MA330
Sathaye

Class discussion of Math. overview

I list below main points of various topics discussed in class. More parts would be added as needed.

Spring 2017

1. Fundamental parts of Mathematics: numbers, logic.
2. Discussion about the concept of "truth" in Mathematics. A proper mathematical statement only claims that a certain assertion is true, provided the set axioms hold.
3. Examples of axioms. Euclid's 5th postulate and non euclidean geometries.
4. Which axiom systems are "valid"? Ans: If they have a "model".
5. Well order and the principle of induction. (Homework exercises will appear on the web page.)

Topics from previous semester.

1. Numbers and shapes form the Mathematical language. Numbers have two main models: Greeks: lengths of line segments, Indians: counting money (or wealth). The Indian model leads to easy understanding of zero and negatives.

   These distinct models perhaps lead to Calculus and Algebra respectively.

2. Modern construction of number systems.
   - We discussed Peano axioms and construction of natural numbers starting with just an empty set. I recommend googling for "Peano Axioms" for a deeper understanding.
   - Then we discussed a way to "build" the integers by considering equivalence classes of pairs of natural numbers. I illustrated how the same consstruction leads to other constructions of more numbers.
   - Thus we constructed rationals, algebraic numbers and so on. A general construction of adjoining a "root " of any polynomial due to Kronecker.

     We discussed other possible extensions of number systems which lose many desirable properties. These are the quaternions and Cayley-Dickson algebras.
   - The next discussion is that of real numbers. Their description is, in some sense easy but as a set it is too big!

     The set of all real numbers can be described as decimal expaansions $\{a_0\dot{a}_1\, a_2 \cdots , a_n \cdots\}$ where all a_i are integers, the numbers from a_1 onwards are in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. In addition, to avoid two different expansions giving the "same" number, (for example $0.100 \cdots 0 \cdots = 0.099 \cdots 9 \cdots$) we require that the sequence shall not have all digits as 9 from some position onwards.
   - We discussed the concept of cardinality $c(A)$ of a set A due to Cantor. For a finite set A it denotes the number of elements in A. However, for infinite sets the definition is rather subtle.
   - Thus for any two sets A, B we say that $c(A) \le c(B)$ if we can find an injective map from A into B, or alternatively a surjective map from B onto A.

     It is a theorem that if $c(A) \le c(B)$ and $c(B) \le c(A)$ both hold, then there exists a bijective map between A and B and we declare that $c(A) = c(B)$.

     Further, we declare $c(A) < c(B)$ when $c(A) \le c(B)$ but $c(A) \ne c(B)$.

- **Cantor's Theorem** Given any set A, Cantor proved that $c(A) < c(\mathcal{P}(A))$ where $\mathcal{P}(A)$ denotes the power set (i.e. the set of all subsets) of A.
- We call a set A countably infinite if $c(A) = c(\mathbb{N})$ where $\mathbb{N}$ denotes the set of natural numbers. We then explicitly argued that $c(\mathbb{R}) > c(\mathbb{N})$ where $\mathbb{R}$ is the set of reals.
- We discussed several known and unknown results in this connection.

$\square$ More details to come ...