MA330
Sathaye

A universal divisibility test

Given below is a certain universal divisibility test invented by Bhāratī Kṛṣṇa Tīrtha, who claimed (in 1950's) that he found it in ancient vedic scriptures. While the claim is very doubtful, the test is a nice alteration of the usual modern techniques.

The test is for checking if a given natural number n is divisible by a given natural number $d > 1$.

1. First assume that d is coprime with 10, since for numbers having common factors with 10, the possible common factors $2, 5$ can be factored out from d. The divisibility test by 2 is very easy: n must be even, i.e. the units digit is among $0, 2, 4, 6, 8$.

   The divisibility test by 5 is easier: n must have the units digit 0 or 5.

2. It is easy to find a number m such that $10m \equiv s \pmod{d}$, where s is 1 or -1. (See more details below.) The resulting product sm shall be called a multiplier for d.

   Write $n = 10u + v$.

3. Then the divisibility test says that $d|n = 10u + v$ iff $d|u + smv$. The number $u + smv$ is usually much smaller than n and we can repeat the test until divisibility becomes evident.

4. Here is an illustration.

   Let $d = 13$. Note that $(3)13 + 1 = (4)10$, so we can take $s = 1$ and $m = 4$. Start with $n = 34351 = 10(3435) + 1$, so $u = 3435$ and $v = 1$. So we can replece $n = 34351$ by $u + smv = 3435 + (4)1 = 3439$. Taking $n = 3439 = (10)343 + 9$, we next replace it by $343 + (4)9 = 379$. Next replacement shall be $37 + (4)9 = 73$ and this is easily seen to be not divisible by 13.

   Usually, we don't spend time in the arguments.

   We start with a given n and a given d. Find a suitable multiplier sm. Use it to reduce n until divisibility is easy.

   Here is another example with $d = 23$.

   $3(23) = 69$, so $(7)10 = 1 + (3)(23)$. Thus, we can take $s = 1, m = 7$. The multiplier is $(1)(7) = 7$.

   Take $n = 314452$. Here are the reductions:

   $$314452 \to 31445+(2)(7) = 31459 \to 3145+(7)(9) = 3208 \to 320+(7)(8) = 376 \to 37+(7)(6) = 79.$$

   The last number is clearly not divisible by 23. (You could make a further reduction if you like:

   $$79 \to 7 + (7)(9) = 70 \to 7 + (7)(0) = 7.$$

5. Here are simple recipies for finding the s, m.

   If d ends in 1, then write $d = 10m + 1$, which defines m and let $s = -1$. Example: $d = 31 = 10(3) + 1$ so $m = 3, s = -1$ and multiplier is -3.

   If d ends in 3, then write $3d + 1 = 10m$, which defines m and set $s = 1$. Example: $d = 43$ so $3d + 1 = 130 = 10(13)$ so $m = 13, s = 1, sm = 13$.

   If d ends in 7, then write $3d = 10m + 1$ thereby defining m and $s = -1$. Example: $d = 37$ so $3d = 111 = 10(11) + 1$ so $m = 11, s = -1, sm = -11$.

If d ends in 9, then write $d + 1 = 10m$ thereby defining m and $s = 1$. Example: $d = 29$ so $d + 1 = 30 = 10(3)$ so $m = 3, s = 1, sm = 3$.

The original recipe of Swamiji stated the following:

Find a multiple of the chosen d which ends in 9. Take the part of the number before 9 and add 1 to it. This is the multiplier sm. For instance, when $d = 17$, we have $(7)(17) = 119$; the part before 9 is 11 so the multiplier is $1 + 11 = 12$.

6. Now you should practice using the above multipliers as well as practicing making your own multipliers.

7. **Proof of the test:** Note that by assumption $10sm \equiv s^2 = 1$ modulo d. Now, $d|n = 10 + v$ iff $d|(sm)n = 10smu + smv$. This last expression is congruent to $u + smv$ modulo d. Hence the test is proved!

8. This reproduces the usual tests for divibility by $3, 9, 11$ since the values of sm come out to be $1, 1, -1$ in these cases respectively. You should verify how this leads to the usual tests.

9. We give one more example for practice. Take $d = 17$. As shown above, the multiplier is 12. This is also -5 modulo 17, so we choose the smaller size multiplier.

   Take $n = 214457$. Here is the work:

   $$214457 \to 21445 - (5)7 = 21410 \to 2141 \to 214 - (5)1 = 209 \to 20 - (5)9 = -25.$$

   The last is clearly non divisible.