2

Problems Leading to Pell's Equation and Preliminary Investigations

The first chapter presented a situation that led to pairs of integers (x, y) that satisfied equations of the form $x^2 - 2y^2 = k$ for some constant k. One of the reasons for the popularity of Pell's equation as a topic for mathematical investigation is the fact that many natural questions that one might ask about integers lead to a quadratic equation in two variables, which in turn can be cast as a Pell's equation. In this chapter we will present a selection of such problems for you to sample.

For each of these you should set up the requisite equation and then try to find numerical solutions. Often, you should have little difficulty in determining at least one and may be able to find several. These exercises should help you gain some experience in handling Pell's equation. Before going on to study more systematic methods of solving them, spend a little bit of time trying to develop your own methods.

While a coherent theory for obtaining and describing the solutions of Pell's equation did not appear until the eighteenth century, the equation was tackled ingeniously by earlier mathematicians, in particular those of India. In the third section, inspired by their methods, we will try to solve Pell's equation.

2.1 Square and Triangular Numbers

The numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, ... , $t_n \equiv \frac{1}{2}n(n+1), \ldots$ are called *triangular*, since the nth number counts the number of dots in an equilateral triangular array with n dots to the side.

It is not difficult to see that the sum of two adjacent triangular numbers is square.

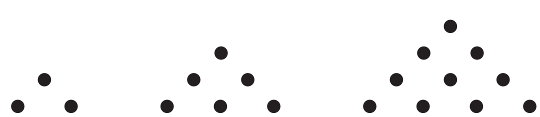

FIGURE 2.1.

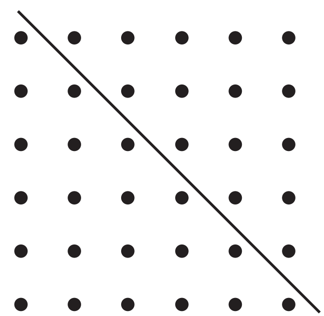

FIGURE 2.2.

But does it often happen that an individual triangular number is square? We will examine this and similar questions.

Exercise 1.1. Verify that the condition that the nth triangular number t_n is equal to the mth square is that $\frac{1}{2}n(n+1) = m^2$. Manipulate this equation into the form

$$(2n+1)^2 - 8m^2 = 1.$$

Thus, we are led to solving the equation $x^2 - 8y^2 = 1$ for integers x and y. It is clear that for any solution, x must be odd (why?), so that we can then find the appropriate values of m and n. Observe that 1 and 36 are included in the list of triangular numbers. What are the corresponding values of x, y, m, n? Use the results of Exercise 1.1.7(c) to generate other solutions.

Exercise 1.2. There are triangular numbers that differ from a square by 1, such as $3 = 2^2 - 1$, $10 = 3^2 + 1$, $15 = 4^2 - 1$, and $120 = 11^2 - 1$. Determine other examples.

Exercise 1.3. Find four sets of three consecutive triangular numbers whose product is a perfect square.

Exercise 1.4. Find four sets of three consecutive triangular numbers that add up to a perfect square.

Exercise 1.5. Determine integers n for which there exists an integer m for which $1 + 2 + 3 + \cdots + m = (m+1) + (m+2) + \cdots + n$.

Exercise 1.6. Determine positive integers m and n for which

$$m + (m+1) + \cdots + (n-1) + n = mn$$

(*International Mathematical Talent Search* 2/31).

Exploration 2.1. The triangular numbers are sums of arithmetic progressions. We can ask similar questions about other arithmetic progressions as well. Determine the smallest four values of n for which the sum of n terms of the arithmetic series $1+5+9+13+\cdots$ is a perfect square. Compare these values of n with the terms of the sequence $\{q_n\}$ listed in Exploration 1.1. Experiment with other initial terms and common differences.

Exploration 2.2. Numbers of the form $n(n+1)$ (twice the triangular numbers) are known as *oblong*, since they represent the area of a rectangle whose sides lengths are consecutive integers. The smallest oblong numbers are

$$2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156.$$

A little experimentation confirms that the product of two consecutive oblong numbers is oblong; can you give a general proof of this result? Look for triples (a, b, c) of oblong numbers a, b, c for which $c = ab$. For each possible value of a, investigate which pairs (b, c) are possible.

An interesting phenomenon is the appearance of related triples of solutions. For example, we have (a, b, c) equal to

$$(14 \times 15, 782 \times 783, 11339 \times 11340),$$

$$(14 \times 15, 13 \times 14, 195 \times 196),$$

$$(13 \times 14, 782 \times 783, 10556 \times 10557),$$

while

$$(11339 \times 11340)(13 \times 14)^2 = (195 \times 196)(10556 \times 10557).$$

Are there other such triples?

2.2 Other Examples Leading to a Pell's Equation

The following exercises also involve Pell's equation. For integers n and k with $1 \le k \le n$, we define

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k} = \frac{n!}{k!(n-k)!}.$$

Also, we define $\binom{n}{0} = 1$ for each positive integer n. Observe that $1+2+\cdots+n = \binom{n+1}{2}$.

Exercise 2.1. Determine nonnegative integers a and b for which

$$\binom{a}{b} = \binom{a-1}{b+1}.$$

Exercise 2.2. Suppose that there are n marbles in a jar with r of them are red and $n - r$ blue. Two marbles are drawn at random (without replacement). The probability that both have the same color is $\frac{1}{2}$. What are the possible values of n and r?

Exercise 2.3. The following problem appeared in the *American Mathematical Monthly* (#10238, 99 (1992), 674):

(a) Show that there exist infinitely many positive integers a such that both $a + 1$ and $3a + 1$ are perfect squares.

(b) Let $\{a_n\}$ be the increasing sequence of all solutions in (a). Show that $a_n a_{n+1} + 1$ is also a perfect square.

Exercise 2.4. Determine positive integers b for which the number $(111\ldots1)_b$ with k digits all equal to 1 when written to base b is a triangular number, regardless of the value of k.

Exercise 2.5. Problem 2185 in *Crux Mathematicorum* (22 (1996), 319) points out that

$$2^2 + 4^2 + 6^2 + 8^2 + 10^2 = 4 \cdot 5 + 5 \cdot 6 + 6 \cdot 7 + 7 \cdot 8 + 8 \cdot 9$$

and asks for other examples for which the sum of the first n even squares is the sum of n consecutive products of pairs of adjacent integers.

Exercise 2.6. Determine integer solutions of the system

$$\begin{aligned} 2uv - xy &= 16, \\ xv - uy &= 12 \end{aligned}$$

(*American Mathematical Monthly* 61 (1954), 126; 62 (1955), 263).

Exercise 2.7. Problem 605 in the *College Mathematics Journal* (28 (1997), 232) asks for positive integer quadruples (x, y, z, w) satisfying $x^2 + y^2 + z^2 = w^2$ for which, in addition, $x = y$ and $z = x \pm 1$. Some examples are (2, 2, 1, 3) and (6, 6, 7, 11). Find others.

Exercise 2.8. The root-mean-square of a set $\{a_1, a_2, \ldots, a_k\}$ of positive integers is equal to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_k^2}{k}}.$$

Is the root-mean-square of the first n positive integers ever an integer? (USAMO, 1986)

Exercise 2.9. Observe that $(1 + 1^2)(1 + 2^2) = (1 + 3^2)$. Find other examples of positive integer triples (x, y, z) for which $(1 + x^2)(1 + y^2) = (1 + z^2)$.

Exercise 2.10. A problem in the *American Mathematical Monthly* (#6628, 98 (1991), 772–774) asks for infinitely many triangles with integer sides whose area is a perfect square. According to one solution, if m is chosen to make $\frac{1}{2}(m^2-1)$ a square, then the triangles with sides $(\frac{1}{2}(m^3+m^2)-1,\ \frac{1}{2}(m^3-m^2)+1,\ m^2)$ and with sides $(m^3-\frac{1}{2}(m-1),\ m^3-\frac{1}{2}(m+1),\ m)$ have square area. Recalling Heron's formula $\sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle with sides (a, b, c) and perimeter $2s$, verify this assertion and give some numerical examples.

Exercise 2.11.

(a) Suppose that the side lengths of a triangle are consecutive integers $t-1, t, t+1$, and that its area is an integer. Prove that $3(t^2-4)$ must be an even perfect square, so that $t = 2x$ for some x. Thus show that $x^2 - 3y^2 = 1$ for some integer y. Determine some examples.

(b) In the situation of (a), prove that the altitude to the side of middle length is an integer and that this altitude partitions the side into two parts of integer length that differ by 4.

(c) Suppose that the sides of a triangle are integers $t-u$, t, and $t+u$. Verify that $3(t^2-4u^2)$ is a square $(3v)^2$ and obtain the equation $t^2-3v^2=4u^2$. Determine some examples with $u \neq 1$.

Exercise 2.12. Here is one approach to constructing triangles with integer sides whose area is an integer. Such a triangle can be had either by slicing one right triangle from another or by juxtaposing two right triangles. (See figure 2.3.)

We suppose that $m, r, a, b = ma + r$ and $c = ma - r$ are integers.

(a) Prove that $4mr = a \pm 2q$ and deduce that $2q$ is an integer.

(b) Prove that $2p$ must be an integer.

(c) By comparing two expressions for the area of the triangle (a, b, c), verify that

$$(4m^2-1)(a^2-4r^2) = 4p^2.$$

Take $a = 2t$ and obtain the equation

$$p^2 - (4m^2-1)t^2 = -(4m^2-1)r^2.$$

(d) Determine some solutions of the equation in (c) and use them to construct some examples of triangles of the desired type.

Exercise 2.13. A Putnam problem (**A2** for the year 2000) asked for a proof that there are infinitely many sets of three consecutive positive integers each of which is the sum of two integer squares. An example of such a triple is $8 = 2^2 + 2^2$, $9 = 0^2 + 3^2$ and $10 = 1^2 + 3^2$.

(a) One way to approach the problem is to let the three integers be $n^2 - 1 = 2m^2$, n^2, and $n^2 + 1$. Derive a suitable Pell's equation for m and n and produce some numerical examples.

(b) However, it is possible to solve this problem without recourse to Pell's equation. Do this.

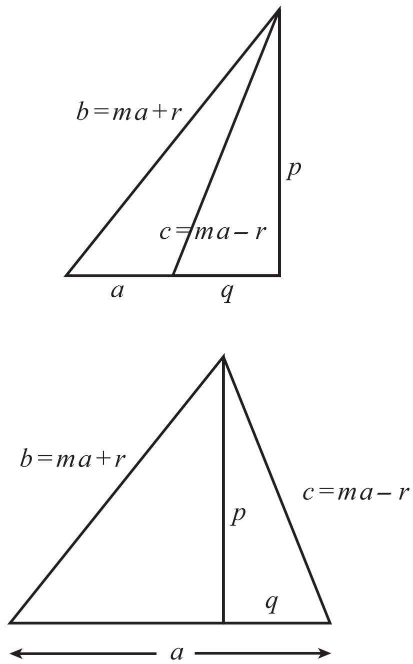

FIGURE 2.3.

Exploration 2.3. Let $\{F_n\}$ be the Fibonacci sequence determined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for each integer n. It turns out that

$$F_{2n-1}^2 + F_{2n+1}^2 + 1 = 3F_{2n-1}F_{2n+1}$$

(can you prove this?), so that (F_{2n-1}, F_{2n+1}) is an example of a pair (a, b) for which $a^2 + b^2 + 1$ is a multiple of ab. Thus we have the instances $(a, b) = (1, 1), (1, 2), (2, 5), (5, 13), (13, 34)$. What other pairs can be found?

Problem #10316 in the *American Mathematical Monthly* (100 (1993), 589; 103 (1996), 905) asks for conditions under which ab divides $a^2 + b^2 + 1$. Suppose for some integer k that $a^2 + b^2 + 1 = kab$. If (a, b) satisfies the equation, then so also do $(b, kb - a)$ and $(a, ka - b)$, so we have a way of generating new solutions from old. Show that the multiple k must exceed 2, and that the Diophantine equation for a and b can be rewritten

$$(2a - kb)^2 - (k^2 - 4)b^2 = -4.$$

We can rule out certain values of k. For example, k must be a multiple of 3, but cannot be twice an odd number.

What are all the solutions for $k = 3$? Are there any solutions for $k = 15$? Are there solutions for any other values of k?

Exploration 2.4. The triple (1, 8, 15) has the interesting property that the three numbers are in arithmetic progression and the product of any two of them plus one is a perfect square. Find other triples that have the same property.

Exploration 2.5. Note that $15^2 + 16^2 + 17^2 = 7 \times (5^2 + 6^2 + 7^2)$ and $8^2 + 9^2 + 10^2 + 11^2 + 12^2 = 2 \times (5^2 + 6^2 + 7^2 + 8^2 + 9^2)$. Determine generalizations.

2.3 Strategies for Solutions and a Little History

It is perverse that equations of the type $x^2 - dy^2 = k$ became associated with the name of Pell. John Pell (c. 1611–1683) was indeed a minor mathematician, but he does not appear to have seriously studied the equation. Kenneth Rosen, on page 459 of his *Elementary Number Theory*, mentions a book in which Pell augmented work of other mathematicians on $x^2 - 12y^2 = n$, and D.E. Smith, in his *Source Book*, says that there is weak indication of his interest in the equation $x^2 = 12y^2 - 33$ is considered in a 1668 algebra book by J. H. Rahn to which Pell may have contributed. H.C. Williams provides a full description of this in his millennial paper on number theory. However, many mathematical historians agree that this is a simple case of misattribution; these equations were ascribed to Pell by Leonhard Euler in a letter to Goldbach on August 10, 1730, and in one of his papers. Since Euler was one of the most influential mathematicians in Europe in the eighteenth century, the name stuck. There were others very interested in the equation, many earlier than Pell. Pierre de Fermat (1601?–1665) was the first Western European mathematician to give the equation serious attention, and he induced his contemporaries John Wallis (1616–1703) and Frénicle de Bessy (1602–1675) to study it. Actually, Pell's equations go back a long way, before the seventeenth century. The Greeks seem to have come across some instances of it; in particular, Archimedes posed a problem about cattle that led to an equation of the type. The Indian mathematician Brahmagupta in the sixth century had a systematic way of generating infinitely many solutions from a particular one, while in the eleventh century, Jayadeva and Bhaskara II had algorithms for finding the first solution.

Exercise 3.1. In our definition of Pell's equation we specified that d had to be positive and nonsquare. Let us see why this restriction is a natural one. First, suppose that d is a negative number, say $-p$.

(a) Consider the equation $x^2 + 3y^2 = 7$. Find all solutions to this equation. How do you know that you have a complete set? In the Cartesian plane, sketch the curve with equation $x^2 + 3y^2 = 7$. Indicate all points on it with integer coordinates. What is this curve?

(b) For a given positive integer p and integer k, sketch the graph of the equation $x^2 + py^2 = k$. Corresponding to every point (x, y) on the graph with integer coordinates there is a solution to the equation. Determine an upper bound on the number of solutions that this equation can possibly have. ♠

Thus, when d is negative, the solutions of $x^2 - dy^2 = k$ in integers x and y are finite in number and can be found by inspection. Characterizing those values of k for which there *is* a solution for a given d is in itself an interesting question, though it is not within the scope of this book.

Exercise 3.2. Consider the equation $x^2 - dy^2 = k$ where $d = q^2$, the square of an integer.

(a) Determine all the solutions in integers that you can for each of the following equations:

$$x^2 - 4y^2 = 45,$$
$$x^2 - y^2 = 6,$$
$$x^2 - 9y^2 = 7.$$

(b) Argue that that equation $x^2 - q^2y^2 = k$ can have at most finitely many solutions in integers x and y. Give an upper bound for this number of solutions in terms of the number of positive integers that divide k evenly.

(c) Sketch the graph of the hyperbola with equation $x^2 - q^2y^2 = k$ along with the graphs of its asymptotes with equations $x + qy = 0$ and $x - qy = 0$. What are the points with integer coordinates lying on the asymptotes? What insight does this give as to why there are so few points on the hyperbola with integer coordinates?

Exercise 3.3. The eleventh century Indian mathematician Bhaskara was able to solve the equation $x^2 - 61y^2 = 1$ for integers x and y. One might think that since it easy to find a solution of $x^2 - 63y^2 = 1$ (do it!), there should not be too much difficulty solving Bhaskara's equation. However, simple trial and error is likely to lead to abject failure, and Bhaskara needed considerable numerical skill to handle the job—this at a time when there were no calculators or even the convenient notation we enjoy today. In this exercise we will indicate the type of strategy followed by Bhaskara, but avail ourselves of modern notation.

(a) Suppose that k, d, x, y are integers for which $x^2 - dy^2 = k$. Show that

$$(mx + dy)^2 - d(ym + x)^2 = k(m^2 - d)$$

for each positive integer m.

(b) Suppose, in (a), that the greatest common divisor of k and y is 1, and that $ym + x$ is a multiple of k. Use the equations

$$(m^2 - d)y^2 = k - (x^2 - m^2y^2) = k - (x + my)(x - my)$$

and

$$mx + dy = m(x + my) - (m^2 - d)y$$

to show that $m^2 - d$ and $mx + dy$ are also multiples of k. Thus, the result in (a) can be rewritten

$$\left(\frac{mx + dy}{k}\right)^2 - d\left(\frac{ym + x}{k}\right)^2 = \frac{m^2 - d}{k},$$

where the quantities in parentheses can be made all integers when m is suitably chosen.

(c) Derive from $8^2 - 61(1)^2 = 3$ the equation

$$\left(\frac{8m + 61}{3}\right)^2 - 61\left(\frac{m + 8}{3}\right)^2 = \frac{m^2 - 61}{3}.$$

Choose m so that $(m + 8)/3$ is an integer and $|(m^2 - 61)/3|$ is as small as possible. Hence derive

$$39^2 - 61(5)^2 = -4.$$

(d) Now obtain the equation

$$\left(\frac{39m + 305}{-4}\right)^2 - 61\left(\frac{5m + 39}{-4}\right)^2 = \frac{m^2 - 61}{-4}.$$

Choose m so that $(5m + 39)/4$ is an integer and $|(m^2 - 61)/4|$ is as small as possible. Using a pocket calculator, if you wish, obtain

$$164^2 - 61(21)^2 = -5.$$

(e) We can continue on in this way to successively derive the following numerical equations. Check the derivation of as many of them as you need to feel comfortable with the process.

$$\begin{aligned}
453^2 - 61(58)^2 &= 5,\\
1523^2 - 61(195)^2 &= 4,\\
5639^2 - 61(722)^2 &= -3,\\
29718^2 - 61(3805)^2 &= -1,\\
469849^2 - 61(60158)^2 &= -3,\\
2319527^2 - 61(296985)^2 &= 4,\\
9747957^2 - 61(1248098)^2 &= 5,\\
26924344^2 - 61(3447309)^2 &= -5,\\
90520989^2 - 61(11590025)^2 &= -4,\\
335159612^2 - 61(42912791)^2 &= 3,\\
1766319049^2 - 61(226153980)^2 &= 1.
\end{aligned}$$

It is interesting to note that the equation $x^2 - 61y^2 = 1$ was proposed by the Frenchman Pierre Fermat to Frénicle in February, 1657. The first European to publish a solution was Leonhard Euler, in 1732.

Exercise 3.4. There are devices known to Bhaskara by which the process can be shortened. One depends on the identity

$$(x^2 - dy^2)(u^2 - dv^2) = (xu + dyv)^2 - d(xv + yu)^2.$$

(a) Verify this identity and draw from it the conclusion that if two integers can be written in the form $x^2 - dy^2$ for integers x and y, then so can their product.
(b) Explain how a solution of $x^2 - dy^2 = -1$ can be used to obtain a solution of $x^2 - dy^2 = +1$.
(c) Determine a solution to the equation $x^2 - 65y^2 = 1$ in integers x and y.
(d) From the identity and the pair of equations, derive

$$x^2 - dy^2 = k, \qquad m^2 - d(1)^2 = m^2 - d,$$

the equation in Exercise 3.3(a).

Exercise 3.5. Refer to Exercise 3.3.
(a) From the numerical equation $39^2 - 61(5)^2 = -4$, deduce that

$$\left(\frac{39}{2}\right)^2 - 61\left(\frac{5}{2}\right)^2 = -1.$$

(b) Substituting $x = u = 39/2$, $y = v = 5/2$, $d = 61$ in the identity of Exercise 3.4(a), derive

$$\left(\frac{1523}{2}\right)^2 - 61\left(\frac{195}{2}\right)^2 = 1.$$

(c) Substituting $x = 39/2,\ \ y = 5/2,\ \ u = 1523/2,\ \ v = 195/2$ in the identity of Exercise 3.4(a), obtain

$$29718^2 - 61(3805)^2 = -1.$$

(d) Now obtain a solution to $x^2 - 61y^2 = 1$ using Exercise 3.4(b).

Exercise 3.6. Another equation solved by Bhaskara was $x^2 - 67y^2 = 1$.
(a) Following the procedure of Exercise 3.3, derive the equations

$$\begin{aligned} 8^2 - 67(1)^2 &= -3, \\ 41^2 - 67(5)^2 &= 6, \\ 90^2 - 67(11)^2 &= -7, \\ 221^2 - 67(27)^2 &= -2. \end{aligned}$$

(b) Using the identity of Exercise 3.4, derive a solution of $x^2 - 67y^2 = 4$ and deduce from this a solution in integers x, y to $x^2 - 67y^2 = 1$.

Exercise 3.7. In 1658, Frénicle claimed that he had found a solution in integers x and y to $x^2 - dy^2 = 1$ for all nonsquare values of d up to 150, but mentioned that he was looking in particular for solutions in the cases $d = 151$ and $d = 313$. In response, John Wallis found that

$$(1728148040)^2 - 151(140634693)^2 = 1,$$

and Lord Brouncker commented that within an hour or two, he had discovered that

$$(126862368)^2 - 313(7170685)^2 = -1.$$

Check that these results are correct. Doing this in this obvious way may not be most efficient, particularly if they lead to overflow of your pocket calculator. A better way may be to set things up so that you can use division rather than multiplication. This might involve manipulating the equation to be checked into forms leading to easy factorization, such as those involving differences of squares. It might involve checking for small prime factors of terms involved. Be creative and use some ingenuity.

Exercise 3.8. How might Wallis and Brouncker have solved a Pell's equation? Consider the example $x^2 - 7y^2 = 1$.

(a) The smallest square exceeding 7 is $9 = 3^2$; we have that $7 = 3^2 - 2$. Deduce from this $7(2)^2 = 6^2 - 8$, $7(3)^2 = 9^2 - 18$, and more generally $7m^2 = (3m)^2 - 2m^2$.

(b) Observe that $9^2 - 18 = (9-1)^2 - 1$, so that $9^2 - 18$ is just 1 shy of being a perfect square. Transform $7(3)^2 = 9^2 - 18$ to $7(3)^2 = (9-1)^2 - 1$ and thence derive a solution to $x^2 - 7y^2 = 1$.

Exercise 3.9. Let us apply the Wallis–Brouncker approach to the general equation $x^2 - dy^2 = 1$. Let positive integers c and k be chosen to satisfy $(c-1)^2 < d < c^2$ and $c^2 - d = k$, so that c^2 is the smallest square exceeding d, and k is the difference between this square and d.

(a) Show that for any integer m, $dm^2 = (cm)^2 - km^2$.

(b) The quantity $dm^2 = (cm)^2 - km^2$ is certainly less than $(cm)^2$, but it may not be less than any smaller square, in particular $(cm-1)^2$. However, as m grows larger, the distance between dm^2 and $(cm)^2$ increases, so that eventually dm^2 will become less than $(cm-1)^2$. This will happen as soon as

$$(cm)^2 - km^2 \leq (cm-1)^2 - 1.$$

Verify that this condition is equivalent to $2c \leq km$.

(The strategy is to select the smallest value of m for which this occurs and hope that $dm^2 - (cm-1)^2 = -1$, in which case $(x, y) = (cm-1, m)$ will satisfy $x^2 - dy^2 = 1$. This, of course, need not occur, and we will need to modify the strategy.)

(c) Start with $13 = 4^2 - 3$ and determine the smallest value of m for which $13m^2 - (4m-1)^2$ has a negative value, and write the numerical equation that evaluates this.

(d) Suppose $dm^2 - (cm-1)^2$ is not equal to -1 when it first becomes negative. It will take larger and larger negative values as m increases (why?). Eventually, there will come a time when $dm^2 - (cm - 2)^2$ will not exceed -1. Verify that this occurs when $4c \leq km + (3/m)$.

(e) This process can be continued. If $dm^2 - (cm - 2)^2$ fails at any point to equal -1, then we can try for a solution with $x = cm - 3$ for some value of m. This process can be continued until (hopefully) a solution is found. Try this in the case that $d = 13$.

(f) Which method do you consider more convenient, this or Bhaskara's?

Exercise 3.10. Here is a systematic way to obtain solutions of $x^2 - dy^2 = 1$ for a great many values of d.

(a) Verify the identity

$$(zy + 1)^2 - \left(z^2 + \frac{2z}{y}\right)y^2 = 1.$$

(b) Suppose that integers y and z are selected so that $2z$ is a multiple of y; let $d = z^2 + (2z/y)$ and $x = zy + 1$. Without loss of generality, we may suppose that $z > 0$ and that y can be either positive or negative. If $1 \leq y \leq 2z$, show that $z^2 + 1 \leq d \leq (z + 1)^2 - 1$, while if $-2z \leq y \leq -1$, show that $(z - 1)^2 - 1 \leq d \leq z^2 + 1$.

(c) Describe how, for a given value of d, one *might* determine solutions to $x^2 - dy^2 = 1$. Apply this method to obtain solutions when $d = 3, 27, 35, 45$.

(d) List values of d up to 50 for which solutions cannot be found using this method.

2.4 Explorations

Exploration 2.6. Archimedes' Cattle Problem. In the eighteenth century, a German dramatist, G.E. Lessing, discovered a problem posed by Archimedes to students in Alexandria. A complete statement of the problem and comments on its history and solution can be found in the following sources:

H.W. Lenstra, Jr., Solving the Pell's equation. *Notices of the American Mathematical Society* 49:2 (February, 2002), 182-192.

James R. Newman (*ed.*), *The World of Mathematics, Volume 1* (Simon & Schuster, New York, 1956) pages 197–198, 105–106.

H.L. Nelson, A solution to Archimedes' cattle problem, *Journal of Recreational Mathematics* 13:3 (1980–81), 162–176.

Ilan Vardi, Archimedes' Cattle Problem, *American Mathematical Monthly* 106 (1998), 305–319.

The paper of H.C. Williams on solving Pell's equation, delivered to the Millennial Conference on Number Theory in 2002 and listed in the historical references, discusses the cattle problem and lists additional references by P. Schreiber and W.

Waterhouse. In modern symbolism, this problem amounts to finding eight positive integers to satisfy the conditions

$$W = \tfrac{5}{6}X + Z, \quad X = \tfrac{9}{20}Y + Z, \quad Y = \tfrac{13}{42}W + Z,$$
$$w = \tfrac{7}{12}(X + x), \quad x = \tfrac{9}{20}(Y + y), \quad y = \tfrac{11}{30}(Z + z), \quad z = \tfrac{13}{42}(W + w),$$

with the additional requirements that $W + X$ is to be square and $Y + Z$ triangular. Solving this problem involves obtaining a solution to the Pell's equation $p^2 - 4{,}729{,}494q^2 = 1$, a feat that was not accomplished until 1965. Now, of course, we have sophisticated software available to do the job. Can you find a solution to the equation?

Exploration 2.7. There are certain values of d for that it is easy to find a solution of $x^2 - dy^2 = 1$. One does not have to look very far to solve $x^2 - 3y^2 = 1$ or $x^2 - 8y^2 = 1$. Indeed, there are categories of values of d for which some formula for a solution can be given. For example, 3 and 8 are both of the form $t^2 - 1$; what would a solution of $x^2 - (t^2 - 1)y^2 = 1$ be for an arbitrary value of the parameter t? Can you find more than one solution?

Determine the smallest pair (x, y) of positive integers that satisfies $x^2 - dy^2 = 1$ in each of the following special cases.

(a) $d = 2, 5, 10, 17, 26, \ldots, t^2 + 1, \ldots$.
(b) $d = 3, 6, 11, 18, 27, \ldots, t^2 + 2, \ldots$.
(c) $d = 2, 7, 14, 23, \ldots, t^2 - 2, \ldots$.
(d) $d = 2, 6, 12, 20, 30, \ldots, t^2 + t, \ldots$.
(e) $d = 7, 32, 75, \ldots, t^2 + (4t + 1)/3, \ldots$ (where t is 1 less than a multiple of 3.
(f) $d = 3, 14, 33, \ldots, t^2 + (3t + 1)/2, \ldots$ (where t is odd) .
Now we come to some tougher cases that do not seem to follow an easy pattern. Find at least one solution in positive integers to each of the following:
(g) $x^2 - 21y^2 = 1$.
(h) $x^2 - 22y^2 = 1$.
(i) $x^2 - 28y^2 = 1$.
(j) $x^2 - 19y^2 = 1$.
(k) $x^2 - 13y^2 = 1$.
(l) $x^2 - 29y^2 = 1$.
(m) $x^2 - 31y^2 = 1$.

Exploration 2.8. For which values of the integer d is $x^2 - dy^2 = -1$ solvable? In particular, is there a solution when d is a prime exceeding a multiple of 4 by 1? Do not look at the discussion for this exploration until you have completed working through Chapter 5.

Exploration 2.9. Which integers can be written in the form $x^2 - y^2$? $x^2 - 2y^2$? $x^2 - 3y^2$? $x^2 - dy^2$?

Exploration 2.10. Let $x_n = a + (n-1)d$ be the nth term of an arithmetic progression with initial term a and common difference d. The quantity $s_n = x_1 + x_2 + \cdots + x_n$ is called a partial sum of the series $x_1 + x_2 + x_3 + \cdots$. Must there be at least one partial sum that is a square? Even if the a and d are coprime? For which progressions is it true that every partial sum is a square? Suppose that there is one square partial sum; must there be infinitely many more?

Exploration 2.11. In Exploration 1.4 the equation $x^2 - 3y^2 = 1$ was considered. Its solutions are given by

$$(x, y) = (1, 0), (2, 1), (7, 4), (26, 15), (97, 56), (362, 209), \ldots .$$

What is special about the numbers 2, 26, and 362? The solutions for $x^2 - 6y^2 = 1$ are

$$(x, y) = (1, 0), (5, 2), (49, 20), (485, 198), (4801, 1960), \ldots .$$

Note the appearance of 5 and 485. You may also wish in this context to look at the solutions of $x^2 - 7y^2 = 1$ and $x^2 - 8y^2 = 1$. Are there other values of d for which the solutions of $x^2 - dy^2 = 1$ exhibit similar behavior?

2.5 Historical References

There are several books and papers concerned with the history of Pell's equation:

David M. Burton, *The History of Mathematics: An Introduction*. Allyn and Bacon, Newton, MA, 1985 [pp. 243, 250, 504].

Bibhutibhusan Datta and Avadhesh Narayan Singh, *History of Hindu Mathematics*, a source book, Asia Publishing House, Bombay, 1962.

Leonard Eugene Dickson, *History of the Theory of Numbers*, *Volume II: Diophantine Analysis*. Chelsea, New York, 1952 (reprint of 1920 edition) [Chapter XII].

Victor J. Katz, *A History of Mathematics: An Introduction*. (Harper-Collins, New York, 1993) [pp. 208–211, 555–556].

Morris Kline, *Mathematical Thought from Ancient to Modern Times*. Oxford University Press, New York, 1972 [pp. 278, 610, 611].

James R. Newman *editor*, *The World of Mathematics*, *Volume 1*. Simon and Schuster, New York, 1956 [pp. 197–198].

C.O. Selenius, Rationale of the Chakravala process of Jayadeva and Bhaskara II, *Historia Mathematica* 2 (1975), 167–184.

David E. Smith *ed.*, *A Source Book in Mathematics, Volume One* Dover, 1959 [pp. 214–216].

D.J. Struik *ed.*, *A Source Book in Mathematics, 1200–1800*, Harvard University Press, Cambridge, MA, 1969 [pages 29–31].

André Weil, *Number Theory: An Approach Through History from Hammurapi to Legendre*. Birkhäuser, Boston, 1983.

H.C. Williams, Solving the Pell's equation. *Proceedings of the Millennial Conference on Number Theory* (Urbana, IL, 2000) (M.A. Bennett *et al.*, editors), A.K. Peters, Boston, 2002.

Weil describes how rational approximations to the square root of 3 involved obtaining some solutions to Pell's equation. Burton, Dickson (pp. 342–345), and Newman mention the Archimedean cattle problem. Datta and Singh, Dickson (pp. 346–350), Katz, and Weil give quite a bit of attention to Indian mathematics, with Selenius giving an analysis of their method. Dickson and Weil give a lot of detail on European developments in the seventeenth and eighteenth centuries. Smith and Struik document a 1657 letter of Fermat in which he asserts that given any number not a square, there are infinitely many squares that when multiplied by the given number are one less than a square.

2.10, 2.11. See A.R. Beauregard and E.R. Suryanarayan, Arithmetic triangle, *Mathematics Magazine* (1997) 106–116.

5.9. *Oeuvres de Fermat* III, 457-480, 490-503; Dickson, p. 352.

2.6 Hints

2.3(a). If the two numbers are y^2 and x^2, what is $x^2 - 3y^2$?

2.4. In particular, $1 + b + b^2 = \frac{1}{2}v(v + 1)$ for some integer v.

2.5. Suppose that $2^2 + 4^2 + \cdots + (2n)^2 = m(m+1) + \cdots + (m+n-1)(m+n)$. The left side can be summed using the formula for the sum of the first r squares: $\frac{1}{6}r(r + 1)(2r + 1)$. The right side can be summed by expressing each term as a difference: $3x(x + 1) = x(x + 1)(x + 2) - (x - 1)x(x + 1)$. Show that this equation leads to $(n + 1)^2 = m(n + m)$, which can be rewritten as a Pythagorean equation: $n^2 + [2(n + 1)]^2 = (2m + n)^2$. At this point you can use the general formula for Pythagorean triples to get an equation of the form $x^2 - 5y^2 = 4$.

2.6. Square each equation and eliminate terms that are linear in each variable.

3.7. Rearrange the terms in the equation involving 151 to obtain a difference of squares on one side. The equation involving 313 is trickier. It may help to observe that $313 = 12^2 + 13^2$. It is easy to check divisibility of factors by powers of 2. Casting out 9's will help check divisibility by powers of 3. Check also for divisibility of factors by other small primes. Another way to compare divisors is as follows. Suppose we wish to show that $ab = cd$. We might look for common divisors of the two sides; one such would be the greatest common divisor of a and c. Finding such a greatest common divisor need not involve knowing the prime-power decomposition of the numbers. The Euclidean algorithm can be used.

Suppose $a > c$. Divide c into a and get a remainder $r = a - cq$, where q is the quotient. Then $\gcd(a, c) = \gcd(c, r)$. Now we have a smaller pair of numbers to work with. We can continue the process with c and r. Eventually, we will come to a pair of numbers one of which divides the other.

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