentaries by later astronomers of the Kerala School, whose mathematical contributions are yet to be studied in detail. We shall here cite only one result due to Acyuta Piṣāraṭi (c. 1550–1621), a disciple of Jyeṣṭhadeva, on the instantaneous velocity of a planet, which involves the evaluation of the derivative of the ratio of two functions.

9 Nīlakaṇṭha's discussion of irrationality of π

In the context of discussing the procedure for finding the approximate square root of a non-square number, by multiplying it by a large square number (the method given in *Triśatikā* of Śrīdhara referred to earlier in Section 3.3), Nīlakaṇṭha observes in his *Āryabhaṭīya-bhāṣya*:[53]

[52] *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī, Ed. with *Yukti-dīpikā* of Śaṅkara Vāriyar by K. V. Sarma, Hoshiarpur 1977, p. 77. The same acknowledgement appears at the end of the subsequent chapters also.

[53] *Āryabhaṭīya* of Āryabhaṭa, Ed. with *Āryabhaṭīya-bhāṣya* of Nīlakaṇṭha Somayājī by K. Sāmbaśiva Śāstrī, Trivandrum Sanskrit Series 101, Trivandrum 1930, comm. on *Gaṇitapāda* 4, p. 14.

> एवं कृतोऽप्यासन्नमेव मूलं स्यात्। न पुनः करणीमूलस्य तत्त्वतः परिच्छेदः कर्तुं शक्य इत्यभिप्रायः। ततो यावदपेक्षम् अंशानां सूक्ष्मत्वाय महता वर्गेण हननमुक्तम्।
>
> Even if we were to proceed this way, the square root obtained will only be approximate. The idea [that is being conveyed] is, that it is actually not possible to exactly de-limit (*paricchedaḥ*) the square root of a non-square number. Precisely for this reason, multiplication by a large square was stated (recommended) in order to get as much accuracy as desired.

Regarding the choice of the large number that must be made, it is mentioned that one may choose any number—as large a number as possible—that gives the desired accuracy.[54]

> तत्र यावता महता गुणने बुद्ध्यावलंभावः स्यात् तावता हन्यात्। महत्त्वस्य आपेक्षिकत्वात् क्वचिदपि न परिसमाप्तिरिति भावः।
>
> You can multiply by whichever large number you want upto your satisfaction (*buddhāvalaṃbhāvaḥ*). Since largeness is a relative notion, it may be understood that the process is an unending one.

In this context, Nīlakaṇṭha cites the verse given by Āryabhaṭa specifying the ratio of the circumference to the diameter of a circle (value of π), particularly drawing our attention to the fact that Āryabhaṭa refers to this value as "approximate".[55]

> वक्ष्यति च – 'अयुतद्वयविष्कम्भस्य आसन्नो वृत्तपरिणाहः' इति। तत्र व्यासेन परिधिज्ञाने अनुमानपरम्परा स्यात्। तत्कर्मण्यपि मूलीकरणस्य अन्तर्भावादेव तस्य आसन्नत्वम्। तत्सर्वं तदवसरे एव प्रतिपादयिष्यामः।
>
> As will be stated [by the author himself] – 'this is [only] an approximate measure of the circumferene of the circle whose diameter is twenty-thousand.' In finding the circumference from the diameter, a series of inferences are involved. The approximate nature of this also stems from the fact that it involves finding square roots. All this will be explained later at the appropriate context.

Addressing the issue—later in his commentary, as promised earlier—while discussing the value of π Nīlakaṇṭha observes:[56]

> परिधिव्यासयोः सङ्ख्यासम्बन्धः प्रदर्शितः।...आसन्नः, आसन्नतयैव अयुतद्वयसङ्ख्याविष्कम्भस्य इयं परिधिसङ्ख्या उक्ता। कुतः पुनः वास्तवीं सङ्ख्याम् उत्सृज्य आसन्नैव इहोक्ता ? उच्यते। तस्या वक्तुमशक्यत्वात्। कुतः ?

[54] Ibid.
[55] Ibid.
[56] Ibid., comm. on *Gaṇitapāda* 10, p. 41.

> The relation between the circumference and the diameter has been presented. …Approximate: This value (62,832) has been stated as only an aproximation to the circumference of a circle having a diameter of 20,000. "Why then has an approximate value been mentioned here instead of the actual value?" It is explained [as follows]. Because it (the exact value) cannot be expressed. Why?

Explaining as to why the exact value cannot be presented, Nīlakaṇṭha continues:[57]

> येन मानेन मीयमानो व्यासः निरवयवः स्यात्, तेनैव मीयमानः परिधिः पुनः सावयव एव स्यात्। येन च मीयमानः परिधिः निरवयवः तेनैव मीयमानो व्यासोऽपि सावयव एव; इति एकेनैव मानेन मीयमानयोः उभयोः क्वापि न निरवयवत्वं स्यात्। महान्तम् अध्वानं गत्वापि अल्पावयवत्वम् एव लभ्यम्। निरवयवत्वं तु क्वापि न लभ्यम् इति भावः।

> Given a certain unit of measurement (*māna*) in terms of which the diameter (*vyāsa*) specified [is just an integer and] has no [fractional] part (*niravayava*), the same measure when employed to specify the circumference (*paridhi*) will certainly have a [fractional] part (*sāvayava*) [and cannot be just an integer]. Again if in terms of certain [other] measure the circumference has no [fractional] part, then employing the same measure the diameter will certainly have a [fractional] part [and cannot be an integer]. Thus when both [the diameter and the circumference] are measured by the same unit, they cannot both be specified [as integers] without [fractional] parts. Even if you go a long way (i.e., keep on reducing the measure of the unit employed), the fractional part [in specifying one of them] will only become very small. A situation in which there will be no [fractional] part (i.e, both the diameter and circumference can be specified in terms of integers) is impossible, and this is what is the import [of the expression *āsanna*].

Evidently, what Nīlakaṇṭha is trying to explain here is the incommensurability of the circumference and the diameter of a circle. Particularly, the last line of the above quote—where Nīlakaṇṭha clearly mentions that, however small you may choose your unit of measurement to be, the two quantities will never become commensurate—is noteworthy.

10 Nīlakaṇṭha's discussion of the sum of an infinite geometric series

In his *Āryabhaṭīya-bhāṣya*, while deriving an interesting approximation for the arc of a circle in terms of the *jyā* (Rsine) and the *śara* (Rversine), Nīlakaṇṭha presents

[57] Ibid., pp. 41–42.

a detailed demonstration of how to sum an infinite geometric series. The specific geometric series that arises in this context is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}\ .$$

We shall now present an outline of Nīlakaṇṭha's argument that gives an idea of how the notion of limit was understood in the Indian mathematical tradition.

10.1 Nīlakaṇṭha's approximate formula for the arc in terms of *jyā* and *śara*

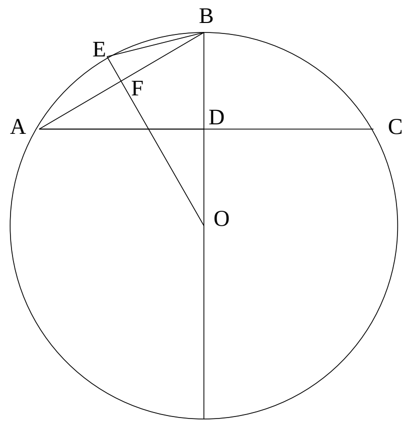

Figure 7: Arc-length in terms of *jyā* and *śara*.

In Figure 7, AB is the arc whose length (assumed to be small) is to be determined in terms of the chord lengths AD and BD. In the Indian mathematical literature, the arc AB, the semi-chord AD and the segment BD are referred to as the *cāpa*, *jyārdha* and *śara* respectively. As can be easily seen from the figure, this terminology arises from the fact that these geometrical objects look like a bow, a string and an arrow respectively. Denoting them by c, j, and s, the expression for the arc given by Nīlakaṇṭha may be written as:

$$c \approx \sqrt{\left(1 + \frac{1}{3}\right)\ s^2 + j^2}. \tag{74}$$

Nīlakaṇṭha's proof of the above equation has been discussed in detail by Sarasvati Amma.[58] It may also be mentioned that the above approximation actually does not form a part of the text *Āryabhaṭīya*; but nevertheless it is introduced by Nīlakaṇṭha while commenting upon a verse in *Āryabhaṭīya* that gives the arc in terms of the chords in a circle.[59] The verse that succinctly presents the above equation goes as

[58] T. A. Sarasvati Amma, cited above (fn. 21), pp. 179–182.

[59] वृत्ते शरसंवर्गः अर्धज्यावर्गः स खलु धनुषोः। (*Āryabhaṭīya*, *Gaṇitapāda*, verse 17).

follows:[60]

> सत्र्यंशादिषुवर्गात् ज्यावर्गाढ्यात् पदं धनुः प्रायः।
>
> The arc is nearly (*prāyaḥ*) equal to the square root of the sum of the square of the *śara* added to one-thirds of it, and the square of the *jyā*.

The proof of (74) given by Nīlakaṇṭha involves:

1. Repeated halving of the arc-bit, *cāpa* c to get $c_1 \ldots c_i \ldots$.
2. Finding the corresponding semi-chords, *jyā* (j_i) and the Rversines, *śara* (s_i).
3. Estimating the difference between the *cāpa* and *jyā* at each step.

If δ_i denotes the difference between the *cāpa* and *jyā* at the i^{th} step, that is,

$$\delta_i = c_i - j_i,$$

then it is seen that this difference decreases as the size of the *cāpa* decreases. Having made this observation, Nīlakaṇṭha proceeds with the argument that

- Generating successive values of the j_i-s and s_i-s is an 'unending' process (*na kvacidapi paryavasyati*) as one can keep on dividing the *cāpa* into half *ad infinitum* (*ānantyāt vibhāgasya*).
- It would therefore be appropriate to proceed upto a stage where the difference δ_i becomes negligible (*śūnyaprāya*) and make an 'intelligent approximation', to obtain the value of the difference between c and j approximately.

The original passage in *Āryabhaṭīya-bhāṣya* which presents the above argument reads as follows:[61]

> तत्र ज्याचापयोरन्तरस्य पुनः पुनः न्यूनत्वं चापपरिमाणाल्पत्वक्रमेणेति तत्तदर्धचापानाम् अर्धज्यापरम्परा शरपरम्परा च आनीयमाना न क्वचिदपि पर्यवस्यति आनन्त्याद् विभागस्य।
>
> ततः कियन्तञ्चित् प्रदेशं गत्वा चापस्य जीवायाश्च अल्पीयस्त्वम् आपाद्य चापज्यान्तरं च शून्यप्रायं लब्ध्वा पुनरपि कल्प्यमानमन्तरम् अत्यल्पमपि कौशलात् ज्ञेयम्।

[60] *Āryabhaṭīya-bhāṣya* on *Āryabhaṭīya*, cited above (fn. 50), comm. on *Gaṇitapāda* 12 and 17, p. 63 and p. 110. That the verse cited is from another work of his, namely *Golasāra*, has been alluded to by Nīlakaṇṭha in both the instances of citation.

[61] Ibid., comm. on *Gaṇitapāda* 17, pp. 104–05.

10.2 Nīlakaṇṭha's summation of the infinite geometric series

The question that Nīlakaṇṭha poses as he commences his detailed discussion on the sum of geometric series is very important and arises quite naturally whenever one encounters the sum of an infinite series:[62]

> कथं पुनः तावदेव वर्धते तावद्वर्धते च ?
>
> How do you know that [the sum of the series] increases only upto that [limiting value] and that it certainly increases upto that [limiting value]?

Proceeding to answer the above question, Nīlakaṇṭha first states the general result

$$a\left[\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2+\left(\frac{1}{r}\right)^3+\ldots\right]=\frac{a}{r-1}\ .$$

Here, the left hand side is an infinite geometric series with the successive terms being obtained by dividing by a common divisor, r, known as *cheda*, whose value is assumed to be greater than 1. He further notes that this result is best demonstrated by considering a particular case, say $r = 4$. In his own words:[63]

> उच्यते। एवं यः तुल्यच्छेदपरभागपरम्परायाः अनन्तायाः अपि संयोगः तस्य अनन्तानामपि कल्प्यमानस्य योगस्य आद्यावयविनः परम्परांशच्छेदात् एकोनच्छेदांशसाम्यं सर्वत्र समानमेव। तद्यथा – चतुरंशपरम्परायामेव तावत् प्रथमं प्रतिपाद्यते।
>
> It is being explained. Thus, in an infinite (*ananta*) geometrical series (*tulyaccheda-parabhāga-paramparā*) the sum of all the infinite number of terms considered will always be equal to the value obtained by dividing by a factor which is one less than the common factor of the series. That this is so will be demonstrated by first considering the series obtained with one-fourth (*caturaṃśa-paramparā*).

What is intended to be demonstrated is

$$a\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\ldots\right]=\frac{a}{3}\ . \tag{75}$$

Besides the multiplying factor a, it is noted that, one-fourth and one-third are the only terms appearing in the above equation. Nīlakaṇṭha first defines these numbers in terms of one-twelfth of the multiplier a referred to by the word *rāśi*. For the sake of simplicity we take the *rāśi* to be unity.

$$3\times\frac{1}{12}=\frac{1}{4};\qquad\qquad 4\times\frac{1}{12}=\frac{1}{3}.$$

[62] Ibid., p. 106.
[63] Ibid., pp. 106–07.

Having defined them, Nīlakaṇṭha first obtains the sequence of results,

$$
\begin{aligned}
\frac{1}{3} &= \frac{1}{4} + \frac{1}{(4.3)}\,, \\
\frac{1}{(4.3)} &= \frac{1}{(4.4)} + \frac{1}{(4.4.3)}\,, \\
\frac{1}{(4.4.3)} &= \frac{1}{(4.4.4)} + \frac{1}{(4.4.4.3)}\,,
\end{aligned}
$$

and so on, which leads to the general result,

$$
\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right). \tag{76}
$$

Nīlakaṇṭha then goes on to present the following crucial argument to derive the sum of the infinite geometric series: As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$ (as given by the right hand side of the above equation), becomes extremely small, but never zero. Only when we take all the terms of the infinite series together do we obtain the equality

$$
\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}\,. \tag{77}
$$

A brief extract from the text presenting the above argument is given below:[64]

> ये राशेर्द्वादशांशाः तेषां त्रिकं हि चतुरंशः। चतुष्कं च त्र्यंशः। तच्चतुष्टये त्र्यंशात्मके भागत्रयं चतुरंशेनापूर्णम्। यः पुनः तस्य चतुर्थोऽम्शः तस्यापि पादत्रयं चतुरंशस्य चतुरंशेनापूर्णम्। द्वादशांशानां त्रयाणां ...
>
> तस्य पुनः पुनरतिसूक्ष्मत्वादेव न केवलं त्र्यंशत्वेन अङ्गीकारः, निरूप्यमाणस्य वा क्रियमाणस्य वा आनन्त्यात्। आनन्त्यादेव शिष्टत्वादेव कर्मणस्तस्य अपरिपूर्तिर्भाति। एवं सर्वदापि सावशेषाणां कर्मणां परम्परायां कात्स्न्र्येनाकृष्यात्र सन्निहितायां परिपूर्तिः स्यादेवेति निश्चीयते चतुर्गुणोत्तरे गुणोत्तराख्ये गणितेऽपि।
>
> Three times one-twelfth of a *rāśi* is one-fourth (*caturaṃśa*) [of that *rāśi*]. Four times that is one-third (*tryaṃśa*). [Considering] four times that [one-twelfth of the *rāśi*] which is one-third, three by fourth of that falls short by one-fourth [of one-third of the *rāśi*]. Three-fourths of that [i.e., of $\frac{1}{4.3}$ of the *rāśi*] which is one-fourth of that (*tryaṃśa*), again falls short [of the same] by one-fourth of one-fourth [of one-third of the *rāśi*] ...
>
> Since the result to be demonstrated or the process to be carried out is never ending (*ānantyāt*) and the difference though very small (*atisūkṣmatvāt*) [still exists and the sum of the series] cannot be simply taken to be one-third. It seems that the process is incomplete since always something remains because of its never ending nature. In fact, since in all the problems involving [infinite] series, by bringing in all the terms and placing them together, the process would [in principle] become complete, here, in the mathematics involving repeated multiplication of one-fourth, a similar conclusion may be drawn.

[64] Ibid., p. 107.

11 Derivation of binomial series expansion

Yuktibhāṣā presents a very interesting derivation of the binomial series for $(1+x)^{-1}$ by making iterative substitutions in an algebric identity. The method given in the text may be summarized as follows.

Consider the product $a\left(\frac{c}{b}\right)$, where some quantity a is multiplied by the multiplier c, and divided by the divisor b. Here, a is called *guṇya*, c the *guṇaka* and b the *hāra*, which are all assumed to be positive. Now the above product can be rewritten as:

$$a\ \left(\frac{c}{b}\right) = a - a\ \frac{(b-c)}{b}. \tag{78}$$

In the expression $a\frac{(b-c)}{b}$ in (78) above, if we want to replace the division by b (the divisor) by division by c (the multiplier), then we have to make a subtractive correction (called *śodhya-phala*) which amounts to the following equation.

$$a\frac{(b-c)}{b} = a\frac{(b-c)}{c} - \left(a\frac{(b-c)}{c} \times \frac{(b-c)}{b}\right). \tag{79}$$

Now, in the second term (inside parenthesis) in (79)—which is what we referred to as *śodhya-phala*, which literally means a quantity to be subtracted—if we again replace the division by the divisor b by the multiplier c, then we have to employ the relation (79) once again to get another subtractive term

$$\begin{aligned} a\frac{c}{b} &= a - \left[a\frac{(b-c)}{c} - a\frac{(b-c)}{c} \times \frac{(b-c)}{b}\right] \\ &= a - \left[a\frac{(b-c)}{c} - a\frac{(b-c)}{c} \times \frac{(b-c)}{c} \times \frac{c}{b}\right] \\ &= a - \left[a\frac{(b-c)}{c} - \left[a\frac{(b-c)^2}{c^2} - \left(a\frac{(b-c)^2}{c^2} \times \frac{(b-c)}{b}\right)\right]\right] \end{aligned} \tag{80}$$

Here, the quantity $a\frac{(b-c)^2}{c^2}$ is called *dvitīya-phala* or simply *dvitīya* and the one subtracted from that is *dvitīya-śodhya-phala*. If we carry out the same set of operations, the m^{th} *śodhya-phala* subtracted from the m^{th} term will be of the form

$$a\left[\frac{(b-c)}{c}\right]^m - a\left[\frac{(b-c)}{c}\right]^m \times \frac{(b-c)}{b}.$$

Since the successive *śodhya-phalas* are subtracted from their immediately preceding term, we will end up with a series in which all the odd terms (leaving out the *guṇya*, a) are negative and the even ones positive. Thus, after taking m *śodhya-phalas* we get

$$\begin{aligned} a\frac{c}{b} &= a - a\frac{(b-c)}{c} + a\left[\frac{(b-c)}{c}\right]^2 - \ldots + (-1)^m a\left[\frac{(b-c)}{c}\right]^m \\ &\quad +(-1)^{m+1} a\left[\frac{(b-c)}{c}\right]^m \frac{(b-c)}{b}. \end{aligned} \tag{81}$$

Regarding the question of termination of the process, both the texts *Yuktibhāṣā* and *Kriyākramakarī* clearly mention that logically there is no end to the process of generating *śodhya-phalas*. We may thus write our result as:[65]

$$
\begin{aligned}
a\frac{c}{b} \quad = \quad & a - a\frac{(b-c)}{c} + a\left[\frac{(b-c)}{c}\right]^2 - \ldots + (-1)^{m-1}a\left[\frac{(b-c)}{c}\right]^{m-1} \\
& + (-1)^m a\left[\frac{(b-c)}{c}\right]^m + \ldots . \qquad (82)
\end{aligned}
$$

It is also noted that the process may be terminated after having obtained the desired accuracy by neglecting the subsequent *phalās* as their magnitudes become smaller and smaller. In fact, *Kriyākramakarī* explicitly mentions the condition under which the succeeding *phalās* will become smaller and smaller:[66]

> एवं मुहुः फलानयने कृतेऽपि युक्तितः क्वापि न समाप्तिः। तथापि यावदपेक्षं सूक्ष्मतामापाद्य पाश्चात्यान्युपेक्ष्य फलानयनं समापनीयम्। इहोत्तरोत्तरफलानां न्यूनत्वं तु गुणहारान्तरे गुणकाराल्न्यून एव स्यात्।
>
> Thus, even if we keep finding the *phalās* repeatedly, logically there is no end to the process. Even then, having carried on the process to the desired accuracy (*yāvadapekṣaṃ sūkṣmatāmāpādya*), one should terminate computing the *phalās* by [simply] neglecting the terms that may be obtained further (*pāścātyānyupekṣya*). Here, the succeeding *phalas* will become smaller and smaller only when the difference between the *guṇaka* and *hāra* is smaller than *guṇaka*, [that is $(b \sim c) < c$].

12 Estimation of sums of $1^k + 2^k + \ldots n^k$ for large n

As mentioned in section 4.1, Āryabhaṭa has given the explicit formula for the summation of squares and cubes of integers. The word employed in the Indian mathematical literature for summation is *saṅkalita*. The formulae given by Āryabhaṭa for the *saṅkalitas* are as follows:

$$
\begin{aligned}
S_n^{(1)} &= 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \\
S_n^{(2)} &= 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \\
S_n^{(3)} &= 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 . \qquad (83)
\end{aligned}
$$

[65] It may be noted that if we set $\frac{(b-c)}{c} = x$, then $\frac{c}{b} = \frac{1}{(1+x)}$. Hence, the series (82) is none other than the well known binomial series

$$\frac{a}{1+x} = a - ax + ax^2 - \ldots + (-1)^m ax^m + \ldots ,$$

which is convergent for $-1 < x < 1$.

[66] *Kriyākramakarī* on *Līlāvatī*, cited above (fn. 14), comm. on verse 199, p. 385.

From these, it is easy to estimate these sums when n is large. *Yuktibhāṣā* gives a general method of estimating the *sama-ghāta-saṅkalita* [67]

$$S_n^{(k)} = 1^k + 2^k + \cdots + n^k, \tag{84}$$

when n is large. Actually the text presents a general method of estimation, which does not make use of the actual value of the sum. In fact, the same argument is repeated even for $k = 1, 2, 3$, although the result of summation is well known in these cases.

12.1 The sum of natural numbers (*Mūla-saṅkalita*)

Yuktibhāṣā takes up the discussion on *saṅkalitas* in the context of evaluating the circumference of a circle which is conceived to be inscribed in a square. It is half the side of this square that is being referred to by the word *bhujā* in both the citations as well as explanations offered below. Half of the side of the square (equal to the radius) is divided into n equal bits, known as *bhujā-khaṇḍas*. It is these *bhujā-khaṇḍas* $\left(\frac{r}{n}\right),\ 2\left(\frac{r}{n}\right)\cdots$ whose powers are summed.

To start with, *Yuktibhāṣā* discusses just the basic summation of *bhujā-khaṇḍas* called *Mūla-saṅkalita*. We now cite the following from the translation of *Yuktibhāṣā*:[68]

> Now is described the methods of making the summations (referred to in the earlier sections). At first, the simple arithmetical progression (*kevala-saṅkalita*) is described. This is followed by the summation of the products of equal numbers (squares). ...
>
> Here, in this *mūla-saṅkalita* (basic arithmetical progression), the final *bhujā* is equal to the radius. The term before that will be one segment (*khaṇḍa*) less. The next one will be two segments less. Here, if all the terms (*bhujās*) had been equal to the radius, the result of the summation would be obtained by multiplying the radius by the number of *bhujās*. However, here, only one *bhujā* is equal to the radius. And, from that *bhujā*, those associated with the smaller hypotenuses are less by one segment each, in order. Now, suppose the radius to be of the same number of units as the number of segments to which it has been divided, in order to facilitate remembering (their number). Then, the number associated with the penultimate *bhujā* will be less by one (from the number of units in the radius); the number of the next one, will be less by two from the number of units in the radius. This reduction (in the number of segments) will increase by one (at each step). The last reduction will practically be equal to the measure of the radius, for it will be less only by one segment. In other words, when the reductions are all added, the sum thereof will practically (*prāyeṇa*) be equal to the summation of the series from 1 to the number of units in the radius; it will be less only by one radius length. Hence, the summation will be equal to the product of the number of units in the radius with the number of segments plus one, and divided by 2.

[67] The compound *sama-ghāta* in this context means product of a number with itself same number of times.

[68] *Gaṇita-yukti-bhāṣā*, cited above, Section 6.4, pp. 61–67, 192–97, 382–88.

> The summation of all the *bhujās* of the different hypotenuses is called *bhujā-saṅkalita*.
>
> Now, the smaller the segments, the more accurate (*sūkṣma*) will be the result. Hence, do the summation also by taking each segment as small as an atom (*aṇu*). Here, if it (namely, the *bhujā* or the radius) is divided into *parārdha* (a very large number) parts, to the *bhujā* obtained by multiplying by *parārdha* add one part in *parārdha* and multiply by the radius and divide by 2, and then divide by *parārdha*. For, the result will practically be the square of the radius divided by two. ...

The first summation, the *bhujā-saṅkalita*, may be written in the reverse order from the final *bhujā* to the first *bhujā* as

$$S_n^{(1)} = \left(\frac{nr}{n}\right) + \left(\frac{(n-1)r}{n}\right) + ... + \left(\frac{r}{n}\right). \tag{85}$$

Now, conceive of the *bhujā-khaṇḍa* $\frac{r}{n}$ as being infinitesimal (*aṇu*) and at the same time as of unit-measure (*rūpa*), so that the radius will be the measure of n, the *pada*, or the number of terms. Then

$$S_n^{(1)} = n + (n-1) + ... + 1. \tag{86}$$

If each of the terms were of the measure of radius (n) then the sum would be nothing but n^2, the square of the radius. But only the first term is of the measure of radius, the next is deficient by one segment (*khaṇḍa*), the next by two segments and so on till the last term which is deficient by an amount equal to radius-minus-one segment. In other words,

$$\begin{aligned} S_n^{(1)} &= n + [n-1] + [n-2] .... + [n-(n-2)] + [n-(n-1)] \\ &= n.n - [1+2+...+(n-1)]. \end{aligned} \tag{87}$$

When n is very large, the quantity to be subtracted from n^2 is practically (*prāyeṇa*) the same as $S_n^{(1)}$, thus leading to the estimate

$$S_n^{(1)} \approx n^2 - S_n^{(1)}, \tag{88}$$

or, equivalently

$$S_n^{(1)} \approx \frac{n^2}{2}. \tag{89}$$

It is stated that the result is more accurate, when the size of the segments are small (or equivalently, the value of n is large).[69]

If instead of making the approximation as in (88), we proceed with (87) as it is, we get $S_n^{(1)} = n^2 - (S_n^{(1)} - n)$, which leads to the well-known exact value of the sum of the first n natural numbers

$$S_n^{(1)} = \frac{n(n+1)}{2}, \tag{90}$$

With the convention that the $\frac{r}{n}$ is of unit-measure, the above estimate (89) is stated in the form that the *bhujā-saṅkalita* is half the square of the radius.

[69] Śaṅkara Vāriyar also emphasizes the same idea, in his discussion of the estimation of *saṅkalita*-s in his commentary *Kriyākramakarī* on *Līlāvatī* (cited above (fn. 14), comm. on verse 199, p. 382.):

> खण्डस्याल्पत्वे सत्येव लब्धस्य सूक्ष्मता च स्यात्।
>
> Only when the segment is small (*khaṇḍasyālpatve*) the result obtained would be accurate.

12.2 Summation of squares (*Varga-saṅkalita*)

We now cite the following from the translation of *Yuktibhāṣā*:[70]

> Now is explained the summation of squares (*varga-saṅkalita*). Obviously, the squares of the *bhujās*, which are summed up above, are the *bhujās* each multiplied by itself. Here, if the *bhujās* which are all multipliers, had all been equal to the radius, their sum, (*saṅkalita* derived above), multiplied by the radius would have been the summation of their squares. Here, however, only one multiplier happens to be equal to the radius, and that is the last one. The one before that will have the number of segments one less than in the radius. (Hence) if that, (i.e., the second one), is multiplied by the radius, it would mean that one multiplied by the penultimate *bhujā* would have been the increase in the summation of the squares. Then (the segment) next below is the third. That will be less than the radius by two segments. If that is multiplied by the radius, it will mean that, the summation of the squares will increase by the product of the *bhujā* by two (segments). In this manner, the summation in which the multiplication is done by the radius (instead of the *bhujās*) would be larger than the summation of squares by terms which involve the successively smaller *bhujās* multiplied by successively higher numbers. If (all these additions) are duly subtracted from the summation where the radius is used as the multiplier, the summation of squares (*varga-saṅkalita*) will result.
>
> Now, the *bhujā* next to the east-west line is less than the radius by one (segment). So if all the excesses are summed up and added, it would be the summation of the basic summation (*mūla-saṅkalita-saṅkalita*). Because, the sums of the summations is verily the 'summation of summations' (*saṅkalita-saṅkalita*). There, the last sum has (the summation of) all the *bhujās*. The penultimate sum is next lower summation to the last. This penultimate sum is the summation of all the *bhujās* except the last *bhujā*. Next to it is the third sum which is the sum of all the *bhujās* except the last two. Thus, each sum of the *bhujās* commencing from any *bhujā* which is taken to be the last one in the series, will be less by one *bhujā* from the sum (of the *bhujās*) before that.
>
> Thus, the longest *bhujā* is included only in one sum. But the *bhujā* next lower than the last (*bhujā*) is included both in the last sum and also in the next lower sum. The *bhujās* below that are included in the three, four etc. sums below it. Hence, it would result that the successively smaller *bhujās* commencing from the one next to the last, which have been multiplied by numbers commencing from 1 and added together, would be summation of summations (*saṅkalita-saṅkalita*). Now, it has been stated earlier that the summation (*saṅkalita*) of (the segments constituting) a *bhujā* which has been very minutely divided, will be equal to half the square of the last *bhujā*. Hence, it follows that, in order to obtain the summation (*saṅkalita*) of the *bhujās* ending in any particular *bhujā*, we will have to square each of the *bhujās* and halve it. Thus, the summation of summations (*saṅkalita-saṅkalita*) would be half the summation of the squares

[70] *Gaṇita-yukti-bhāṣā*, cited above, Section 6.4, pp. 61–67, 192–97, 382–88.

> of all the *bhujās*. In other words, half the summation of the squares is the summation of the basic summation. So, when the summation is multiplied by the radius, it would be one and a half times the summation of the squares. This fact can be expressed by stating that this contains half more of the summation of squares. Therefore, when the square of the radius divided by two is multiplied by the radius and one-third of it subtracted from it, the remainder will be one-third of the whole. Thus it follows that one-third of the cube of the radius will be the summation of squares (*varga-saṅkalita*).

With the same convention that $\frac{r}{n}$ is the measure of the unit, the *bhujā-varga-saṅkalita* (the sum of the squares of the *bhujās*) will be

$$S_n^{(2)} = n^2 + (n-1)^2 + ... + 1^2. \tag{91}$$

In above expression, each *bhujā* is multiplied by itself. If instead, we consider that each *bhujā* is multiplied by the radius (n in our units), then that would give raise to the sum

$$n\,[n + (n-1) + ... + 1] = n\; S_n^{(1)}. \tag{92}$$

This sum is exceeds the *bhujā-varga-saṅkalita* by the amount

$$nS_n^{(1)} - S_n^{(2)} = 1.(n-1) + 2.(n-2) + 3.(n-3) + \ldots + (n-1).1.$$

This may be written as

$$\begin{aligned} nS_n^{(1)} - S_n^{(2)} = (n-1) + (n-2) + (n-3) &\quad + \ldots \quad +1 \\ +(n-2) + (n-3) &\quad + \ldots \quad +1 \\ +(n-3) &\quad + \ldots \quad +1 \\ &\quad + \ldots \quad . \end{aligned} \tag{93}$$

Thus,

$$nS_n^{(1)} - S_n^{(2)} = S_{n-1}^{(1)} + S_{n-2}^{(1)} + S_{n-3}^{(1)} + \ldots. \tag{94}$$

The right hand side of (94) is called the *saṅkalita-saṅkalita* (or *saṅkalitaikya*), the repeated sum of the sums $S_i^{(1)}$ (here taken in the order $i = n-1,\ n-2, \ldots 1$). These are defined also by Śaṅkara Vāriyar in *Kriyākramakarī* as follows:[71]

> तथा हि सङ्कलितानां योगो हि सङ्कलितसङ्कलितमुच्यते। तत्र अन्त्यसङ्कलितं सर्वासां भुजानां योगः। उपान्त्यसङ्कलितं तु अन्त्यभुजाव्यतिरिक्तानामितरेषां योगः। उपान्त्यात् पूर्वस्य सङ्कलितं पुनस्तदवधिकानामेव भुजानां योगः। एवं पूर्वसङ्कलितानि स्वोत्तरात् सङ्कलितात् एकैकेन भुजेन विरहितानि भवन्ति।
>
> The sum of the summations is called as *saṅkalita-saṅkalita*. Of them the last *saṅkalita* is the sum all the *bhujā*-s. The penultimate *saṅkalita* is the sum of all the *bhujā*-s other than the last one. The *saṅkalita* of the one preceding the penultimate is the sum of the *bhujā*-s ending with that. Thus, all the preceding *saṅkalita*-s will fall short by a *bhujā* from the succeeding *saṅkalita*.

[71] *Kriyākramakarī* on *Līlāvatī*, cited above (fn. 14), comm. on verse 199, pp. 382–83.

For large n, we have already estimated in (89) that $S_n^{(1)} \approx \frac{n^2}{2}$. Thus, for large n

$$nS_n^{(1)} - S_n^{(2)} \approx \frac{(n-1)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-3)^2}{2} + \ldots. \tag{95}$$

Thus, the right hand side of (94) (the *saṅkalita-saṅkalita* or the excess of $nS_n^{(1)}$ over $S_n^{(2)}$) is essentially $\frac{S_n^{(2)}}{2}$ for large n, so that we obtain

$$nS_n^{(1)} - S_n^{(2)} \approx \frac{S_n^{(2)}}{2}. \tag{96}$$

Again, using the earlier estimate (89) for $S_n^{(1)}$, we obtain the result

$$S_n^{(2)} \approx \frac{n^3}{3}. \tag{97}$$

Thus *bhujā-varga-saṅkalita* is one-third the cube of the radius.

12.3 *Sama-ghāta-saṅkalita*

We now cite the following from the translation of *Yuktibhāṣā*:[72]

> Now, the square of the square (of a number) is multiplied by itself, it is called *sama-pañca-ghāta* (number multiplied by itself five times). The successive higher order summations are called *sama-pañcādi-ghāta-saṅkalita* (and will be the summations of powers of five and above). Among them if the summation (*saṅkalita*) of powers of some order is multiplied by the radius, then the product is the summation of summations (*saṅkalita-saṅkalita*) of the (powers of the) multiplicand (of the given order), together with the summation of powers (*sama-ghāta-saṅkalita*) of the next order. Hence, to derive the summation of the successive higher powers: Multiply each summation by the radius. Divide it by the next higher number and subtract the result from the summation got before. The result will be the required summation to the higher order.
>
> Thus, divide by two the square of the radius. If it is the cube of the radius, divide by three. If it is the radius raised to the power of four, divide by four. If it is (the radius) raised to the power of five, divide by five. In this manner, for powers rising one by one, divide by numbers increasing one by one. The results will be, in order, the summations of powers of numbers (*sama-ghāta-saṅkalita*). Here, the basic summation is obtained from the square, the summation of squares from the cube, the summation of cubes from the square of the square. In this manner, if the numbers are multiplied by themselves a certain number of times (i.e., raised to a certain degree) and divided by the same number, that will be the summation of the order one below that. Thus (has been stated) the method of deriving the summations of (natural) numbers, (their) squares etc.

[72] *Gaṇita-yukti-bhāṣā*, cited above, Section 6.4, pp. 61–67, 192–97, 382–88.

In the case of a general *samaghāta-saṅkalita*, (summation of equal powers) given by

$$S_n^{(k)} = n^k + (n-1)^k + \ldots + 1^k, \tag{98}$$

the procedure followed to estimate its behavior for large n is essentially the same as that followed in the case of *vargasaṅkalita*. We first compute the excess of $nS_n^{(k-1)}$ over $S_n^{(k)}$ to be a *saṅkalita-saṅkalita* or repeated sum of the lower order *saṅkalitas* $S_r^{(k-1)}$

$$nS_n^{(k-1)} - S_n^{(k)} = S_{n-1}^{(k-1)} + S_{n-2}^{(k-1)} + S_{n-3}^{(k-1)} + \ldots \quad . \tag{99}$$

If the lower order *saṅkalita* $S_n^{(k-1)}$ has already been estimated to be, say,

$$S_n^{(k-1)} \approx \frac{n^k}{k}, \tag{100}$$

then, the above relation (99) leads to[73]

$$\begin{aligned} nS_n^{(k-1)} - S_n^{(k)} &\approx \frac{(n-1)^k}{k} + \frac{(n-2)^k}{k} + \frac{(n-3)^k}{k} + \ldots \\ &\approx \left(\frac{1}{k}\right) S_n^{(k)}. \end{aligned} \tag{101}$$

Rewriting the above equation we have[74]

$$S_n^{(k)} \approx nS_n^{(k-1)} - \left(\frac{1}{k}\right) S_n^{(k)}. \tag{102}$$

Using (100), we obtain the estimate

$$S_n^{(k)} \approx \frac{n^{k+1}}{(k+1)}. \tag{103}$$

12.4 Repeated summations (*Saṅkalita-saṅkalita*)

After having estimated the sum of powers of natural numbers *samaghāta-saṅkalita* *Yuktibhāṣā* goes on to derive an estimate for the repeated summation (*saṅkalita-saṅkalita* or *saṅkalitaikya* or *vārasaṅkalita*) of the natural number $1, 2, \cdots, n$.[75]

[73] As one of the reviewers has pointed out, this argument leading to (101) is indeed similar to the derivation of the following relation, which is based on the interchange of order in iterated integrals:

$$\int_0^1 (1-x)x^{k-1}dx \quad = \quad \int_0^1 x^{k-1} \int_x^1 dy\, dx = \int_0^1 y \int_0^y x^{k-1} dx\, dy = \int_0^1 \frac{y^k}{k} dy.$$

[74] As Śaṅkara Vāriyar states in his *Kriyākramakarī* on *Līlāvatī* (cited above (fn. 14), p. 383):

> अत उत्तरोत्तरसङ्कलितानयनाय तत्तत्सङ्कलितस्य व्यासार्धगुणनम् एकैकाधिकसङ्ख्याप्त-
> स्वांशशोधनं च कार्यम् इति स्थितम्।
>
> Therefore it is established that, for obtaining the sum of the next order, the previous sum, has to be multiplied by the radius and the present sum, divided by one more than the previous [order], has to be diminished [from that product].

[75] *Gaṇita-yukti-bhāṣā*, cited above, Section 6.4, pp. 61–67, 192–97, 382–88.

> Now, are explained the first, second and further summations: The first summation ($\bar{a}dya$-$sa\dot{n}kalita$) is the basic summation ($m\bar{u}la$-$sa\dot{n}kalita$) itself. It has already been stated (that this is) half the product of the square of the number of terms ($pada$-$varg\bar{a}rdha$). The second ($dvit\bar{\imath}ya$-$sa\dot{n}kalita$) is the summation of the basic summation ($m\bar{u}la$-$sa\dot{n}kalitaikya$). It has been stated earlier that it is equal to half the summation of squares. And that will be one-sixth of the cube of the number of terms.
>
> Now, the third summation: For this, take the second summation as the last term ($antya$); subtract one from the number of terms, and calculate the summation of summations as before. Treat this as the penultimate. Then subtract two from the number of terms and calculate the summation of summations. That will be the next lower term. In order to calculate the summation of summations of numbers in the descending order, the sums of one-sixths of the cubes of numbers in descending order would have to be calculated. That will be the summation of one-sixth of the cubes. And that will be one-sixth of the summation of cubes. As has been enunciated earlier, the summation of cubes is one-fourth the square of the square. Hence, one-sixth of one-fourth the square of the square will be the summation of one-sixth of the cubes. Hence, one-twenty-fourth of the square of the square will be the summation of one-sixth of the cubes. Then, the fourth summation will be, according to the above principle, the summation of one-twenty-fourths of the square of squares. This will also be equal to one-twenty-fourth of one-fifth of the fifth power. Hence, when the number of terms has been multiplied by itself a certain number of times, (i.e., raised to a certain degree), and divided by the product of one, two, three etc. up to that index number, the result will be the summation up to that index number amongst the first, second etc. summations ($\bar{a}dya$-$dvit\bar{\imath}y\bar{a}di$-$sa\dot{n}kalita$).

The first summation ($\bar{a}dya$-$sa\dot{n}kalita$) $V_n^{(1)}$ is just the $m\bar{u}la$-$sa\dot{n}kalita$ or the basic summation of natural numbers, which has already been estimated in (89)

$$V_n^{(1)} = S_n^{(1)} = n + (n-1) + (n-2) + \ldots + 1 \approx \frac{n^2}{2}. \tag{104}$$

The second summation ($dvit\bar{\imath}ya$-$sa\dot{n}kalita$ or $sa\dot{n}kalita$-$sa\dot{n}kalita$ or $sa\dot{n}kalitaikya$) is given by

$$\begin{aligned} V_n^{(2)} &= V_n^{(1)} + V_{n-1}^{(1)} + V_{n-2}^{(1)} + \ldots \\ &= S_n^{(1)} + S_{n-1}^{(1)} + S_{n-2}^{(1)} + \ldots \quad . \end{aligned} \tag{105}$$

As was done earlier, this second summation can be estimated using the estimate for $S_n^{(1)}$

$$V_n^{(2)} \approx \frac{n^2}{2} + \frac{(n-1)^2}{2} + \frac{(n-2)^2}{2} + \ldots \quad . \tag{106}$$

Therefore

$$V_n^{(2)} \approx \left(\frac{1}{2}\right) S_n^{(2)}. \tag{107}$$

Using the earlier estimate (97) for $S_n^{(2)}$, we get an estimate for the *dvitīya-saṅkalita*

$$V_n^{(2)} \approx \frac{n^3}{6}. \tag{108}$$

Now the next repeated summation can be found in the same way

$$\begin{aligned} V_n^{(3)} &= V_n^{(2)} + V_{n-1}^{(2)} + V_{n-2}^{(2)} + \dots \\ &\approx \frac{n^3}{6} + \frac{(n-1)^3}{6} + \frac{(n-2)^3}{6} + \dots \\ &\approx \left(\frac{1}{6}\right) S_n^{(3)} \\ &\approx \frac{n^4}{24}. \end{aligned} \tag{109}$$

It is noted that proceeding this way we can estimate repeated summation $V_n^{(k)}$ of order k, for large n, to be[76]

$$\begin{aligned} V_n^{(k)} &= V_n^{(k-1)} + V_{n-1}^{(k-1)} + V_{n-2}^{(k-1)} + \dots \\ &\approx \frac{n^{k+1}}{1.2.3. \dots (k+1)}. \end{aligned} \tag{110}$$

13 Derivation of the Mādhava series for π

The following accurate value of π (correct to 11 decimal places), given by Mādhava, has been cited by Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya* and by Śaṅkara Vāriyar in his *Kriyākramakarī*.[77]

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः ।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः ॥

The π value given above is:

$$\pi \approx \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592... \tag{111}$$

The 13 digit number appearing in the numerator has been specified using *bhūta-saṅkhya* system, whereas the denominator is specified by word numerals.[78]

[76] These are again estimates for large n. As mentioned in Section 4, exact expressions for the first two summations, $V_n^{(1)}$ and $V_n^{(2)}$, are given in *Āryabhaṭīya, Gaṇitapāda* 21; and the exact expression for the k-th order repeated summation $V_n^{(k)}$ has been given (under the name *vāra-saṅkalita*), by Nārāyaṇa Paṇḍita (c. 1350) in his *Gaṇitakaumudī*, 3.19. This exact expression for $V_n^{(k)}$ is also noted in section 7.5.3 of *Yuktibhāṣā*.

[77] *Āryabhaṭīya-bhāṣya* on *Āryabhaṭīya*, cited above (fn. 53), comm. on *Gaṇitapāda* 10, p. 42; *Kriyākramakarī* on *Līlāvatī*, cited above (fn. 14), comm. on verse 199, p. 377.

[78] In the *bhūta-saṅkhyā* system, *vibudha* =33, *netra* =2, *gaja* =8, *ahi* =8, *hutāśana* =3, *triguṇa* =3, *veda* =4, *bha* =27, *vāraṇa* =8, *bāhu* =2. In word numerals, *nikharva* represents 10^{11}. Hence, *nava-nikharva* =9×10^{11}.

13.1 Infinite series for π

The infinite series for π attributed to Mādhava is cited by Śaṅkara Vāriyar in his commentaries *Kriyākramakarī* and *Yukti-dīpikā*. Mādhava's verse quoted runs as follows:[79]

> व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
> त्रिशरादिविषमसङ्ख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात्॥

> The diameter multiplied by four and divided by unity [is found and saved]. Again the products of the diameter and four are divided by the odd numbers like three, five, etc., and the results are subtracted and added in order [to the earlier result saved].

The series given by the verse may be represented as

$$Paridhi = 4 \times Vy\bar{a}sa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right). \tag{112}$$

The words *paridhi* and *vyāsa*[80] in the above equation refer to the circumference and diameter respectively. Hence the equation may be rewritten as

$$\frac{\pi}{4} = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \right). \tag{113}$$

We shall now present the derivation of the above result as outlined in *Yuktibhāṣā* of Jyeṣṭhadeva and *Kriyākramakarī* of Śaṅkara Vāriyar. For this purpose, let us consider the quadrant OP_0P_nS of the square circumscribing the given circle (see Figure 8). Divide the side P_0P_n into n equal parts (n very large). P_0P_i's are the *bhujās* and OP_i's are the *karṇas* denoted by k_i. The points of intersection of these *karṇas* and the circle are marked as A_is.

The *bhujās* P_0P_i, the *karṇas* k_i and the east-west line OP_0 form right-angled triangles whose hypotenuses are given by

$$k_i^2 = r^2 + \left(\frac{ir}{n}\right)^2, \tag{114}$$

where r is the radius of the circle.

The feet of perpendiculars from the points A_{i-1} and P_{i-1} along the i^{th} *karṇa* are denoted by B_i and C_i. The triangles $OP_{i-1}C_i$ and $OA_{i-1}B_i$ are similar. Hence,

$$\frac{A_{i-1}B_i}{OA_{i-1}} = \frac{P_{i-1}C_i}{OP_{i-1}}. \tag{115}$$

Similarly triangles $P_{i-1}C_iP_i$ and P_0OP_i are similar. Hence,

$$\frac{P_{i-1}C_i}{P_{i-1}P_i} = \frac{OP_0}{OP_i}. \tag{116}$$

[79] op. cit., p. 379.

[80] Nīlakaṇṭha, in his *Āryabhaṭīya-bhāṣya*, presents the etymological derivation of the word *vyāsa* as 'the one which splits the circle into two halves': व्यासेन हि वृत्तं व्यस्यते। (*Āryabhaṭīya-bhāṣya*, cited above (fn. 53), comm. on *Gaṇitapāda* 11, p. 43).

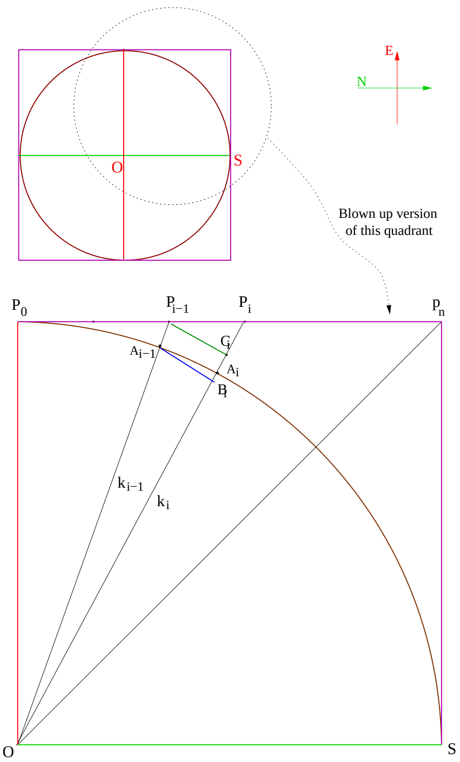

Figure 8: Geometrical construction used in the proof of the infinite series for π.

From these two relations we have,

$$\begin{aligned} A_{i-1}B_i &= \frac{OA_{i-1}.OP_0.P_{i-1}P_i}{OP_{i-1}.OP_i} \\ &= P_{i-1}P_i \times \frac{OA_{i-1}}{OP_{i-1}} \times \frac{OP_0}{OP_i} \\ &= \left(\frac{r}{n}\right) \times \frac{r}{k_{i-1}} \times \frac{r}{k_i} \\ &= \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right). \end{aligned} \tag{117}$$

It is then noted that when n is large, the Rsines $A_{i-1}B_i$ can be taken as the arc-bits themselves.

परिधिखण्डस्यार्धज्या $\rightarrow$ परिध्यंश

$$\text{i.e.,} \qquad A_{i-1}B_i \rightarrow \widehat{A_{i-1}A_i}\,.$$

Thus, $\frac{1}{8}$th of the circumference of the circle can be written as sum of the contributions given by (117). That is

$$\frac{C}{8} \approx \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0k_1}\right) + \left(\frac{r^2}{k_1k_2}\right) + \left(\frac{r^2}{k_2k_3}\right) + \cdots + \left(\frac{r^2}{k_{n-1}k_n}\right)\right]. \tag{118}$$

Though this is the expression that actually needs to be evaluated, the text mentions that there may not be much difference in approximating it by either of the following expressions:

$$\left[\frac{C}{8}\right]_{left} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0^2}\right)+\left(\frac{r^2}{k_1^2}\right)+\left(\frac{r^2}{k_2^2}\right)+\cdots+\left(\frac{r^2}{k_{n-1}^2}\right)\right] \tag{119}$$

or

$$\left[\frac{C}{8}\right]_{right} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_1^2}\right)+\left(\frac{r^2}{k_2^2}\right)+\left(\frac{r^2}{k_3^2}\right)+\cdots+\left(\frac{r^2}{k_n^2}\right)\right]. \tag{120}$$

It can be easily seen that

$$\left[\frac{C}{8}\right]_{right} < \frac{C}{8} < \left[\frac{C}{8}\right]_{left}. \tag{121}$$

In other words, though the actual value of the circumference lies inbetween the values given by (120) and (119) what is being said is that there will not be much difference if we divide by the square of either of the *karṇa*-s rather than by the product of two successive ones. Actually, the difference between (120) and (119) is given by

$$\begin{aligned}\left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0^2}\right)-\left(\frac{r^2}{k_n^2}\right)\right] &= \left(\frac{r}{n}\right)\left[1-\left(\frac{1}{2}\right)\right] \qquad (\text{since } k_0^2, k_n^2 = r^2, 2r^2)\\ &= \left(\frac{r}{n}\right)\left(\frac{1}{2}\right) \qquad (122)\end{aligned}$$

Evidently this difference approaches zero as n becomes very large, as noted in both the texts *Yuktibhāṣā* and *Kriyākramakarī*.

The terms in (120) are evaluated using the *śodhya-phala* technique (binomial series, discussed earlier in Section 11) and each one of them may be re-written in the form[81]

$$\frac{r}{n}\left(\frac{r^2}{k_i^2}\right) = \frac{r}{n} - \frac{r}{n}\left(\frac{k_i^2-r^2}{r^2}\right) + \frac{r}{n}\left(\frac{k_i^2-r^2}{r^2}\right)^2 - \ldots \tag{123}$$

Using (114) and (123) in (120), we obtain:

$$\begin{aligned}\frac{C}{8} &= \sum_{i=1}^{n}\frac{r}{n}\left(\frac{r^2}{k_i^2}\right)\\ &= \sum_{i=1}^{n}\left(\frac{r}{n}\right)\left(\frac{r^2}{r^2+\left(\frac{ir}{n}\right)^2}\right) \qquad (124)\\ &= \sum_{i=1}^{n}\left[\frac{r}{n} - \frac{r}{n}\left(\frac{\left(\frac{ir}{n}\right)^2}{r^2}\right) + \frac{r}{n}\left(\frac{\left(\frac{ir}{n}\right)^2}{r^2}\right)^2 - \ldots\right] \qquad (125)\\ &= \left(\frac{r}{n}\right)[1+1+\ldots+1]\\ &\quad -\left(\frac{r}{n}\right)\left(\frac{1}{r^2}\right)\left[\left(\frac{r}{n}\right)^2+\left(\frac{2r}{n}\right)^2+\ldots+\left(\frac{nr}{n}\right)^2\right]\end{aligned}$$

[81]It may be noted that this series is convergent since $k_i^2 = r^2 + \left(\frac{ir}{n}\right)^2$ and $0 \leq (k_i^2 - r^2) < r^2$ for $i < n$.

$$+\left(\frac{r}{n}\right)\left(\frac{1}{r^4}\right)\left[\left(\frac{r}{n}\right)^4+\left(\frac{2r}{n}\right)^4+\ldots+\left(\frac{nr}{n}\right)^4\right]$$
$$-\left(\frac{r}{n}\right)\left(\frac{1}{r^6}\right)\left[\left(\frac{r}{n}\right)^6+\left(\frac{2r}{n}\right)^6+\ldots+\left(\frac{nr}{n}\right)^6\right]$$
$$+\ldots\ . \tag{126}$$

Each of the terms in (126) is a sum of results (*phala-yoga*) which we need to estimate when n is very large, and we have a series of them (*phala-paramparā*) which are alternatively positive and negative. Clearly the first term is just the sum of the *bhujā-khaṇḍas*.

The *bhujās* themselves are given by the integral multiples of *bhujā-khaṇḍa*, namely, $\frac{r}{n}, \frac{2r}{n}, \ldots \frac{nr}{n}$. In the series expression for the circumference given above, we thus have the *saṅkalitas* or summations of even powers of the *bhujās*, such as the *bhujā-varga-saṅkalita*, $\left(\frac{r}{n}\right)^2+\left(\frac{2r}{n}\right)^2+....+\left(\frac{nr}{n}\right)^2$, *bhujā-varga-varga-saṅkalita*, $\left(\frac{r}{n}\right)^4+\left(\frac{2r}{n}\right)^4+ ..... +\left(\frac{nr}{n}\right)^4$, and so on.

If we take out the powers of *bhujā-khaṇḍa* $\frac{r}{n}$, the summations involved are that of even powers of the natural numbers, namely *edādyekottara-varga-saṅkalita*, $1^2+2^2+...+n^2$, *edādyekottara-varga-varga-saṅkalita*, $1^4+2^4+...+n^4$, and so on.

Now, recalling the estimates that were obtained earlier for these *saṅkalita*-s, when n is large,

$$\sum_{i=1}^{n} i^k \approx \frac{n^{k+1}}{k+1}, \tag{127}$$

we arrive at the result[82]

$$\frac{C}{8} = r\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right), \tag{128}$$

which is same as (112).

14 Derivation of end-correction terms (*Antya-saṃskāra*)

It is well known that the series given by (112) for $\frac{\pi}{4}$ is an extremely slowly converging series. It is so slow that even for obtaining the value of π correct to 2 decimal places one has to find the sum of hundreds of terms and for getting it correct to 4-5 decimal places we need to consider millions of terms. Mādhava seems to have found an ingenious way to circumvent this problem. The technique employed by Mādhava is known as *antya-saṃskāra*. The nomenclature stems from the fact that a correction (*saṃskara*) is applied towards the end (*anta*) of the series, when it is terminated after considering only a certain number of terms from the beginning.

[82] In modern terminology, the above derivation amounts to the evaluation of the following integral

$$\frac{C}{8} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{r}{n}\right)\left(\frac{r^2}{r^2+\left(\frac{ir}{n}\right)^2}\right) = r\int_0^1 \frac{dx}{1+x^2}.$$

14.1 The criterion for *antya-saṃskāra* to yield accurate result

The discussion on *antya-saṃskāra* in both *Yuktibhāṣā* and *Kriyākramakarī* commences with the question:

> How is it that one obtains the value of the circumference more accurately by doing *antya-saṃskara*, instead of repeatedly dividing by odd numbers? [83]

The argument adduced in favor of terminating the series at any desired term, still ensuring the accuracy, is as follows. Let the series for $\frac{\pi}{4}$ be written as

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots + (-1)^{\frac{p-3}{2}} \frac{1}{p-2} + (-1)^{\frac{p-1}{2}} \frac{1}{a_{p-2}}, \tag{129}$$

where $\frac{1}{a_{p-2}}$ is the correction term applied after odd denominator $p-2$. On the other hand, if the correction term $\frac{l}{a_p}$, is applied after the odd denominator p, then

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots + (-1)^{\frac{p-3}{2}} \frac{1}{p-2} + (-1)^{\frac{p-1}{2}} \frac{1}{p} + (-1)^{\frac{p+1}{2}} \frac{1}{a_p}. \tag{130}$$

If the correction terms indeed lead to the exact result, then both the series (129) and (130) should yield the same result. That is,

$$\frac{1}{a_{p-2}} = \frac{1}{p} - \frac{1}{a_p} \qquad \text{or} \qquad \frac{1}{a_{p-2}} + \frac{1}{a_p} = \frac{1}{p}, \tag{131}$$

is the criterion that must be satisfied for the end-correction (*antya-saṃskāra*) to lead to the exact result.

14.2 Successive approximations to get more accurate correction-terms

The criterion given by (131) is trivially satisfied when we choose $a_{p-2} = a_p = 2p$. However, this value $2p$ cannot be assigned to both the correction-divisors[84] a_{p-2} and a_p because both the corrections should follow the same rule. That is,

$$\begin{aligned} a_{p-2} &= 2p, \quad \Rightarrow \quad a_p = 2(p+2) \\ \text{or,} \qquad a_p &= 2p, \quad \Rightarrow \quad a_{p-2} = 2(p-2). \end{aligned}$$

We can, however, have both a_{p-2} and a_p close to $2p$ by taking $a_{p-2} = 2p-2$ and $a_p = 2p+2$, as there will always persist this much difference between $p-2$ and p when they are doubled. Hence, the first (order) estimate of the correction divisor is given as, “double the even number above the last odd-number divisor p”,

$$a_p = 2(p+1). \tag{132}$$

[83] कथं पुनरत्र मुहुर्विषमसङ्ख्याहरणेन लभ्यस्य परिधेः आसन्नत्वम् अन्त्यसंस्कारेण आपाद्यते । उच्यते ।… (*Kriyākramakarī* on *Līlāvatī*, cited above (fn. 14), comm. on verse 199, p. 386.)

[84] By the term correction-divisor (*saṃskāra-hāraka*) is meant the divisor of the correction term.

But, it can be seen right away that, with this value of the correction divisor, the condition for accuracy (131), stated above, is not exactly satisfied. Therefore a measure of inaccuracy ($sthaulya$) $E(p)$ is introduced

$$E(p) = \left[\frac{1}{a_{p-2}} + \frac{1}{a_p}\right] - \frac{1}{p} \ . \tag{133}$$

Now, since the error cannot be eliminated, the objective is to find the correction denominators a_p such that the inaccuracy $E(p)$ is minimised. When we set $a_p = 2(p+1)$, the inaccuracy will be

$$\begin{aligned} E(p) &= \left[\frac{1}{(2p-2)} + \frac{1}{(2p+2)}\right] - \frac{1}{p} \\ &= \frac{1}{(p^3 - p)} \ . \end{aligned} \tag{134}$$

This estimate of the inaccuracy, E_p being positive, shows that the correction has been over done and hence there has to be a reduction in the correction. This means that the correction-divisor has to be increased. If we take $a_p = 2p+3$, thereby leading to $a_{p-2} = 2p-1$, we have

$$\begin{aligned} E(p) &= \left[\frac{1}{(2p-1)} + \frac{1}{(2p+3)}\right] - \frac{1}{p} \\ &= \frac{(-2p+3)}{(4p^3 + 4p^2 - 3p)} \ . \end{aligned} \tag{135}$$

Now, the inaccuracy happens to be negative. But, more importantly, it has a term proportional to p in the numerator. Hence, for large p, $E(p)$ given by (135) varies inversely as p^2, while for the divisor given by (132), $E(p)$ as given by (134) varied inversely as p^3.[85]

From (134) and (135) it is obvious that, if we want to reduce the inaccuracy and thereby obtain a better correction, then a number less than 1 has to be added to the correction-divisor (132) given above. If we try adding $r\bar{u}pa$ (unity) divided by the correction divisor itself, i.e., if we set $a_p = 2p + 2 + \frac{1}{(2p+2)}$, the contributions from the correction-divisors get multiplied essentially by $\left(\frac{1}{2p}\right)$. Hence, to get rid of the higher order contributions, we need an extra factor of 4, which will be achieved if we take the correction divisor to be

$$a_p = (2p+2) + \frac{4}{(2p+2)} = \frac{(2p+2)^2 + 4}{(2p+2)} \ . \tag{136}$$

Then, correspondingly, we have

$$a_{p-2} = (2p-2) + \frac{4}{(2p-2)} = \frac{(2p-2)^2 + 4}{(2p-2)} \ . \tag{137}$$

We can then calculate the inaccuracy to be

$$E(p) = \left[\frac{1}{(2p-2) + \dfrac{4}{2p-2}} + \frac{1}{(2p+2) + \dfrac{4}{2p+2}}\right] - \left(\frac{1}{p}\right)$$

[85]It may be noted that among all possible correction divisors of the type $a_p = 2p + m$, where m is an integer, the choice of $m = 2$ is optimal, as in all other cases there will arise a term proportional to p in the numerator of the inaccuracy $E(p)$.

$$
\begin{aligned}
&= \left[\frac{(4p^3)}{(4p^4+16)}\right] - \frac{(16p^4+64)}{4p(4p^4+16)} \\
&= \frac{-4}{(p^5+4p)} \,. \qquad (138)
\end{aligned}
$$

Clearly, the *sthaulya* with this (second order) correction divisor has improved considerably, in that it is now proportional to the inverse fifth power of the odd number.[86]

At this stage, we may display the result obtained for the circumference with the correction term as follows. If only the first order correction (132) is employed, we have

$$
C = 4d\left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}}\frac{1}{p} + (-1)^{\frac{(p+1)}{2}}\frac{1}{(2p+2)}\right]. \qquad (139)
$$

If the second order correction (136) is taken into account, we have

$$
\begin{aligned}
C &= 4d\left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}}\frac{1}{p} + (-1)^{\frac{(p+1)}{2}}\frac{1}{(2p+2) + \frac{4}{(2p+2)}}\right] \\
&= 4d\left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}}\frac{1}{p} + (-1)^{\frac{(p+1)}{2}}\frac{\frac{(p+1)}{2}}{(p+1)^2+1}\right]. \qquad (140)
\end{aligned}
$$

The verse due to Mādhava that we cited earlier as defining the infinite series for $\frac{\pi}{4}$ is, in fact, the first of a group of four verses that present the series along with the above end-correction.[87]

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
त्रिशरादिविषमसङ्ख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात्॥१॥

यत्सङ्ख्ययाऽत्र हरणे कृते निवृत्ता हृतिस्तु जामितया।
तस्या ऊर्ध्वगताया समसङ्ख्या तद्दलं गुणोऽन्ते स्यात्॥२॥

तद्वर्गो रूपयुतो हारो व्यासाब्धिघाततः प्राग्वत्।
ताभ्यामाप्तं स्वमृणे कृते धने क्षेप एव करणीयः॥३॥
लब्धः परिधिः सूक्ष्मो बहुकृत्वो हरणतोऽतिसूक्ष्मः स्यात्॥४॥

The diameter multiplied by four and divided by unity. Again the products of the diameter and four are divided by the odd numbers like three, five, etc., and the results are subtracted and added in order.

Take half of the succeeding even number as the multipler at whichever [odd] number the division process is stopped, because of boredom. The

[86]It may be noted that if we take any other correction-divisor $a_p = 2p + 2 + \frac{m}{(2p+2)}$, where m is an integer, we will end up having a contribution proportional to p^2 in the numerator of the inaccuracy $E(p)$, unless $m = 4$. Thus the above form (136) is the optimal second order choice for the correction-divisor.

[87]*Kriyākramakarī* on *līlāvatī*, cited above (fn. 14), comm. on verse 199, p. 379.

> square of that [even number] added to unity is the divisor. Their ratio has to be multiplied by the product of the diameter and four as earlier.
>
> The result obtained has to be added if the earlier term [in the series] has been subtracted and subtracted if the earlier term has been added. The resulting circumference is very accurate; in fact more accurate than the one which may be obtained by continuing the division process [with large number of terms in the series].

Continuing this process further, *Yuktibhāṣā* presents the next order correction-term which is said to be even more accurate:[88]

> अन्ते समसङ्ख्यादलवर्गः सैको गुणः स एव पुनः ॥
> युगगुणितो रूपयुतः समसङ्ख्यादलहतो भवेद् हारः ।
>
> At the end, [i.e., after terminating the series at some point, apply the correction term with] the multiplier being square of half of the [next] even number plus 1, and the divisor being four times the same multiplier with 1 added and multiplied by half the even number.

In other words,[89]

$$
\begin{aligned}
\frac{1}{a_p} &= \frac{\left(\dfrac{p+1}{2}\right)^2 + 1}{[(p+1)^2 + 4 + 1]\left(\dfrac{p+1}{2}\right)} \\
&= \cfrac{1}{(2p+2) + \cfrac{4}{2p+2+\cfrac{16}{2p+2}}} \, . \qquad (141)
\end{aligned}
$$

[88] *Gaṇita-yukti-bhāṣā*, cited above, p. 82; Also cited in *Yukti-dīpikā* on *Tantrasaṅgraha*, cited above (fn. 49), comm. on verse 2.1, p. 103.

[89] The inaccuracy or *sthaulya* associated with this correction can be calculated to be

$$
E(p) = \frac{2304}{(64p^7 + 448p^5 + 1792p^3 - 2304p)} \, .
$$

The inaccuracy now is proportional to the inverse seventh power of the odd-number. Again it can be shown that the number 16 in (141) is optimally chosen, in that any other choice would introduce a term proportional to p^2 in the numerator of $E(p)$, given above.

In fact, it has been noted by C. T. Rajagopal and M. S. Rangachari that D. T. Whiteside has shown (personal communication of D. T. Whiteside cited in C. T. Rajagopal and M. S. Rangachari, 'On an untapped source of medieval Kerala mathematics', Arch. for Hist. Sc. 35(2), 89–102, 1978), that the end correction-term can be exactly represented by the following continued fraction

$$
\frac{1}{a_p} = \cfrac{1}{(2p+2) + \cfrac{2^2}{(2p+2) + \cfrac{4^2}{(2p+2) + \cfrac{6^2}{(2p+2) + \ldots}}}} \, .
$$

Hence, a much better approximation for $\frac{\pi}{4}$ is:[90]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{p} - \frac{\left(\frac{p+1}{2}\right)^2 + 1}{[(p+1)^2 + 4 + 1]\left(\frac{p+1}{2}\right)}. \tag{142}$$

15 Transforming the Mādhava series for better convergence

After the estimation of end-correction terms, *Yuktibhāṣā* goes on to outline a method of transforming the Mādhava series (by making use of the above end-correction terms) to obtain new series that have much better convergence properties. We now reproduce the following from the English translation of *Yuktibhāṣā*:[91]

> Therefore, the circumference (of a circle) can be derived in taking into consideration what has been stated above. A method for that is stated in the verse
>
> > समपञ्चाहतयो या रूपाद्ययुजां चतुर्घ्नमूलयुताः ताभिः।
> > षोडशगुणितात् व्यासाद् पृथगाहतेषु विषमयुतेः।
> > समफलयुतिमपहाय स्यादिष्टव्याससंभवः परिधिः ॥ (I)
> >
> > The fifth powers of the odd numbers (1, 3, 5 etc.) are increased by 4 times themselves. The diameter is multiplied by 16 and it is successively divided by the (series of) numbers obtained (as above). The odd (first, third etc.) quotients obtained are added and are subtracted from the sum of the even (the second, fourth etc.) quotients. The result is the circumference corresponding to the given diameter.
>
> Herein above is stated a method for deriving the circumference. If the correction term is applied to an approximate circumference and the amount of inaccuracy (*sthaulya*) is found, and if it is additive, then the result is higher. Then it will become more accurate when the correction term obtained from the next higher odd number is subtracted. Since it happens that (an approximate circumference) becomes more and more accurate by making corrections in succeeding terms, if the corrections are applied right from the beginning itself, then the circumference will come out accurate. This is the rationale for this (above-stated result).
>
> When it is presumed that the correction-divisor is just double the odd number, the following is a method to obtain the (accurate) circumference by a correction for the corresponding inaccuracy (*sthaulyāṃśa-parihāra*), which is given by the verse:

[90] It may be noted that this correction term leads to a value of π, which is accurate up to 11 decimal places, when we merely evaluate terms up to $n = 50$ in the series (142). Incidentally the value of π, given in the rule *vibudhanetra...*, attributed to Mādhava that was cited in the beginning of Section 13, is also accurate up to 11 decimal places.

[91] *Gaṇita-yukti-bhāṣā*, cited above, Section 6.9, pp. 80–82, 205–07, 402–04.

व्यासाद् वारिधिनिहतात् पृथगाप्तं त्र्याद्ययुग्विमूलघनैः।
त्रिघ्नव्यासे स्वमृणं क्रमशः कृत्वा परिधिरानेयः॥ (II)

The diameter is multiplied by 4 and is divided, successively, by the cubes of the odd numbers beginning from 3, which are diminished by these numbers themselves. The diameter is now multiplied by three, and the quotients obtained above, are added to or subtracted from, alternatively. The circumference is to be obtained thus.

If, however, it is taken that half the result (of dividing) by the last even number is taken as the correction, there is a method to derive the circumference by that way also, as given by the verse

द्व्यादियुजां वा कृतयोः व्येका हाराद् द्विनिघ्नविष्कम्भे।
धनम् ऋणमन्तेऽन्त्योर्ध्वगतौजकृतिर्द्विसहिता हरस्यार्धम्॥ (III)

The squares of even numbers commencing from 2, diminished by one, are the divisors for four times the diameter. (Make the several divisions). The quotients got by (the division) are alternately added to or subtracted from twice the diameter. In the end, divide four times the diameter by twice the result of squaring the odd number following the last even number to which is added 2.

The method of *sthaulya-parihāra*, outlined above, essentially involves incorporating the correction terms into the series from the beginning itself. Let us recall that inaccuracy or *sthaulya* at each stage is given by

$$E(p) = \frac{1}{a_{p-2}} + \frac{1}{a_p} - \left(\frac{1}{p}\right). \tag{143}$$

The series for the circumference (112) can be expressed in terms of these *sthaulyas* as follows:

$$\begin{aligned} C &= 4d\left[\left(1-\frac{1}{a_1}\right)+\left(\frac{1}{a_1}+\frac{1}{a_3}-\frac{1}{3}\right)-\left(\frac{1}{a_3}+\frac{1}{a_5}-\frac{1}{5}\right)-\ldots\right] \\ &= 4d\left[\left(1-\frac{1}{a_1}\right)+E(3)-E(5)+E(7)-\ldots\right]. \end{aligned} \tag{144}$$

Now, by choosing different correction-divisors a_p in (144), we get several transformed series which have better convergence properties. If we consider the correction-divisor (136), then using the expression (138) for the *sthaulyas*, we get

$$\begin{aligned} C &= 4d\left(1-\frac{1}{5}\right)-16d\left[\frac{1}{(3^5+4.3)}-\frac{1}{(5^5+4.5)}+\frac{1}{(7^5+4.7)}-\ldots\right] \\ &= 16d\left[\frac{1}{(1^5+4.1)}-\frac{1}{(3^5+4.3)}+\frac{1}{(5^5+4.5)}-\ldots\right]. \end{aligned} \tag{145}$$

The above series is given in the verse *samapañcāhatayoḥ* …(I). Note that each term in the above series involves the fifth power of the odd number in the denominator, unlike the original series which only involved the first power of the odd number. Clearly, this transformed series gives more accurate results with fewer terms.

If we had used only the lowest order correction (132) and the associated $sthaulya$ (134), instead of the correction employed above, then the transformed series is the one given in the verse $vy\bar{a}s\bar{a}d\ v\bar{a}ridhinihat\bar{a}t\ldots$(II)

$$C = 4d\left[\frac{3}{4} + \frac{1}{(3^3-3)} - \frac{1}{(5^3-5)} + \frac{1}{(7^3-7)} - \ldots\right]. \tag{146}$$

Note that the denominators in the above transformed series are proportional to the third power of the odd number.

Even if we take non-optimal correction-divisors, we often end-up obtaining interesting series. For instance, if we take a non-optimal correction-divisor, say of the form $a_p = 2p$, then the $sthaulya$ is given by

$$\begin{aligned} E(p) &= \frac{1}{(2p-4)} + \frac{1}{2p} - \frac{1}{p} \\ &= \frac{1}{(p^2-2p)} \\ &= \frac{1}{(p-1)^2-1}. \end{aligned} \tag{147}$$

Then, the transformed series will be the one given in the verse $dvy\bar{a}diyuj\bar{a}\d{m}\ v\bar{a}\ k\d{r}tayo\ldots$(III)[92]

$$C = 4d\left[\frac{1}{2} + \frac{1}{(2^2-1)} - \frac{1}{(4^2-1)} + \frac{1}{(6^2-1)} + \ldots\right]. \tag{148}$$

16 Derivation of the Mādhava series for Rsine and Rversine

16.1 First and second order differences of Rsines

We shall now outline the derivation of Mādhava series for Rsine (*bhujā-jyā*) and Rversine (*śara*), as given in *Yuktibhāṣā*.[93] *Yuktibhāṣā* begins with a discussion of the first and second order Rsine-differences and derives an exact form of the result of Āryabhaṭa that the second-order Rsine-differences are proportional to the Rsines themselves. We had briefly indicated this proof in Section 5.3.

Here we are interested in obtaining the Mādhava series for the *jyā* and *śara* of an arc of length s indicated by EC in Figure 9. This arc is divided into n equal arc bits, where n is large. If the arc length $s = R\theta$, then the j-th *piṇḍa-jyā*, B_j is given by[94]

$$B_j = jy\bar{a}\left(\frac{js}{n}\right) = R\sin\left(\frac{j\theta}{n}\right). \tag{149}$$

[92] The verse III in fact presents the series (148) along with an end correction-term of the form $(-1)^p \frac{4d}{2(p+1)^2+2}$.

[93] *Yuktibhāṣā*, cited earlier, Vol. I Section 16.5, pp. 94–103, 221–233, 417–427.

[94] Figure 9 is essentialy the same as Figure 3 considered in section 5 except that the *piṇḍajyās* B_j are Rsines assotiated with multiples of the arc-bit $\frac{s}{n}$ into which the arc $EC = s$ is divided. In Figure 3, the B_j's are the tabular Rsines associated with multiplies of $225'$.

The corresponding *koṭi-jyā* K_j, and the *śara* S_j, are given by

$$K_j = koṭi\left(\frac{js}{n}\right) = R\cos\left(\frac{j\theta}{n}\right), \tag{150}$$

$$S_j = śara\left(\frac{js}{n}\right) = R\left[1 - \cos\left(\frac{j\theta}{n}\right)\right]. \tag{151}$$

Now, C_jC_{j+1} represents the $(j+1)$-th arc bit. Then, for the arc $EC_j = \frac{js}{n}$, its *piṇḍa-jyā* is $B_j = C_jP_j$, and the corresponding *koṭi-jyā* and *śara* are $K_j = C_jT_j,\ S_j = EP_j$. Similarly we have

$$B_{j+1} = C_{j+1}P_{j+1},\ K_{j+1} = C_{j+1}T_{j+1} \text{ and } S_{j+1} = EP_{j+1}. \tag{152}$$

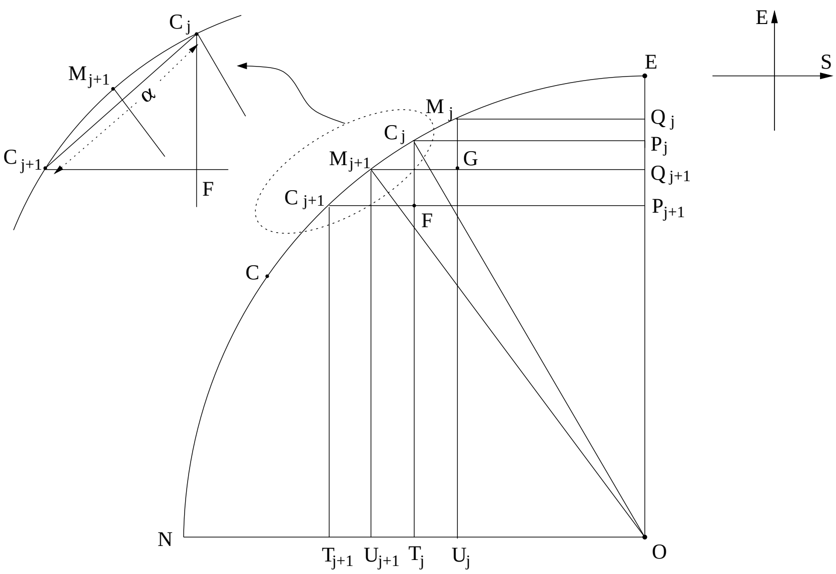

Figure 9: Computation of *Jyā* and *Śara* by *Saṅkalitas*.

Let M_{j+1} be the mid-point of the arc-bit C_jC_{j+1} and similarly M_j the mid-point of the previous (j-th) arc-bit. We shall denote the *piṇḍa-jyā* of the arc EM_{j+1} as $B_{j+\frac{1}{2}}$ and clearly

$$B_{j+\frac{1}{2}} = M_{j+1}Q_{j+1}\ .$$

The corresponding *koṭi-jyā* and *śara* are

$$K_{j+\frac{1}{2}} = M_{j+1}U_{j+1} \quad \text{and} \quad S_{j+\frac{1}{2}} = EQ_{j+1}\ .$$

Similarly,

$$B_{j-\frac{1}{2}} = M_jQ_j,\ K_{j-\frac{1}{2}} = M_jU_j \text{ and } S_{j-\frac{1}{2}} = EQ_j\ . \tag{153}$$

Let α be the chord corresponding to the equal arc-bits $\frac{s}{n}$ as indicated in Figure 9. That is, $C_jC_{j+1} = M_jM_{j+1} = \alpha$.

Let F be the intersection of C_jT_j and $C_{j+1}P_{j+1}$, and G of M_jU_j and $M_{j+1}Q_{j+1}$. The triangles $C_{j+1}FC_j$ and $OQ_{j+1}M_{j+1}$ are similar, as their sides are mutually perpendicular. Thus we have

$$\frac{C_{j+1}C_j}{OM_{j+1}} = \frac{C_{j+1}F}{OQ_{j+1}} = \frac{FC_j}{Q_{j+1}M_{j+1}} \ . \tag{154}$$

Hence we obtain

$$B_{j+1} - B_j = \left(\frac{\alpha}{R}\right) K_{j+\frac{1}{2}}, \tag{155}$$

$$K_j - K_{j+1} = S_{j+1} - S_j = \left(\frac{\alpha}{R}\right) B_{j+\frac{1}{2}}. \tag{156}$$

Similarly, the triangles $M_{j+1}GM_j$ and OP_jC_j are similar and we get

$$\frac{M_{j+1}M_j}{OC_j} = \frac{M_{j+1}G}{OP_j} = \frac{GM_j}{P_jC_j}. \tag{157}$$

Thus we obtain

$$B_{j+\frac{1}{2}} - B_{j-\frac{1}{2}} = \left(\frac{\alpha}{R}\right) K_j, \tag{158}$$

$$K_{j-\frac{1}{2}} - K_{j+\frac{1}{2}} = S_{j+\frac{1}{2}} - S_{j-\frac{1}{2}} = \left(\frac{\alpha}{R}\right) B_j \ . \tag{159}$$

We define the Rsine-differences (*khaṇḍa-jyā*) Δ_j by

$$\Delta_j = B_j - B_{j-1} \ , \tag{160}$$

with the convention that $\Delta_1 = B_1$. From (155), we have

$$\Delta_j = \left(\frac{\alpha}{R}\right) K_{j-\frac{1}{2}} \ . \tag{161}$$

From (159) and (161), we also get the second order Rsine-differences (the differences of the Rsine-differences called *khaṇḍa-jyāntara*):

$$\begin{aligned} \Delta_j - \Delta_{j+1} &= (B_j - B_{j-1}) - (B_{j+1} - B_j) \\ &= \left(\frac{\alpha}{R}\right)\left(K_{j-\frac{1}{2}} - K_{j+\frac{1}{2}}\right) \\ &= \left(\frac{\alpha}{R}\right)\left(S_{j+\frac{1}{2}} - S_{j-\frac{1}{2}}\right) \\ &= \left(\frac{\alpha}{R}\right)^2 B_j \ . \end{aligned} \tag{162}$$

Now, if the sum of the second-order Rsine-differences, is subtracted from the first Rsine-difference, then we get any desired Rsine-difference. That is

$$\Delta_1 - [(\Delta_1 - \Delta_2) + (\Delta_2 - \Delta_3) + \ldots + (\Delta_{j-1} - \Delta_j)] = \Delta_j \ . \tag{163}$$

From (162) and (163) we conclude that

$$\Delta_1 - \left(\frac{\alpha}{R}\right)^2 (B_1 + B_2 + \ldots + B_{j-1}) = \Delta_j \ . \tag{164}$$

16.2 Rsines and Rversines from *Jyā-saṅkalita*

We can sum up the Rversine-differences (159), to obtain the *śara*, Rversine, at the midpoint of the last arc-bit as follows:

$$\begin{aligned} S_{n-\frac{1}{2}} - S_{\frac{1}{2}} &= \left(S_{n-\frac{1}{2}} - S_{n-\frac{3}{2}}\right) + \ldots\ldots \left(S_{\frac{3}{2}} - S_{\frac{1}{2}}\right) \\ &= \left(\frac{\alpha}{R}\right)(B_{n-1} + B_{n-2} + \ldots + B_1). \end{aligned} \tag{165}$$

Using (162), the right hand side of (165) can also be expressed as a summation of the second order differences. From (164) and (165) it follows that the Rversine at the midpoint of the last arc-bit is also given by

$$\left(\frac{\alpha}{R}\right)\left(S_{n-\frac{1}{2}} - S_{\frac{1}{2}}\right) = (\Delta_1 - \Delta_n). \tag{166}$$

Now, since the first Rsine-difference $\Delta_1 = B_1$, any desired Rsine can be obtained by adding the Rsine-differences; these Rsine-differences have been obtained in (164). Now, by making use of (164), the last *piṇḍa-jyā* can be expressed as follows:

$$\begin{aligned} B_n &= \Delta_n + \Delta_{n-1} + \ldots + \Delta_1 \\ &= n\Delta_1 - \left(\frac{\alpha}{R}\right)^2 [(B_1 + B_2 \ldots + B_{n-1}) + (B_1 + B_2 \ldots + B_{n-2}) + \ldots + B_1] \\ &= nB_1 - \left(\frac{\alpha}{R}\right)^2 [B_{n-1} + 2B_{n-2} + \ldots + (n-1)B_1]. \end{aligned} \tag{167}$$

The results (158) – (167), obtained so far, involve no approximations. It is now shown how better and better approximations to the Rsine and Rversine can be obtained by taking n to be very large or, equivalently, the arc-bit $\frac{s}{n}$ to be very small. Then, we can approximate the full-chord and the Rsine of the arc-bit by the length of the arc-bit $\frac{s}{n}$ itself. Also, as a first approximation, we can approximate the *piṇḍa-jyās* B_j in the equations (164), (165) or (167) by the corresponding arcs themselves. That is

$$B_j \approx \frac{js}{n}. \tag{168}$$

The result for the Rsine obtained this way is again used to obtain a better approximation for the *piṇḍa-jyās* B_j which is again substituted back into the equations (165) and (167) and thus by a process of iteration successive better approximations are obtained for the Rsine and Rversine. Now, once we take $B_j \approx \frac{js}{n}$, we will be led to estimate the sums and repeated sums of natural numbers (*ekādyekottara-saṅkalita*), when the number of terms is very large.

16.3 Derivation of Mādhava series by iterative corrections to *Jyā* and *Śara*

As we noted earlier, these relations given by (165) and (167) are exact. But now we shall show how better and better approximations to the Rsine and Rversine of any desired arc can be obtained by taking n to be very large or, equivalently, taking the arc-bit $\frac{s}{n}$ to be very small. Then both the full-chord α, and the first Rsine B_1 (the

Rsine of the arc-bit), can be approximated by the arc-bit $\frac{s}{n}$ itself, and the Rversine $S_{n-\frac{1}{2}}$ can be taken as S_n and the Rversine $S_{\frac{1}{2}}$ may be treated as negligible. Thus the above relations (165), (167) become[95]

$$\begin{aligned} S = S_n &\approx \left(\frac{s}{nR}\right)(B_{n-1} + B_{n-2} + \ldots + B_1), &(169)\\ B = B_n &\approx s - \left(\frac{s}{nR}\right)^2 [(B_1 + B_2 + \ldots + B_{n-1}) \\ &\qquad + (B_1 + B_2 \ldots + B_{n-2}) + \ldots + B_1], &(170)\end{aligned}$$

where B and S are the Rsine and Rversine of the desired arc of length s and the results will be more accurate, larger the value of n.

Now, as a first approximation, we take each *piṇḍa-jyā* B_j in (169) and (170) to be equal to the corresponding arc itself, that is

$$B_j \approx \frac{js}{n} \,. \tag{171}$$

Then we obtain for the Rversine

$$\begin{aligned} S &\approx \left(\frac{s}{nR}\right)\left[(n-1)\left(\frac{s}{n}\right) + (n-2)\left(\frac{s}{n}\right) + \ldots\right] \\ &= \left(\frac{1}{R}\right)\left(\frac{s}{n}\right)^2 [(n-1) + (n-2) + \ldots]. &(172)\end{aligned}$$

For large n, we can use the estimate (89) for the sum of integers. Hence (172) reduces to

$$S \approx \left(\frac{1}{R}\right)\frac{s^2}{2} \,. \tag{173}$$

Equation (173) is the first *śara-saṃskāra*, correction to the Rversine. We now substitute our first approximation (171) to the *piṇḍa-jyās* B_j in (170), which gives the Rsine of the desired are as a second order repeated sum of the *piṇḍa-jyās* B_j. We then obtain

$$B \approx s - \left(\frac{1}{R}\right)^2 \left(\frac{s}{n}\right)^3 [(1+2+\ldots+(n-1)) + (1+2+\ldots(n-2)) + \ldots]. \tag{174}$$

The second term in (174) is a *dvitīya-saṅkalita*, the second order repeated sum, and using the estimate (108), we obtain

$$B \approx s - \left(\frac{1}{R}\right)^2 \frac{s^3}{1.2.3} \,. \tag{175}$$

Thus we see that the first correction obtained in (175) to the Rsine-arc-difference (*jyā-cāpāntara-saṃskāra*), is equal to the earlier correction to the Rversine (*śara-saṃskāra*) given in (173) multiplied by the arc and divided by the radius and 3.

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[95] As has been pointed out by one of the reveiwers, in the following derivation instead of using the relation (170), which involves repeated summation of *piṇḍajyās*, one could use the much simpler relation

$$B = B_n \approx s - \frac{s}{nR}(S_{n-1} + S_{n-2} + \ldots + S_1),$$

which essentially follows from (165) and (170). Then we can iterate between the above equation and (169) which involve considering only sums of powers of integers. *Yuktibhāṣā*, however, employes successive iteration between (169) and (170), which involves consideration of repeated sums of integers.

It is noted that the results (173) and (175) are only approximate (*prāyika*), since, instead of the *saṅkalita* of the *piṇḍa-jyās* in (169) and (170), we have only carried out *saṅkalita* of the arc-bits. Now that (175) gives a correction to the difference btween the Rsine and the arc (*jyā-cāpāntara-saṃskāra*), we can use that to correct the values of the *piṇḍa-jyās* and thus obtain the next corrections to the Rversine and Rsine.

Following (175), the *piṇḍa-jyās* may now be taken as

$$B_j \approx \frac{js}{n} - \left(\frac{1}{R}\right)^2 \left[\frac{\left(\frac{js}{n}\right)^3}{1.2.3}\right]. \qquad (176)$$

If we introduce (176) in (169), we obtain

$$\begin{aligned} S &\approx \left(\frac{1}{R}\right)\left(\frac{s}{n}\right)^2 [(n-1)+(n-2)+\ldots] \\ &- \left(\frac{s}{nR}\right)\left(\frac{1}{R}\right)^2\left(\frac{s}{n}\right)^3\left(\frac{1}{1.2.3}\right)[(n-1)^3+(n-2)^3+\ldots]. \end{aligned} \qquad (177)$$

The first term in (177) was already evaluated while deriving (173). The second term in (177) can either be estimated as a summation of cubes (*ghana-saṅkalita*), or as a *tṛtīya-saṅkalita*, third order (repeated) summation, because each individual term there has been obtained by doing a second-order (repeated) summation. Hence, recollecting our earlier estimate (110) for these *saṅkalitas*, we get

$$S \approx \left(\frac{1}{R}\right)\frac{s^2}{1.2} - \left(\frac{1}{R}\right)^3 \frac{s^4}{1.2.3.4}. \qquad (178)$$

Equation (178) gives a correction (*śara-saṃskāra*) to the earlier value (173) of the Rversine, which is nothing but the earlier correction to the Rsine-arc difference (*jyā-cāpāntara-saṃskāra*) given in (175) multiplied by the arc and divided by the radius and 4.

Again, if we use the corrected *piṇḍa-jyās* (176) in the expression (170) for the Rsine, we obtain

$$\begin{aligned} B &\approx s - \left(\frac{1}{R}\right)^2\left(\frac{s}{n}\right)^3 [(1+2+..+(n-1))+(1+2+..+(n-2))+..] \\ &\quad + \left(\frac{1}{R}\right)^4\left(\frac{s}{n}\right)^5 \\ &\quad \times \left(\frac{1}{1.2.3}\right)\left[(1^3+2^3+...+(n-1)^3)+(1^3+2^3+...+(n-2)^3)+..\right] \\ &\approx s - \left(\frac{1}{R}\right)^2 \frac{s^3}{1.2.3} + \left(\frac{1}{R}\right)^4 \frac{s^5}{1.2.3.4.5}. \end{aligned} \qquad (179)$$

The above process can be repeated to obtain successive higher order corrections for the Rversine and Rsine: By first finding a correction (*jyā-cāpāntara-saṃskāra*) for the difference between the Rsine and the arc, using this correction to correct the *piṇḍa-jyās* B_j, and using them in equations (169) and (170) get the next correction (*śara-saṃskāra*) for the Rversines, and the next correction (*jyā-cāpāntara-saṃskāra*) for the Rsine-arc-difference itself, which is then employed to get further corrections iteratively. In this way we are led to the Mādhava series for *jyā* and *śara* given by

$$\begin{aligned} B = R\sin(s) &= s - \left(\frac{1}{R}\right)^2 \frac{s^3}{(1.2.3)} + \left(\frac{1}{R}\right)^4 \frac{s^5}{(1.2.3.4.5)} \\ &\qquad - \left(\frac{1}{R}\right)^6 \frac{s^7}{(1.2.3.4.5.7)} + \ldots, \\ S = R\ \text{vers}(s) &= \left(\frac{1}{R}\right) \frac{s^2}{2} - \left(\frac{1}{R}\right)^3 \frac{s^4}{(1.2.3.4)} + \left(\frac{1}{R}\right)^5 \frac{s^6}{(1.2.3.4.6)} - \ldots \end{aligned} \tag{180}$$

That is,

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{(1.2.3)} + \frac{\theta^5}{(1.2.3.4.5)} - \frac{\theta^7}{(1.2.3.4.5.6.7)} + \ldots\ , \\ \text{vers}\ \theta &= \frac{\theta^2}{(1.2)} - \frac{\theta^4}{(1.2.3.4)} + \frac{\theta^6}{(1.2..4.5.6)} - \ldots\ . \end{aligned} \tag{181}$$

17 Instantaneous velocity and derivatives

As we saw in Section 6.1, the *mandaphala* or the equation of centre for a planet $\Delta\mu$ is given by

$$R\sin(\Delta\mu) = \left(\frac{r_0}{R}\right) R\sin(M-\alpha), \tag{182}$$

where r_0 is the mean epicycle radius, M is the mean longitude of the planet and α the longitude of the apogee. Further as we noted earlier, Muñjāla, Āryabhaṭa II and Bhāskara II used the approximation

$$R\sin(\Delta\mu) \approx \Delta\mu, \tag{183}$$

in (182) and obtained the following expression as correction to the instantaneous velocity of the planet:

$$\frac{d}{dt}(\Delta\mu) = \left(\frac{r_0}{R}\right) R\cos(M-\alpha)\frac{d}{dt}(M-\alpha). \tag{184}$$

Actually the instantaneous velocity of the planet has to be evaluated from the more accurate relation

$$\Delta\mu = R\sin^{-1}\left[\left(\frac{r_0}{R}\right) R\sin(M-\alpha)\right]. \tag{185}$$

The correct expression for the instantaneous velocity which involves the derivative of arc-sine function has been given by Nīlakaṇṭha in his *Tantrasaṅgraha*.[96]

चन्द्रबाहुफलवर्गशोधितत्रिज्यकाकृतिपदेन संहरेत्।
तत्र कोँटिफललिप्तिकाहतां केन्द्रभुक्तिरिह यच्च लभ्यते॥

[96] *Tantrasaṅgraha*, cited above (fn. 52), verses 2.53–54, pp.169–170. Elsewhere, Nīlakaṇṭha has ascribed these verses to his teacher Dāmodara (*Jyotirmīmāṃsā*, Ed. by K. V. Sarma, VVRI, Hoshiarpur 1977, p. 40).

तद्विशोध्य मृगादिके गतेः क्षिप्यतामिह तु कर्कटादिके।
तद्भवेत्स्फुटतरा गतिर्विधोः अस्य तत्समयजा रवेरपि॥

Let the product of the *koṭiphala* $[r_0 \cos(M - \alpha)]$ in minutes and the daily motion of the *manda-kendra* $\left(\frac{d(M-\alpha)}{dt}\right)$ be divided by the square root of the square of the *bāhuphala* subtracted from the square of *trijyā* $\left(\sqrt{R^2 - r_0^2 \sin^2(M - \alpha)}\right)$. The result thus obtained has to be subtracted form the daily motion of the Moon if the *manda-kendra* lies within six signs beginning from *Mṛga* and added if it lies within six signs beginning from *Karkaṭaka.* The result gives a more accurate value of the Moon's angular velocity. In fact, the procedure for finding the instantaneous velocity of the Sun is same as this.

If $(M - \alpha)$ be the *manda-kendra*, then the content of the above verse can be expressed as

$$\frac{d}{dt}\left[\sin^{-1}\left(\frac{r_0}{R}\sin(M-\alpha)\right)\right] = \frac{r_0 \cos(M-\alpha)\,\dfrac{d(M-\alpha)}{dt}}{\sqrt{R^2 - r_0^2 \sin^2(M-\alpha)}}. \qquad (186)$$

The instantaneous velocity of the planet is given by

$$\frac{d}{dt}\mu = \frac{d}{dt}(M-\alpha) - \frac{r_0 \cos(M-\alpha)\,\dfrac{d(M-\alpha)}{dt}}{\sqrt{R^2 - r_0^2 \sin^2(M-\alpha)}}. \qquad (187)$$

Here, the first term in the RHS represents the mean velocity of the planet and the second term the rate of change in the *mandaphala* given by (186).

In his *Āryabhaṭīya-bhāṣya*, Nīlakaṇṭha explains how his result is more correct than the traditional result of Muñjāla and Bhāskarācārya:[97]

अतः फलसाम्यं कुतः? …पुनरपि यो विशेषः तत्र कोटिज्यागुणितस्य त्रिज्यया हरणमुक्तम्, इह कोटिफलगुणितस्य केन्द्रभोगस्य दोःफलकोट्या हरणमुक्तम् इति। तेन तत्फलं चापीकृतं भुजाफलगतिः स्यात्। कथम् ?

चापगतिसम्बन्धिज्यागत्यानयने यत् त्रैराशिकमुक्तं, ज्यागत्या चापगत्यानयने तद्विपरीतं कर्म कार्यम्। तत्र पूर्वोक्ते कर्मणि त्रैराशिकद्वयेन या दोःफलगतिः आनीता तां व्यासार्धेन हत्वा दोःफलकोट्या हृत्वा तच्चापगतिर्लभ्या। तत्रेदं त्रैराशिकम् …

Hence, how can the results be equal? …Again the distinction being: there it was prescribed that the multiplier *koṭi-jyā* was to be divided by *trijyā*, [but] here it has been prescribed that the product of *koṭiphala*

[97] *Āryabhaṭīya* of Āryabhaṭa, Ed. with *Āryabhaṭīya-bhāṣya* of Nīlakaṇṭha Somayājī by K. Sāmbaśiva Śāstrī, Trivandrum Sanskrit Series 110, Trivandrum 1931, comm. on *Kālakriyāpāda* 22–25, pp. 62–63.

and the rate of change of *kendra* be divided by *koṭi* of the *doḥphala* (*doḥphalakoṭyā*).[98] ...

17.1 Acyuta's expression for instantaneous velocity involving the derivative of ratio of two functions

In the third chapter of his *Sphuṭanirṇayatantra*, Acyuta Piṣāraṭi (c. 1550–1621), a disciple of Jyeṣṭhadeva, discusses various results for the instantaneous velocity of a planet depending on the form of equation of centre (*manda-saṃskāra*). He first presents the formula involving the derivative of arc-sine function given by Nīlakaṇṭha (in the name of (*manda*)-*sphuṭagati*) as follows:[99]

कोटिफलाहतकेन्द्रगतेर्यद् दोःफलकोटिकयाप्तमनेन।
हीनयुतामृगकर्कटकाद्दोर्मध्यगतिर्भवति स्फुटभुक्तिः॥

Acyuta also gives the formula for the instantaneous velocity of a planet if one were to follow a different model proposed by *Munjāla* for the equation of centre, according to which *mandaphala* is given by

$$\Delta\mu = \frac{\frac{r_0}{R}\sin(M-\alpha)}{\left(1 - \frac{r_0}{R}\cos(M-\alpha)\right)}, \tag{188}$$

instead of (182), where $\Delta\mu$ is small. If one were to use this formula for *mandaphala* for finding the true longitude of the planet, then it may be noted that the instantaneous velocity will involve the derivative of the ratio of two functions both varying with time. Taking note of this, Acyuta observes:[100]

कृत्स्नस्य मान्दपरिधेर्निजकर्णतुल्यौ वृद्धिक्षयाविति मते कथितः क्रमोऽयम्।
अर्धस्य मान्दपरिधेः क्षयवृद्धिपक्षे, युक्तं क्रियामथ प्रतिपादयामः॥

> The procedure that was prescribed earlier is with reference to the School that conceives of the increase and decrease in the circumference of the *manda-vṛtta* in accordance with the *karṇa*. With reference to the School that conceives of increase and decrease only to the half [of it], now we prescribe the appropriate procedure to be adopted.

Acyuta then proceeds to give the correct expression for the instantaneous velocity of a planet in Munjāla's model:[101]

कृतकोटिफलं त्रिजीवया विहृतं दोःफलवर्गतस्तु यत्।
मृगकर्कटकादिकेऽमुना युतहीनं फलमत्रकोटिजम्॥

[98] The terms *doḥphala* and *koṭiphala* refer to $\frac{r_0}{R}\sin(M-\alpha)$ and $\frac{r_0}{R}\cos(M-\alpha)$ respectively. Hence, the term *doḥphalakoṭi* refers to $\sqrt{1-(\frac{r_0}{R}\sin(M-\alpha))^2}$.

[99] *Sphuṭanirṇayatantra* of Acyuta Piṣāraṭi, Ed. by K. V. Sarma, VVRI, Hoshiarpur 1974, p. 19.

[100] Ibid., p. 20.

[101] Ibid., p. 21.

दिनकेन्द्रगतिघ्नमुद्धरेत् कृतकोटीफलया त्रिजीवया।
फलपूर्वफलैकतो दलं दिनभुक्तेरपि संस्कृतिर्भवेत्॥

> Having applied the *koṭiphala* to *trijyā* [positively or negatively depending upon the *mandakendra*], let the square of the *doḥphala* be divided by that. This may be added to or subtracted from the *koṭiphala* depending on if it is *Mṛgādi* or *Karkyādi*. The product of this [result thus obtained] and the daily motion of the *manda-kendra* divided by the *koṭiphala* and applied to *trijyā* will be the correction to the daily motion.

Thus according to Acyuta, the correction to the mean velocity of a planet to obtain its instantaneous velocity is given by

$$\frac{\left(\frac{r_0}{R}\cos(M-\alpha)\right)+\frac{\left(\frac{r_0}{R}\sin(M-\alpha)\right)^2}{\left(1-\frac{r_0}{R}\cos(M-\alpha)\right)}}{\left(1-\frac{r_0}{R}\cos(M-\alpha)\right)}\;\frac{d(M-\alpha)}{dt}, \qquad (189)$$

which is nothing but the derivative of the expression given in (188).

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